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INTUITIONSITIC FUZZY G-CONGRUENCES

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Abstract

We introduce the concept of intuitionistic fuzzy G-equivalence relations (congruence), and we obtain some results. Furthermore, we prove that $IFC_G(K)$ is isomorphic to $IFN^*(K)$ for any group K. Also, we prove that $(IFC_{G,(\lambda,\mu)}/\sim,*)$ and $(IFNG_{(\lambda,\mu)}(K),\circ)$ are isomorphic.

Key words : intuitionistic fuzzy set, intuitionistic fuzzy G-equivalence relation, intuitionistic fuzzy G-congruence, intuitionistic fuzzy right (left) conformable.

1. 0.Introduction

The concept of a fuzzy sets was introduced by Zadeh[21] in 1965, and since then these has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behavior studies. In particular, many researchers [7,17,19,20,22] applied the notion of a fuzzy set to relations and congruences.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov[1] in 1986. After that time, various researchers [2-6,9-12,14] applied the notion of intuitionistic fuzzy sets to relation, group theory and topology. In particular, Hur and his colleagues [13,15] introduce the notion of intuitionistic fuzzy congruences on a lattice and a semigroup, and investigate some of it's properties, respectively. Moreover, Hur and his colleagues [16] studied intuitionistic fuzzy congruences in the sense of lattice.

In this paper, we introduce the concept of intuitionistic fuzzy G-equivalence relations (congruence), and we obtain some results. Furthermore, we prove that $IFC_G(K)$ is isomorphic to $IFN^*(K)$ for any group K, where $IFC_G(K)$ [resp. $IFN^*(K)$] denotes the set of all intuitionistic fuzzy G-congruences on K [resp. intuitionistic fuzzy nonempty normal subgroups of G]. Also, we prove that $(IFC_{G,(\lambda,\mu)}/\sim,*)$ and $(IFNG_{(\lambda,\mu)}(K),\circ)$ are isomorphic.

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2. Preliminaries

In this section, we list some basic concepts and one result which are needed in the later sections.

For sets X, Y and Z, $f=(f_1,f_2): X\to Y\times Z$ is called a *complex mapping* if $f_1: X\to Y$ and $f_2: X\to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I.

Definition 2.1[1,5]. Let X be a nonempty set. A complex mapping $A=(\mu_A,\nu_A): X\to I\times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x)+\nu_A(x)\leq 1$ for each $x\in X$, where the mappings $\mu_A: X\to I$ and $\nu_A: X\to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x\in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x)=(0,1)$ and $1_\sim(x)=(1,0)$ for each $x\in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 2.2[1]. Let X be a nonempty set and let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be IFSs in X. Then:

(1) $A\subset B$ iff $\mu_A\leq\mu_B$ and $\nu_A\geq\nu_B$.

(2) $A=B$ iff $A\subset B$ and $B\subset A$.

(3) $A^c=(\nu_A,\mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 2.3[5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then:
(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 2.4[4]. Let X be a set. Then a complex mapping $R = (\mu_R, \nu_R) : X \times X \rightarrow I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times X$, i.e., $R \in$ IFS$(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 2.5[4]. Let $R \in$ IFR(X). Then the *inverse* of R, R^{-1} is defined as by $R^{-1}(x, y) = R(y, x)$ for any $x, y \in X$.

Definition 2.6[4,6]. Let X be a set and let $R, Q \in$ IFR(X). Then the *composition* of R and Q, $Q \circ R$, is defined as follows: For any $x, y \in X$,

$$\mu_{Q \circ R}(x, y) = \bigvee_{z \in X}[\mu_R(x, z) \wedge \mu_Q(z, y)]$$

and

$$\nu_{Q \circ R}(x, y) = \bigwedge_{z \in X}[\nu_R(x, z) \vee \nu_Q(z, y)].$$

Definition 2.7[4]. An Intuitionistic fuzzy Relation R on a set X is called an *intuitionistic fuzzy equivalence relation* (in short, *IFER*) on X if it satisfies the following conditions:
(i) it is *intuitionistic fuzzy reflexive*, i.e., $R(x, x) = (1, 0)$ for each $x \in X$.
(ii) it is *intuitionistic fuzzy symmetric*, i.e., $R^{-1} = R$.
(iii) it is *intuitionistic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \rightarrow I \times I$ as follows: For each $x \in X$,

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in$ IFS(X). The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by X/R.

