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TOPOLOGICAL MAGNITUDE OF A SPECIAL SUBSET IN A SELF-SIMILAR CANTOR SET

IN-SOO BAEK

ABSTRACT. We study the topological magnitude of a special subset from the distribution subsets in a self-similar Cantor set. The special subset whose every element has no accumulation point of a frequency sequence as some number related to the similarity dimension of the self-similar Cantor set is of the first category in the self-similar Cantor set.

1. INTRODUCTION

Recently the Hausdorff and packing dimensions of multifractal subsets by a self-similar measure on a self-similar Cantor set (cf. [1, 2, 3, 4, 9, 14, 15]) were studied ([5, 7, 8, 11]) for the investigation of the sizes of subsets of fixed local dimension. Further the self-similar Cantor set can be completely decomposed into a class of lower(upper) distribution sets deduced from a frequency sequence ([4, 6, 12, 13]). The class of lower(upper) distribution sets was used for investigating the Hausdorff and packing dimension information of the subset having same local dimension of a self-similar measure on the self-similar Cantor set. It sometimes gives rich information for the structure of the self-similar set. In particular, the packing dimension will give some information of the topological magnitude about some special subset from the distribution subsets in a self-similar Cantor set. In this paper, the term of the topological magnitude of a test subset in a self-similar Cantor set will be used for the test set to be of first category or of second category in the self-similar Cantor set. If the test set is of first category in the Cantor set, it can be considered as small for its topological magnitude. In the paper ([4]), singular value related to the similarity dimension of the self-similar Cantor set has a critical role to decide phase

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transition. Similarly in this paper, we will show that it has an important role for a subset to have a small topological magnitude or not. In this paper, we note that packing dimension is closely related to the topological magnitude. Further we will have our results from the facts that countable union of some subsets cannot increase its topological magnitude from the definition whereas their packing dimension and Hausdorff dimension increase their values. We also note that Olsen conjectured that the manipulation from his result ([12]) may give a better result than ours. However this paper gives a heuristic proof for the relation between packing dimension and topological magnitude together with the singular value related to the similarity dimension of the self-similar Cantor set.

2. PRELIMINARIES

We denote F a self-similar Cantor set, which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0,1]$ with $a > 0$, $b > 0$ and $1-(a+b) > 0$. Let $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{1,2\}$ and $1 \le j \le k$. We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). If $x \in F$ and $x \in I_\sigma$ where $\sigma \in \{1,2\}^k$, $c_k(x)$ denotes I_σ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \cdots$. Let $p \in (0,1)$ and we denote γ_p a self-similar Borel probability measure on F satisfying $\gamma_p(I_1) = p$ (cf. [8]). $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E ([8]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E ([8]). We denote $n_1(x|k)$ the number of times the digit 1 occurs in the first k places of $x = \sigma$ (cf. [10]).

For $r \in [0,1]$, we define lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 1 in proportion r by

$$\underline{F}(r) = \left\{ x \in F : \liminf_{k \to \infty} \frac{n_1(x|k)}{k} = r \right\},$$
$$\overline{F}(r) = \left\{ x \in F : \limsup_{k \to \infty} \frac{n_1(x|k)}{k} = r \right\}.$$

We write $\underline{E}_\alpha^{(p)}$ $(\overline{E}_\alpha^{(p)})$ for the set of points at which the lower(upper) local dimension of γ_p on F is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \left\{ x : \liminf_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\},$$

$$\overline{E}_\alpha^{(p)} = \left\{ x : \limsup_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\}.$$

Let $p \in (0,1)$ and consider a self-similar measure γ_p on F and let $r \in [0,1]$ and $g(r,p) = \frac{r \log p + (1-r)\log(1-p)}{r \log a + (1-r) \log b}$. We note that there exists a real number s satisfying $a^s + b^s = 1$, which is called the self-similarity dimension of a self-similar Cantor set F.

We note that each $x \in F$ has its frequency sequence $(n_1(x|k))_k$. In this paper, we will show that the subset T of points whose frequency sequence does not have a^s as an accumulation point is of first category in F.

3. Main Results

We note that a self-similar Cantor set F is of second category in itself since F is closed in a complete metric space R.

Lemma 1. *Let K be a subset in a real line and assume that $Dim(B_r(x) \cap K) = Dim(K)$ for any $x \in K$ and any radius $r > 0$. Then $E \subset K$ with $Dim(E) < Dim(K)$ must be of first category in K.*

