퍼지 및 지능시스템학회 논문지 2007, Vol. 17, No. 1, pp. 143-147

Some properties of the convergence of sequences of fuzzy points in a fuzzy normed linear space

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Abstract

With a new ordinary norm as an analogy of Krishna and Sarma[5] and Bag and Samanta[1], we will characterize the notions of the convergence of the sequences of fuzzy points, the fuzzy α-Cauchy sequence and fuzzy completeness.

key words: fuzzy normed linear space, convergence of sequence of fuzzy points, α- Cauchy sequence of fuzzy points, α- fuzzy completeness and fuzzy completeness

1. 서 론

Katsaras and Liu [2] introduced the notions of fuzzy vector spaces and fuzzy topological vector spaces. These ideas were modified by Katsaras [3] and, in [4] Katsaras defined the fuzzy norm on a vector space. In [5] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Krishna and Sarma[6] introduced the notion of the convergence of sequence of fuzzy points in a fuzzy normed linear space and Rhie et al. [7] introduced the concepts of the α-Cauchy sequence of fuzzy points and fuzzy completeness in a fuzzy normed linear space.

In this paper, as an analogy of Krishna and Sarma[5] and Bag and Samanta[1], we observe that if ρ is a fuzzy norm on a linear space X, then for each ε $(0<\varepsilon<1)$,

$$N_\varepsilon(x) = \wedge\{t>0 \mid t\rho(x) \geq \varepsilon\}$$

gives an ordinary norm on X. With this norm we will characterize the notions of the convergence of sequences of fuzzy points, the fuzzy α-Cauchy sequence and fuzzy completeness in a fuzzy normed linear space. Also we prove a useful sufficient condition that a fuzzy norm on a linear space is fuzzy complete.

2. Preliminaries.

Throughout this paper, X is a vector space over the field $K(\mathbb{R}\ or\ \mathbb{C})$. Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of a set A. By a fuzzy point μ we mean a fuzzy subset

$\mu: X \to [0, 1]$ such that

$$\mu(z) = \begin{cases} \alpha, & \text{if } z = x \\ 0, & otherwise \end{cases}$$

here $\alpha \in (0, 1)$, and I^X denotes the set $\{\mu \mid \mu: X \to [0, 1]\}$. We usually denote the fuzzy point with support x and value α by (x, α)

Definition 2.1. For two fuzzy subset μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by $(\mu_1+\mu_2)(x) = sup_{x_1+x_2=x} \min\{\mu_1(x_1), \mu_2(x_2)\}$.

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(\frac{x}{t}) & \text{if } t \neq 0 \\ 0 & \text{if } t=0 \text{ and } x \neq 0 \\ sup_{y\in X}\mu(y) & \text{if } t=0 \text{ and } x=0 \end{cases}$$

Definition 2.2 [2]. $\mu \in I^X$ is said to be

i) convex if $t\mu + (1-t)\mu \leq \mu$ for each $t \in [0, 1]$

ii) balanced if $t\mu \leq \mu$ for each $t \in K$ with $|t| \leq 1$

iii) absorbing if $sup_{t>0} t\mu(x) = 1$ for all $x \in X$.

Definition 2.3 [3]. Let (X, γ) be a topological space and $\omega(\gamma) = \{f: X \to [0, 1] \mid lowersemicontinuous\}$, Then $\omega(\gamma)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by γ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

Definition 2.4 [2]. A fuzzy linear topology on a vector space X over K is a fuzzy topology on X such that the two mappings

$+ : X \times X \to X,\ (x, y) \to x + y$

$\cdot : K \times K \to X,\ (t, x) \to tx$

접수일자 : 2006년 11월 20일
완료일자 : 2006년 12월 19일

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a fuzzy linear space or a fuzzy topological vector space.

Definition 2.5[6]. Let (X, γ) be a fuzzy topological space. A fuzzy subset μ in X is called a neighbourhood of (x, α) if there exists $\phi\in\gamma$ with $\phi(x)\geq\alpha$ and $\phi\leq\mu$.

Definition 2.6[4]. A fuzzy seminorm on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $inf_{t>0}t\rho(x)=0$ for every nonzero x, then ρ is called a fuzzy norm.