Result 2.A[14, Theorem 2.15]. Let R be an intuitionistic fuzzy equivalence relation on a set X. Then the followings hold:
(1) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.
(2) $R(a, b) = (0, 1)$ if and only if $Ra \cap Rb = 0_{\sim}$ for any $a, b \in X$.
(3) $\bigcup_{a \in X} Ra = 1_{\sim}$.
(4) There exists the surjection $p : X \rightarrow X/R$ defined by $p(x) = Rx$ for each $x \in X$.

3. Intuitionistic fuzzy G-equivalence relations

Definition 3.1. Let R be an intuitionistic fuzzy relation on a set X. Then R is said to be *G-reflexive* if for any $x, y \in X$ with $x \neq y$,
(i) $\mu_R(x, x) > 0$ and $\nu_R(x, x) < 1$,
(ii) $\mu_R(x, y) \leq \delta_1(R)$ and $\nu_R(x, y) \geq \delta_2(R)$, where $\delta_1(R) = \bigwedge_{t \in X} \mu_R(t, t)$ and $\delta_2(R) = \bigvee_{t \in X} \nu_R(t, t)$.

An intuitionistic fuzzy G-reflexive and transitive relation on X is called an *intuitionistic fuzzy G-preorder* on S. An intuitionistic fuzzy symmetric G-preorder on X is called an *intuitionistic fuzzy G-equivalence relation* on X. We will denote the set of all intuitionistic fuzzy G-equivalence relations on X as $IFE_G(X)$.

Proposition 3.2. (1) If H and K are intuitionistic fuzzy G-reflexive relations on a set X, then $(K \circ H)(x, x) = (H \cap K)(x, x)$ for each $x \in X$.

(2) If R is an intuitionistic fuzzy G-preorder on a set X, then $R \circ R = R$.

Proof. (1) Let $x \in X$. Then

$$\mu_{H \circ K}(x, x) = \bigvee_{t \in X}[\mu_K(x, t) \wedge \mu_H(t, x)]$$
$$= \mu_K(x, x) \wedge \mu_H(x, x) \text{ (Since } H \text{ and } K \text{ are intuitionistic fuzzy } G\text{-reflexive)}$$
$$= \mu_{H \cap K}(x, x)$$

and

$$\nu_{H \circ K}(x, x) = \bigwedge_{t \in X}[\nu_K(x, t) \vee \nu_H(t, x)]$$
$$= \nu_K(x, x) \vee \nu_H(x, x)$$
$$= \nu_{H \cap K}(x, x).$$

Hence $(K \circ H)(x, x) = H \cap K(x, x)$ for each $x \in X$.

(2) Since R is intuitionistic fuzzy transitive, $R \circ R \subset R$. Let $x, y \in X$. Then

$$\mu_{R \circ R}(x, y) = \bigvee_{t \in X}[\mu_R(x, t) \wedge \mu_R(t, y)]$$
$$\geq \mu_R(x, x) \wedge \mu_R(x, y) \text{(Since } R \text{ is intuitionistic fuzzy } G\text{-reflexive)}$$
$$= \mu_R(x, y)$$

and

$$\nu_{R \circ R}(x, y) = \bigwedge_{t \in X}[\nu_R(x, t) \vee \nu_R(t, y)]$$
$$\leq \nu_R(x, x) \vee \nu_R(x, y)$$
$$= \nu_R(x, y).$$

Thus $R \subset R \circ R$. Hence $R \circ R = R$. □

Proposition 3.3 If H and K are intuitionistic fuzzy G-equivalence relations on a set X, then $H \cap K$ is so on X.

Proof. It is clear that $H \cap K$ is intuitionistic fuzzy G-reflexive and intuitionistic fuzzy symmetric. Let $x, y \in X$.