Proof. Assume that E is of second category in K. If $E = \cup_n E_n$, there is some integer n such that $(\overline{E_n})^o \neq \phi$. Therefore there exists some $r > 0$ such that $B_r(x) \cap K \subset (\overline{E_n})^o$. Then from the definition of a dimensional index Δ ([16]),

$$\begin{aligned} Dim(E) &= \inf_{\cup E_n = E} (\sup_{(E_n)} \Delta(E_n)) = \inf_{\cup E_n = E} (\sup_{(E_n)} \Delta(\overline{E_n})_R) \\ &\geq \inf_{\cup E_n = E} (\sup_{(E_n)} \Delta(\overline{E_n})) = Dim(K), \end{aligned}$$

where $\Delta(\overline{A})_R$ means the dimensional index of the closure of A in R whereas $\Delta(\overline{A})$ means the dimensional index of the closure of A in F. It follows from the assumption with the fact that $\Delta(\overline{E_n}) \geq Dim(\overline{E_n}) \geq Dim(B_r(x) \cap K) = Dim(K)$. □

Example 1. The self-similar Cantor set F is an example K satisfying the above condition since the neighborhood of each point in F has a fundamental interval of some stage to construct F, which means it has the full structure of F.

Example 2. $\underline{F}(r), \overline{F}(r)$ are also the example K satisfying the above condition.

Theorem 2. *Let $\underline{H}(t) = \cup_{t \leq r \leq 1} \underline{F}(r)$ and $\overline{G}(t) = \cup_{0 \leq r \leq t} \overline{F}(r)$. Then $\underline{H}(t)$ where $a^s < t \leq 1$ is of first category. Similarly $\overline{G}(t)$ where $0 \leq t < a^s$ is of first category.*

Proof. It follows from Theorem 2 in [4] and the above Lemma. □

Theorem 3. $\dim(\underline{H}(t)) = Dim(\underline{H}(t)) = g(t,t)$ *where* $a^s < t \leq 1$. *Similarly* $\dim(\overline{G}(t)) = Dim(\overline{G}(t)) = g(t,t)$ *where* $0 \leq t < a^s$.

Proof. Let $x \in \overline{G}(t))$ where $0 \leq t < a^s$. Then $x \in \overline{F}(r)$ where $0 \leq r \leq t$. Since $\overline{F}(r) = \overline{E}^{(t)}_{g(r,t)}$ by the theorem 2 in [4], noting $g(r,t)$ is an increasing function for r where $0 \leq t < a^s$, we have

$$\limsup_{n\to\infty} \frac{\log \gamma_t(c_n(x))}{\log |c_n(x)|} = g(r,t) \leq g(t,t) < g(a^s, a^s) = s.$$

Since $g(t,t) < s$, $Dim(\overline{G}(t)) \leq g(t,t) < s$. Now $\dim(\overline{G}(t)) \geq \dim(\overline{F}(t)) = g(t,t)$. Hence $\dim(\overline{G}(t)) = Dim(\overline{G}(t)) \leq g(t,t) < s$. Dually we can prove that $\dim(\underline{H}(t)) = Dim(\underline{H}(t)) = g(t,t)$ where $a^s < t \leq 1$. □

Theorem 4. *Let* $T = \left[\cup_{n=1}^{\infty} \underline{H}(a^s + \frac{1}{n})\right] \cup \left[\cup_{n=1}^{\infty} \overline{G}(a^s - \frac{1}{n})\right]$. *Then* $\dim(T) = Dim(T) = s$.

Proof. We note that $\sup_{n\in N} g\big(a^s + \frac{1}{n}, a^s + \frac{1}{n}\big) = s = \sup_{n\in N} g\big(a^s - \frac{1}{n}, a^s - \frac{1}{n}\big)$. It follows from the σ-stability of Hausdorff and packing dimension and the above theorem. □

Theorem 5. *Let* $T = [\cup_{n=1}^{\infty} \underline{H}(a^s + \frac{1}{n})] \cup [\cup_{n=1}^{\infty} \overline{G}(a^s - \frac{1}{n})]$. *Then* T *is of first category in* F.

Proof. It follows since the countable union of subsets of first category is also of first category. □

Lemma 6. *For each* $x \in F$, x *has the accumulation points of the frequency sequence* $(n_1(x|k))_k$ *as* $\left[\liminf_{k\to\infty} \frac{n_1(x|k)}{k}, \limsup_{k\to\infty} \frac{n_1(x|k)}{k}\right]$.

Proof. It easily follows from the lemma 2.3 in [13]. □

Theorem 7. *Let* $T = \left[\cup_{n=1}^{\infty} \underline{H}(a^s + \frac{1}{n})\right] \cup \left[\cup_{n=1}^{\infty} \overline{G}(a^s - \frac{1}{n})\right]$. *Then every point* x *in* T *has no accumulation point of the frequency sequence* $(n_1(x|k))_k$ *as* a^s.

Proof. It follows from the above lemma. □

Remark 1. We note that the above set T has full Hausdorff and packing dimension s of the self-similar set F whereas T has small topological magnitude, which means T is of first category in the self-similar set F.

Remark 2. L. Olsen conjectured that $\underline{F}(0) \cap \overline{F}(1)$ is comeager in F in the view of his previous result ([12]).

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Department of Mathematics, Pusan University of Foreign Studies, Pusan 608-738, Korea
Email address: `isbaek@pufs.ac.kr`