Definition 2.7[4]. If ρ is a fuzzy seminorm on X, then the family

$B_\rho=\{\theta\wedge(t\rho)|0<\theta\leq 1, t>0\}$ is a base for a fuzzy linear topology τ_ρ, where $\theta\wedge(t\rho)$ is the function $X\rightarrow[0,1]$ such that $(\theta\wedge(t\rho))(x)=\min\{\theta, (t\rho)(x)\}$.

Definition 2.8[4]. Let ρ be a seminorm on a linear space. The fuzzy topology τ_ρ in definition 2.7 is called the fuzzy topology induced by the fuzzy seminorm ρ. And a linear space equipped with a fuzzy seminorm (resp.fuzzy norm) is called a fuzzy seminormed (resp.fuzzy normed) linear space.

Theorem 2.9. If ρ is a fuzzy seminorm on X, then the function $N_\varepsilon: X\rightarrow\mathbb{R}^+$ defined by

$N_\varepsilon(x)=inf\{t>0|t\rho(x)\geq\varepsilon\}$ for each $\varepsilon\in(0,1)$ is a seminorm on X. Futher N_ε is a norm on X for each $\varepsilon\in(0,1)$ if and only if ρ is a fuzzy norm on X.

Proof. First,

$$N_\varepsilon(0)=\wedge\{t>0:(t\rho)(0)\geq\varepsilon\}$$
$$=\wedge\{t>0:\rho(0)\geq\varepsilon\}=0.$$

Next, let $\alpha\neq 0$ and $x\in X$. Then

$$N_\varepsilon(\alpha x)=\wedge\{t>0:(t\rho)(\alpha x)\geq\varepsilon\}$$
$$=\wedge\left\{t>0:(\frac{t}{\alpha}\rho)(x)\geq\epsilon\right\},$$

Since ρ is balanced, $\rho(x)=\rho(-x)$ for every $x\in X$. And we get $\frac{t}{\alpha}\rho=\frac{t}{|\alpha|}\rho$ $(\alpha\neq 0)$.

Now $\wedge\left\{t>0:\frac{t}{\alpha}\rho(x)\geq\varepsilon\right\}=\left\{t>0:\frac{t}{|\alpha|}\rho(x)\geq\varepsilon\right\}$

Putting $u=t/|\alpha|$, then

$N_\varepsilon(\alpha x)=\wedge\{|\alpha|u>0:u\rho(x)\geq\varepsilon\}=|\alpha|N_\varepsilon(x)$

For any $x, y\in X$ we define

$A(x;\varepsilon)=\{t>0: t\rho(x)\geq\varepsilon\}$. We claim that $A(x;\varepsilon)+A(y;\varepsilon)\subseteq A(x+y;\varepsilon)$.

If $t\in A(x;\varepsilon)$ and $s\in A(y;\varepsilon)$, then $t\rho(x)\geq\varepsilon$ and $s\rho(y)\geq\varepsilon$.

Also $(t+s)\rho(x+y)=\rho\left(\frac{x+y}{t+s}\right)$

$$=\rho\left(\frac{t}{t+s}\left(\frac{x}{t}\right)+\frac{s}{t+s}\left(\frac{y}{s}\right)\right)\geq\rho\left(\frac{x}{t}\right)\wedge\rho\left(\frac{y}{s}\right)$$

using the convexity of ρ. This shows that $t+s\in A(x+y;\varepsilon)$. The triangle inequality for N_ε now follows by taking infima.

If ρ is a fuzzy norm, $x\in X$ and $N_\varepsilon(x)=0$, then $inf\{t>0:(t\rho)(x)\geq\varepsilon\}=0$ and hence $(t\rho)(x)\geq\varepsilon$ for any $t>0$. This implies that $\wedge_{t>0}(t\rho)(x)\neq 0$ and by the definition of fuzzy norm that $x=0$. And N_ε is a norm on X. Conversely, if N_ε is a norm for each $\varepsilon\in(0,1)$, let $x\in X$ and $x\neq 0$. Then $N_\varepsilon(x)=\delta>0$. From the definition of N_ε it then follows that for any $s\in(0,\delta)$, $s\rho(x)<\varepsilon$. Hence for any $\varepsilon>0$, there exists a $\delta>0$ such that for all $s\in(0,\delta)$, $s\rho(x)<\varepsilon$, implying $\wedge_{t>0}(t\rho)(x)=0$. This shows that ρ is a fuzzy norm on X.