Then:

$$\begin{aligned}
&\mu_{H\cap K}(x,y)\\
&= \mu_H(x,y)\wedge\mu_K(x,y)\\
&\geq \mu_{H\circ H}(x,y)\wedge\mu_{K\circ K}(x,y)\\
&\quad \text{(Since } H \text{ and } K \text{ are intuitionistic fuzzy transitive)}\\
&= (\textstyle\bigvee_{z_1\in X}[\mu_H(x,z_1)\wedge\mu_H(z_1,y)])\\
&\qquad \wedge(\textstyle\bigvee_{z_2\in X}[\mu_K(x,z_2)\wedge\mu_K(z_2,y)])\\
&= \textstyle\bigvee_{(z_1,z_2)\in X\times X}([\mu_H(x,z_1)\wedge\mu_H(z_1,y)]\\
&\qquad \wedge[\mu_K(x,z_2)\wedge\mu_K(z_2,y)])\\
&\geq \textstyle\bigvee_{z_1\in X}([\mu_H(x,z_1)\wedge\mu_H(z_1,y)]\\
&\qquad \wedge[\mu_K(x,z_1)\wedge\mu_K(z_1,y)])\\
&= \textstyle\bigvee_{z_1\in X}([\mu_H(x,z_1)\wedge\mu_K(x,z_1)]\\
&\qquad \wedge[\mu_H(z_1,y)\wedge\mu_K(z_1,y)])\\
&= \textstyle\bigvee_{z_1\in X}[\mu_{H\cap K}(x,z_1)\wedge\mu_{H\cap K}(z_1,y)]\\
&= \mu_{(H\cap K)\circ(H\cap K)}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{H\cap K}(x,y)\\
&= \nu_H(x,y)\vee\nu_K(x,y)\\
&\leq \nu_{H\circ H}(x,y)\vee\nu_{K\circ K}(x,y)\\
&= (\textstyle\bigwedge_{z_1\in X}[\nu_H(x,z_1)\vee\nu_H(z_1,y)])\\
&\qquad \vee(\textstyle\bigwedge_{z_2\in X}[\nu_K(x,z_2)\vee\nu_K(z_2,y)])\\
&= \textstyle\bigwedge_{(z_1,z_2)\in X\times X}([\nu_H(x,z_1)\vee\nu_H(z_1,y)]\\
&\qquad \vee[\nu_K(x,z_2)\vee\nu_K(z_2,y)])\\
&\leq \textstyle\bigwedge_{z_1\in X}([\nu_H(x,z_1)\vee\nu_H(z_1,y)]\\
&\qquad \vee[\nu_K(x,z_1)\vee\nu_K(z_1,y)])\\
&= \textstyle\bigwedge_{z_1\in X}([\nu_H(x,z_1)\vee\nu_K(x,z_1)]\\
&\qquad \vee[\nu_H(z_1,y)\vee\nu_K(z_1,y)])\\
&= \textstyle\bigwedge_{z_1\in X}[\nu_{H\cap K}(x,z_1)\vee\nu_{H\cap K}(z_1,y)]\\
&= \nu_{(H\cap K)\circ(H\cap K)}(x,y).
\end{aligned}$$

Thus $H\cap K$ is intuitionistic fuzzy transitive. Hence $H\cap K$ is an intuitionistic fuzzy G-equivalence relation on X. □

If H and K are intuitionistic fuzzy G-reflexive relation on a set X, then $K\circ H$ may not be intuitionistic fuzzy G-reflexive.

Example 3.4. Let $X=\{a,b\}$. Let H and K be the intuitionistic fuzzy relations defined as follows:

$$H(a,a)=(1,0),\ \ H(b,b)=(\tfrac{1}{3},\tfrac{2}{3}),$$
$$H(a,b)=(\tfrac{1}{4},\tfrac{3}{4}),\ \ H(b,a)=(\tfrac{1}{5},\tfrac{4}{5})$$

and

$$K(a,a)=(1,0),\ \ K(b,b)=(\tfrac{1}{2},\tfrac{1}{2}),$$
$$K(a,b)=(\tfrac{1}{2},\tfrac{1}{2}),\ \ K(b,a)=(\tfrac{1}{4},\tfrac{3}{4}).$$

Then clearly, H and K are both intuitionistic fuzzy G-reflexive on X. But

$$\mu_{K\circ H}(a,b)=\tfrac{1}{2}>\tfrac{1}{3}=\mu_{K\circ H}(b,b)$$

and

$$\nu_{K\circ H}(a,b)=\tfrac{1}{4}<\tfrac{2}{3}=\nu_{K\circ H}(b,b).$$

So $K\circ H$ is not intuitionistic fuzzy G-reflexive on X.

Proposition 3.5. Let H and K be intuitionistic fuzzy G-reflexive relations on a set X such that $\mu_H(x,y)\vee\mu_K(x,y)\leq\delta_1(H)\wedge\delta_1(K)$ and $\nu_H(x,y)\wedge\nu_K(x,y)\geq\delta_2(H)\vee\delta_2(K)$ for any $x,y\in X$ with $x\neq y$. Then $K\circ H$ is intuitionistic fuzzy G-reflexive on X with $\delta_1(K\circ H)=\delta_1(H)\wedge\delta_1(K)$ and $\delta_2(K\circ H)=\delta_2(H)\vee\delta_2(K)$.

Proof. Let $x\in X$. Since H and K are intuitionistic fuzzy G-reflexive, by Proposition 3.2(1), $\mu_{K\circ H}(x,x)=\mu_H(x,x)\wedge\mu_K(x,x)>0$ and $\nu_{K\circ H}(x,x)=\nu_H(x,x)\vee\nu_K(x,x)<1$. Thus

$$\begin{aligned}
\delta_1(K\circ H)&=\textstyle\bigwedge_{t\in X}\mu_{K\circ H}(t,t)\\
&=\textstyle\bigwedge_{t\in X}[\mu_H(t,t)\wedge\mu_K(t,t)]\\
&=(\textstyle\bigwedge_{t\in X}\mu_H(t,t))\wedge(\textstyle\bigwedge_{t\in X}\mu_K(t,t))\\
&=\delta_1(H)\wedge\delta_1(K)
\end{aligned}$$

and

$$\begin{aligned}
\delta_2(K\circ H)&=\textstyle\bigvee_{t\in X}\nu_{K\circ H}(t,t)\\
&=\textstyle\bigvee_{t\in X}[\nu_H(t,t)\vee\nu_K(t,t)]\\
&=(\textstyle\bigvee_{t\in X}\nu_H(t,t))\vee(\textstyle\bigvee_{t\in X}\nu_K(t,t))\\
&=\delta_2(H)\vee\delta_2(K).
\end{aligned}$$

Now let $x,y\in X$ with $x\neq y$, and let $t\in X$ with $x\neq t\neq y$. Since H and K are intuitionistic fuzzy G-reflexive, $\mu_H(x,t)\wedge\mu_K(t,y)\leq\delta_1(H)\wedge\delta_1(K)$ and $\nu_H(x,t)\vee\nu_K(t,y)\geq\delta_2(H)\vee\delta_2(K)$. Also, by the hypothesis,

$$\begin{aligned}
\mu_H(x,x)\wedge\mu_K(x,y)&\leq\mu_K(x,y)\\
&\leq\mu_H(x,y)\vee\mu_K(x,y)\\
&\leq\delta_1(H)\wedge\delta_1(K),\\
\nu_H(x,x)\vee\nu_K(x,y)&\geq\nu_K(x,y)\\
&\geq\nu_H(x,y)\wedge\nu_K(x,y)\\
&\geq\delta_2(H)\vee\delta_2(K),
\end{aligned}$$

and

$$\begin{aligned}
\mu_H(x,y)\wedge\mu_K(y,y)&\leq\mu_H(x,y)\\
&\leq\mu_H(x,y)\vee\mu_K(x,y)\\
&\leq\delta_1(H)\wedge\delta_1(K),\\
\nu_H(x,y)\vee\nu_K(y,y)&\geq\nu_H(x,y)\\
&\geq\nu_H(x,y)\wedge\nu_K(x,y)\\
&\geq\delta_2(H)\vee\delta_2(K).
\end{aligned}$$

So $\mu_{K\circ H}(x,y)\leq\delta_1(H)\wedge\delta_1(K)=\delta_1(K\circ H)$ and $\nu_{K\circ H}(x,y)\geq\delta_2(H)\vee\delta_2(K)=\delta_2(K\circ H)$. Hence $K\circ H$ is intuitionistic fuzzy G-reflexive with $\delta_1(K\circ H)=\delta_1(H)\wedge\delta_1(K)$ and $\delta_2(K\circ H)=\delta_2(H)\vee\delta_2(K)$. □

The following is the immediate result of Proposition 3.5.