3. Fuzzy convergence of sequences of fuzzy points and fuzzy completeness.

In this section, we characterize the convergence of sequences of fuzzy points of Krishna and Ssrma[6] and the fuzzy Cauchy sequence and fuzzy completeness of Rhie et al.[7], and investigate some related properties of fuzzy normed linear spaces.

Definition 3.1[6]. Let $(X, \mathfrak{T})$be a fuzzy topological space,

$\{\mu_n=(x_n, \alpha_n)\}$ be a sequence of fuzzy points in X and μ be a fuzzy point in X. We say that $\{\mu_n\}$ converges to μ, written as $\mu_n\rightarrow\mu$ if and only if for every neighbourhood N of μ there exists a positive integer M such that $n\geq M$ implies $\mu_n\leq N$.

Theorem 3.2. Let (X, ρ) be a fuzzy normed vector space and

$\{\mu_n=(x_n, \alpha_n)\}$ be a sequence of fuzzy points in X and $\mu=(x, \alpha)$ a fuzzy point in X. Then $\mu_n\rightarrow\mu$ if and only if for every $t>0$, there exists $M\in\mathbb{Z}^+$such that for all $n\geq M$, $\alpha_n\leq\alpha$ and $N_{\alpha_n}(x_n-x)<t$.

Proof. Suppose $\mu_n\rightarrow\mu$. Let $t>0$. Write $N=x+\alpha\wedge\frac{1}{2}t\rho$. Then N is a neighbourhood of μ. So by Definition 3.1, there exist $M\in\mathbb{Z}^+$ such that, for all $n\geq M$, $\mu_n\leq x+\alpha\wedge\frac{1}{2}t\rho$; that is for all $n\geq M$, $\alpha_n\leq\alpha\wedge\frac{1}{2}t\rho(x_n-x)$, which means for all $n\geq M$, $\alpha_n\leq\alpha$ and

$\alpha_n\leq\frac{1}{2}t\rho(x_n-x)$, that is for all $n\geq M$, $\alpha_n\leq\alpha$ and

$N_{\alpha_n}(x_n - x) \le \frac{1}{2} t < t$. Hence for every $t>0$, there exists $M \in \mathbb{Z}^+$ such that for all $n \ge M$, $\alpha_n \le \alpha$ and $N_{\alpha_n}(x_n - x) < t$.

Conversely, suppose that for every $t>0$, there exists $M = M(t) \in \mathbb{Z}^+$ such that for all $n \ge M$, $\alpha_n \le \alpha$ and $N_{\alpha_n}(x_n - x) < t$. Let N be a neighbourhood of μ.

Then N contains a neighbourhood of the form $x + \alpha \wedge t\rho$ for some $t>0$.

For this t by hypothesis there exists $M \in \mathbb{Z}^+$ such that for all

$n \ge M$, $\alpha_n \le \alpha$, $N_{\alpha_n}(x_n - x) < t$.

Now, for all $n \ge M$, we have

$$N_{\alpha_n}(x_n - x) < t \Rightarrow t\rho(x_n - x) \ge \alpha_n$$
$$\Rightarrow x + \alpha \wedge t\rho(x_n) \ge \alpha_n$$
$$\Rightarrow \mu_n \le x + \alpha \wedge t\rho$$

for all $n \ge M$. This implies that $\mu_n \le N$ for all $n \ge M$. Hence the sequence $\{\mu_n\}$ converges to μ.

Theorem 3.3. Let (X, ρ) be a fuzzy normed vector space over the filed K.

(a) If $(x_n, \alpha_n) = \mu_n \to \mu = (x, \alpha)$ and $(y_n, \beta_n) = \nu_n \to \nu = (y, \beta)$, then $\mu_n + \nu_n = (x_n + y_n, \alpha_n \wedge \beta_n)$ converges to $((x+y), \alpha \wedge \beta) = \mu + \nu$

(b) If $\{t_n\} \subseteq K$, $t \in K$ and $t_n \to t$, $\mu_n = (x_n, \alpha_n) \to \mu = (x, \alpha)$, then $t_n \mu_n = (t_n x_n, \alpha_n) \to t\mu = (tx, \alpha)$.