Corollary 3.5. Let H and K be intuitionistic fuzzy G-reflexive relation on a set X with $\delta_1(H)=\delta_1(K)$ and $\delta_2(H)=\delta_2(K)$. Then $K\circ H$ is intuitionistic fuzzy G-reflexive with $\delta_1(K\circ H)=\delta_1(H)=\delta_1(K)$ and $\delta_2(K\circ H)=\delta_2(H)=\delta_2(K)$.

Proposition 3.6. Let H and K be intuitionistic fuzzy symmetric relations on a set X. Then $K\circ H$ is intuitionistic fuzzy symmetric if and only if $K\circ H=H\circ K$.

Proof. ($\Rightarrow$): Suppose $K\circ H$ is intuitionistic fuzzy symmetric and let $x,y\in X$. Then

$$\mu_{K\circ H}(x,y)=\mu_{H\circ K}(x,y)$$

$$= \bigvee_{z\in S}[\mu_H(z,y)\wedge\mu_K(z,x)]$$
$$= \bigvee_{z\in S}[\mu_H(y,z)\wedge\mu_K(x,z)]$$
$$= \bigvee_{z\in S}[\mu_K(x,z)\wedge\mu_H(z,y)]$$
$$= \mu_{H\circ K}(x,y)$$

and

$$\nu_{K\circ H}(x,y) = \nu_{H\circ K}(x,y)$$
$$= \bigwedge_{z\in S}[\nu_H(z,y)\vee\nu_K(z,x)]$$
$$= \bigwedge_{z\in S}[\nu_H(y,z)\vee\nu_K(x,z)]$$
$$= \bigwedge_{z\in S}[\nu_K(x,z)\vee\nu_H(z,y)]$$
$$= \nu_{H\circ K}(x,y).$$

Hence $K\circ H = H\circ K$.

($\Leftarrow$): Suppose $K\circ H = H\circ K$ and let $x,y\in X$. Then

$$\mu_{K\circ H}(x,y) = \mu_{H\circ K}(x,y)$$
$$= \bigvee_{z\in S}[\mu_K(x,z)\wedge\mu_H(z,y)]$$
$$= \bigvee_{z\in S}[\mu_K(z,x)\wedge\mu_H(y,z)]$$
$$= \bigvee_{z\in S}[\mu_H(y,z)\wedge\mu_K(z,x)]$$
$$= \mu_{K\circ H}(y,x)$$

and

$$\nu_{K\circ H}(x,y) = \nu_{H\circ K}(x,y)$$
$$= \bigwedge_{z\in S}[\nu_K(x,z)\vee\nu_H(z,y)]$$
$$= \bigwedge_{z\in S}[\nu_K(z,x)\vee\nu_K(y,z)]$$
$$= \bigwedge_{z\in S}[\nu_H(y,z)\vee\nu_K(z,x)]$$
$$= \nu_{K\circ H}(y,x).$$

Hence $K\circ H$ is intuitionistic fuzzy symmetric. □

The following is the immediate result of Corollary 3.5 and Proposition 3.6.

Corollary 3.6. Let H and K be intuitionistic fuzzy G-equivalence relations on a set X with $\delta_1(H)=\delta_1(K)$ and $\delta_2(H)=\delta_2(K)$ such that $K\circ H = H\circ K$. Then $K\circ H$ is an intuitionistic fuzzy G-equivalence relation on X.

4. Intuitionistic fuzzy G-congruences on a groupoid

Definition 4.1[15]. An IFR R on a groupoid S is said to be:

(1) *intuitionistic fuzzy left compatible* if
$\mu_R(x,y)\le\mu_R(zx,zy)$ and $\nu_R(x,y)\ge\nu_R(zx,zy)$, for any $x,y,z\in S$.

(2) *intuitionistic fuzzy right compatible* if
$\mu_R(x,y)\le\mu_R(xz,yz)$ and $\nu_R(x,y)\ge\nu_R(xz,yz)$, for any $x,y,z\in S$.

(3) *intuitionistic fuzzy compatible* if
$\mu_R(x,y)\wedge\mu_R(z,t)\le\mu_R(xz,yt)$ and
$\nu_R(x,y)\vee\nu_R(z,t)\ge\nu_R(xz,yt)$,
for any $x,y,z,t\in S$.

Proposition 4.2. If H and K are intuitionistic fuzzy compatible relations on a groupoid S, then $H\cap K$ is intuitionistic fuzzy compatible on S.

Proof. Let $x,y,a,b\in S$. Then

$$\mu_{H\cap K}(xa,yb) = \mu_H(xa,yb)\wedge\mu_K(xa,yb)$$
$$\ge [\mu_H(x,y)\wedge\mu_H(a,b)]$$
$$\wedge[\mu_K(x,y)\wedge\mu_K(a,b)]$$
$$= [\mu_H(x,y)\wedge\mu_K(x,y)]$$
$$\wedge[\mu_H(a,b)\wedge\mu_K(a,b)]$$
$$= \mu_{H\cap K}(x,y)\wedge\mu_{H\cap K}(a,b)$$

and

$$\nu_{H\cap K}(xa,yb) = \nu_H(xa,yb)\vee\nu_K(xa,yb)$$
$$\le [\nu_H(x,y)\vee\nu_H(a,b)]$$
$$\vee[\nu_K(x,y)\vee\nu_K(a,b)]$$
$$= [\nu_H(x,y)\vee\nu_K(x,y)]$$
$$\vee[\nu_H(a,b)\vee\nu_K(a,b)]$$
$$= \nu_{H\cap K}(x,y)\vee\nu_{H\cap K}(a,b).$$

Hence $H\cap K$ is intuitionistic fuzzy compatible on S. □

Proposition 4.3. If H and K are intuitionistic fuzzy compatible relations on a groupiod S, then $K\circ H$ is intuitionistic fuzzy compatible on S.

Proof. Let $x,y,a,b\in S$. Then

$$\mu_{K\circ H}(xa,yb)$$
$$= \bigvee_{t\in S}[\mu_H(xa,t)\wedge\mu_K(t,yb)]$$
$$\ge \bigvee_{t=zc}[\mu_H(xa,zc)\wedge\mu_K(zc,yb)]$$
$$\ge \bigvee_{(z,c)\in S\times S}[\mu_H(xa,zc)\wedge\mu_K(zc,yb)]$$
$$\ge \bigvee_{(z,c)\in S\times S}[(\mu_H(x,z)\wedge\mu_H(a,c))\wedge(\mu_K(z,y)$$
$$\wedge\mu_K(c,b))]$$
$$= \bigvee_{(z,c)\in S\times S}[(\mu_H(x,z)\wedge\mu_K(z,y))\wedge(\mu_H(a,c)$$
$$\wedge\mu_K(c,b))]$$
$$= (\bigvee_{z\in S}[\mu_H(x,z)\wedge\mu_K(z,y)])$$
$$\wedge(\bigvee_{c\in S}[\mu_H(a,c)\wedge\mu_K(c,b)])$$
$$= \mu_{K\circ H}(x,y)\wedge\mu_{K\circ H}(a,b)$$

and

$$\nu_{K\circ H}(xa,yb)$$
$$= \bigwedge_{t\in S}[\nu_H(xa,t)\vee\nu_K(t,yb)]$$
$$\le \bigwedge_{(z,c)\in S\times S}[\nu_H(xa,zc)\vee\nu_K(zc,yb)]$$
$$\le \bigwedge_{(z,c)\in S\times S}[(\nu_H(x,z)\vee\nu_H(a,c))\vee(\nu_K(z,y)$$
$$\vee\nu_K(c,b))]$$
$$= \bigwedge_{(z,c)\in S\times S}[(\nu_H(x,z)\vee\nu_K(z,y))\vee(\nu_H(a,c)$$
$$\vee\nu_K(c,b))]$$
$$= (\bigwedge_{z\in S}[\nu_H(x,z)\vee\nu_K(z,y)])$$
$$\vee(\bigwedge_{c\in S}[\nu_H(a,c)\vee\nu_K(c,b)])$$
$$= \nu_{K\circ H}(x,y)\vee\nu_{K\circ H}(a,b).$$

Hence $K\circ H$ is intuitionistic fuzzy compatible. □

Definition 4.4. Let R be an intuitionistic fuzzy relation on a groupoid S. Then R is called an *intuitionistic fuzzy G-congruence* on S if $R\in IFE_G(S)$ and R is intuitionistic fuzzy compatible.

We will denote the set of all intuitionistic fuzzy G-congruences on S as $IFC_G(S)$.