Proof. (a) Let $t>0$ be given. So there exists M in $\mathbb{Z}^+$ such that for all $n \ge M$,

$\alpha_n \le \alpha$, $N_{\alpha_n}(x_n - x) < \frac{1}{2} t$,

$\beta_n \le \beta$, $N_{\beta_n}(y_n - y) < \frac{1}{2} t$. Now,

$$N_{(\alpha_n \wedge \beta_n)}(x_n + y_n - \overline{x+y})$$
$$\le N_{\alpha_n}(x_n - x) + N_{\beta_n}(y_n - y).$$

Hence, for all $n \ge M$. We have

$\alpha_n \wedge \beta_n \le \alpha \wedge \beta$ and

$N_{(\alpha_n \wedge \beta_n)}(x_n + y_n - \overline{x+y}) < t$,

which proves (a).

(b) Let $s>0$ be given. Then there exists $M_1 \in \mathbb{Z}^+$ such that for all $n \ge M_1$,

$\alpha_n \le \alpha$, and $N_{\alpha_n}(x_n - x) < \frac{s}{2k}$ where k is a positive number such that

$|t_n| \le k$ for all n. Also, since $t_n \to t$ there exists $M_2 \in \mathbb{Z}^+$ such that for all

$n \ge M_2$, $|t_n - t| \le \frac{s}{[2N_\alpha(x) + 1]}$.

Let $M = \max\{M_1, M_2\}$. Now,

$$N_{\alpha_n}(t_n x_n - tx) = N_{\alpha_n}(t_n(x_n - x) + (t_n - t)x)$$
$$< |t_n| N_{\alpha_n}(x_n - x) + |(t_n - t)| N_{\alpha_n}(x)$$
$$< k \cdot \frac{s}{2k} + \left[\frac{s}{(2N_\alpha(x)+1)}\right] N_{\alpha_n}(x) \quad n \ge M$$
$$< \frac{1}{2} s + \frac{1}{2} s \le s,$$

which proves (b).

Definition 3.4[7]. Let $\alpha \in (0, 1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a fuzzy α-Cauchy sequence in a fuzzy normed linear space (X, ρ) if for each zero neighbourhood N with $N(0) > \alpha$, there exists a positive integer M such that $n, m \ge M$ implies

$$\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \le N.$$

Theorem 3.5. Let (X, ρ) be a fuzzy normed linear space and $\alpha \in (0, 1)$.

Then $\{(x_n, \alpha_n)\}$ is a fuzzy α-Cauchy sequence if and only if for each $t>0$, there exists a positive integer M such that $m, n \ge M$ implies $\alpha_n \wedge \alpha_m \le \alpha$ and $N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$

Proof. Assume that $\{(x_n, \alpha_n)\}$ is a fuzzy α-Cauchy sequence and $t>0$. Let $\beta > \alpha$ and $N = \beta \wedge \frac{1}{2} t\rho$, where $(\beta \wedge \frac{1}{2} t\rho)(x) = \min\left\{\beta \wedge \rho(\frac{2x}{t})\right\}$. Since $N(0) > \alpha$, there exists a positive integer M such that for $n, m \ge M$,

$$\beta \wedge \frac{1}{2} t\rho(x_n - x_m) \ge \alpha_n \wedge \alpha_m$$
$$\Rightarrow \alpha_n \wedge \alpha_m \le \beta \text{ and } \frac{1}{2} t\rho(x_n - x_m) \ge \alpha_n \wedge \alpha_m$$
$$\Rightarrow \alpha_n \wedge \alpha_m \le \beta \text{ and } N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) \le \frac{1}{2} t < t$$

Therefore,

$\alpha_n \wedge \alpha_m \le \alpha$ and $N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$.

For the converse let N be a neighbourhood of zero with $N(0) > \alpha$. Then there exist $t>0$ and $\alpha' > \alpha$ such that $\alpha' \wedge t\rho \le N$. For this t, there exists a positive integer M such that for $n, m \ge M$, $\alpha_n \wedge \alpha_m \le \alpha$ and

$$N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$$
$$\Rightarrow t\rho(x_n - x_m) \ge \alpha_n \wedge \alpha_m$$
$$\Rightarrow \alpha \wedge t\rho(x_n - x_m) \ge \alpha_n \wedge \alpha_m$$
$$\Rightarrow \alpha \wedge t\rho \ge (x_n - x_m, \alpha_n \wedge \alpha_m)$$

Therefore,

$N \ge \alpha' \wedge t\rho \ge \alpha \wedge t\rho \ge (x_n - x_m, \alpha_n \wedge \alpha_m)$.

The proof is completed.

Corollary 3.6. Any subsequence of a fuzzy α-Cauchy sequence is also a fuzzy α-Cauchy sequence.