Example 4.4. Let $K = \{e, a, b, c\}$ be the Klein 4-group, where e is the identity. Let $R = (\mu_R, \nu_R)$ be the intuitionistic fuzzy relation on K defined as follows:

$R(x, y) = (\frac{1}{4}, \frac{1}{2})$ for any $x, y \in K$ with $x \neq y$,
$R(a, a) = R(b, b) = (\frac{1}{3}, \frac{2}{3})$,
$R(c, c) = (\frac{1}{2}, \frac{1}{2})$, $R(e, e) = (1, 0)$.

Then we can see that R is an intuitionistic fuzzy G-congruence on K.

The following is the immediate result of Propositions 4.2 and 3.3.

Proposition 4.5. If H and K are intuitionistic fuzzy G-congruences on a groupoid S, then $H \cap K$ is an intuitionistic fuzzy G-congruence on S.

The following is the immediate result of Corollary 3.6 and Proposition 4.3.

Proposition 4.6. Let H and K are intuitionistic fuzzy G-congruences on a groupoid S with $\delta_1(H) = \delta_1(K)$ and $\delta_2(H) = \delta_2(K)$ such that $K \circ H = H \circ K$. Then $K \circ H$ is an intuitionistic fuzzy G-congruence on S with $\delta_1(K \circ H) = \delta_1(H) = \delta_1(K)$ and $\delta_2(K \circ H) = \delta_2(H) = \delta_2(K)$.

Definition 4.7. Let R be an intuitionistic fuzzy relation on a groupoid S.

(1) R is said to be*right conformable* if for any $a, b, c \in S$, $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$ imply $\mu_R(ac, bc) \geq \mu_R(a, b)$ and $\nu_R(ac, bc) \leq \nu_R(a, b)$.

(2) R is said to be *left conformable* if for any $a, b, c \in S$, $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$ imply $\mu_R(ca, cb) \geq \mu_R(a, b)$ and $\nu_R(ca, cb) \leq \nu_R(a, b)$.

(3) R is called an *intuitionistic fuzzy right* [resp. *left*] *G-congruence* if

(i) it is an intuitionistic fuzzy G-equivalence relation,

(ii) it is intuitoistic fuzzy right [resp. left] conformable.

The following is the immediate result of Definitions 4.1 and 4.7.

Proposition 4.8. Let R be an intuitionistic fuzzy relation on a groupoid S.

(1) If R is intuitionistic fuzzy right [resp. left] compatible, then it is intuitionistic fuzzy right [resp. left] conformable.

(2) If R is both intuitionistic fuzzy reflexive and intuitionistic fuzzy right [resp. left] conformable, then it is intuitionistic fuzzy right [resp. left] compatible.

Proposition 4.9. (1) If R is an intuitionistic fuzzy compatible relation on a groupoid S, then it is both intuitionistic fuzzy right and left conformable.

(2) Let R be an intuitionistic fuzzy G-preorder on a groupoid S. If R is both intuitionistic fuzzy right and left conformable, then it is intuitionistic fuzzy compatible.

Proof. (1) Suppose $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$ for any $a, b, c \in S$. Since R is intuitionistic fuzzy compatible, $\mu_R(ac, bc) \geq \mu_R(a, b) \wedge \mu_R(c, c) = \mu_R(a, b)$ and $\nu_R(ac, bc) \leq \nu_R(a, b) \vee \nu_R(c, c) = \nu_R(a, b)$. Thus R is intuitionistic fuzzy right conformable. Similarly, R is intuitionistic fuzzy left conformable. Hence R is both intuitionistic fuzzy right and left conformable.

(2) Let $a, b, c \in S$. Since R is intuitionistic fuzzy transitive,

$$\begin{aligned}\mu_R(ac, bd) &\geq \mu_{R \circ R}(ac, bd) \\ &= \bigvee_{t \in S}[\mu_R(ac, t) \wedge \mu_R(t, bd)] \\ &\geq \mu_R(ac, bc) \wedge \mu_R(bc, bd) \qquad (4.1)\end{aligned}$$

and

$$\begin{aligned}\nu_R(ac, bd) &\leq \nu_{R \circ R}(ac, bd) \\ &= \bigwedge_{t \in S}[\nu_R(ac, t) \vee \nu_R(t, bd)] \\ &\leq \nu_R(ac, bc) \vee \nu_R(bc, bd). \qquad (4.2)\end{aligned}$$