Note. Let $\alpha < \alpha'$. Every fuzzy α-Cauchy sequence is a α'-Cauchy sequence and
$N_{\alpha'}$-Cauchy sequence is a N_{α}-Cauchy sequence.

Theorem 3.7. If $\{(x_n, \alpha_n)\}$ converges to (X, α) then it is a fuzzy α-Cauchy sequence.

Proof. Since $\{(x_n, \alpha_n)\}$converges to (X, α), for every $t>0$, there exists a positive integer M such that $n \geq M$ implies $\alpha_n \leq \alpha$ and $N_{\alpha_n}(x_n - x) < \frac{1}{2}t$. If $m, n \geq M$, then $\alpha_n \wedge \alpha_m \leq \alpha$ and

$$N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m)$$
$$\leq N_{(\alpha_n \wedge \alpha_m)}(x_n - x) + N_{(\alpha_n \wedge \alpha_m)}(x_m - x)$$
$$\leq N_{\alpha_n}(x_n - x) + N_{\alpha_m}(x_m - x) < \frac{1}{2}t + \frac{1}{2}t = t$$

Therefore, $\{(x_n, \alpha_n)\}$ is a fuzzy α-Cauchy sequence. This completes the proof.

Now, we consider some relations between the fuzzy completeness and ordinary completeness on a linear space X.

Definition 3.8[7]. A fuzzy normed linear space (X, ρ) is said to be fuzzy α-complete if every fuzzy α-Cauchy sequence $\{\mu_n\}$ converges to a fuzzy point $\mu = (x, \alpha)$.

(X, ρ) is said to be fuzzy complete if it is fuzzy α-complete for every $\alpha \in (0, 1)$.

Theorem 3.9. Let $(X, \|\cdot\|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy complete, where $B = \{x \in X \mid \|x\| \leq 1\}$.

Proof. Fix $\alpha \in (0, 1)$ and let $\{(x_n, \alpha_n)\}$ be a fuzzy α-Cauchy sequence of fuzzy points in X. For every $t>0$, there exists a positive integer M such that $m, n \geq M$
implies $\alpha_n \wedge \alpha_m \leq \alpha$ and
$N_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) = \|x_n - x_m\| < t$. That is, $\{x_n\}$ is a crisp Cauchy sequence in X. Since X is complete, there exists an $x \in X$ such that $\|x_n - x\|$ converges to 0. Therefore, for every $t > 0$, there exists a positive integer M such that for $n \geq M$, $\alpha_n \leq \alpha$ and
$N_{\alpha_n}(x_n - x) = \|x_n - x\| < t$.

Hence $\{(x_n, \alpha_n)\}$converges to (x, α) with respect to the topology τ_{χ_B}.

Therefore, (X, χ_B) is fuzzy α-complete for each $\alpha \in (0, 1)$. This completes the proof of the theorem.

corollary 3.10. The field $K(\mathbb{R}\ or \mathbb{C})$ with the fuzzy topology generated by the usual topology on K is a complete fuzzy normed linear space.

Definition 3.11[4]. Two fuzzy
seminorms ρ_1, ρ_2 on X are said to be equivalent if $\tau_{\rho_1} = \tau_{\rho_2}$.

Theorem 3.12[4] The fuzzy seminorms ρ_1, ρ_2 on a linear space X are equivalent if and only if for each $\theta \in (0, 1)$, there exists $t>0$ such that $\theta \wedge \rho_1(tx) \leq \rho_2(x)$ and $\theta \wedge \rho_2(tx) \leq \rho_1(x)$ for all $x \in X$.

Theorem 3.13[7]. Let ρ be a lower semicontinuous fuzzy norm on a normed linear space X, and have the bounded support : $\{x \in X \mid \rho(x) > 0\}$ is bounded. Then ρ is equivalent to the fuzzy norm χ_B, where B is the closed unit ball of X.

Theorem 3.14. If X is a Banach space and ρ is a lower semicontinuous fuzzy norm having the bounded support, then the fuzzy normed linear space (X, ρ) is fuzzy complete.

Proof. From Theorem 3.13, ρ is equivalent to χ_B and so (X, ρ) is fuzzy complete by Theorem 3.9.

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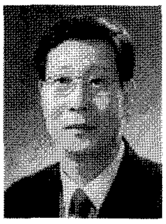

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