Case (i): Suppose $a \neq b$ and $c \neq d$. Since R is intuitionistic fuzzy G-reflexive,

$$\mu_R(c, c) \geq \mu_R(a, b), \quad \nu_R(c, c) \leq \nu_R(a, b)$$

and

$$\mu_R(b, b) \geq \mu_R(c, d), \quad \nu_R(b, b) \leq \nu_R(c, d).$$

Since R is both intuitionistic fuzzy right and left conformable,

$$\mu_R(ac, bc) \geq \mu_R(a, b), \quad \nu_R(ac, bc) \leq \nu_R(a, b)$$

and

$$\mu_R(bc, bd) \geq \mu_R(c, d), \quad \nu_R(bc, bd) \leq \nu_R(c, d).$$

By (4.1) and (4.2), $\mu_R(ac, bd) \geq \mu_R(a, b) \wedge \mu_R(c, d)$ and $\nu_R(ac, bd) \leq \nu_R(a, b) \vee \nu_R(c, d)$.

Case (ii): Suppose $a \neq b$ and $c = d$. Since R is intuitionistic fuzzy G-reflexive, $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$. Since R is intuitionistic fuzzy right conformable,

$$\begin{aligned}\mu_R(ac, bd) &= \mu_R(ac, bc) \geq \mu_R(a, b) \\ &= \mu_R(a, b) \wedge \mu_R(c, c) \\ &= \mu_R(a, b) \wedge \mu_R(c, d)\end{aligned}$$

and

$$\begin{aligned}\nu_R(ac, bd) &= \nu_R(ac, bc) \leq \nu_R(a, b) \\ &= \nu_R(a, b) \vee \nu_R(c, c) \\ &= \nu_R(a, b) \vee \nu_R(c, d).\end{aligned}$$

Case (iii): Suppose $a = b$ and $c \neq d$. By the similar arguments of Case (ii), we have the same result as Case (ii).

Case (iv): Suppose $a = b$ and $c = d$. If $\mu_R(a, a) \geq \mu_R(c, c)$ and $\nu_R(a, a) \leq \nu_R(c, c)$, then, by intuitionistic fuzzy left conformability, we obtain $\mu_R(ac, ac) \geq \mu_R(c, c)$ and $\nu_R(ac, ac) \leq \nu_R(c, c)$. If $\mu_R(c, c) \geq \mu_R(a, a)$ and $\nu_R(c, c) \leq \nu_R(a, a)$, then, by intuitionistic fuzzy right conformability, we obtain $\mu_R(ac, ac) \geq \mu_R(a, a)$ and $\nu_R(ac, ac) \geq \nu_R(a, a)$. So

$$\mu_R(ac, bd) = \mu_R(ac, ac) \geq \mu_R(a, a) \wedge \mu_R(c, c) = \mu_R(a, b) \wedge \mu_R(c, d)$$

and

$$\nu_R(ac, bd) = \nu_R(ac, ac) \leq \nu_R(a, a) \vee \nu_R(c, c) = \nu_R(a, b) \vee \nu_R(c, d).$$

Hence R is intuitionistic fuzzy compatible. This completes the proof. □

Corollary 4.9. Let R be an intuitionistic fuzzy relation on a groupoid S. Then R is an intuitionistic fuzzy G-congruence on S if and only if it is both an intuitionistic fuzzy right and left G-congruence on S.

5. Intuitionistic fuzzy G-congruences on a group

Proposition 5.1. Let R be an intuitionistic fuzzy G-reflexive relation on a group K.

(1) If R is intuitionistic right conformable, then $R(ac, bc) = R(a, b)$, whenever $a \neq b$, $c \in K$.

(2) If R is intuitionistic left conformable, then $R(ca, cb) = R(a, b)$, whenever $a \neq b$, $c \in K$.

(3) If R is both intuitionistic right and left conformable, then $R(cad, cbd) = R(ad, bd) = R(ca, cb) = R(a, b)$, whenever $a \neq b$, $c, d \in K$.

Proof. (1) Let $a \neq b$, $c \in K$. Since R is intuitionistic G-reflexive,

$$\mu_R(c, c) \geq \mu_R(a, b), \quad \nu_R(c, c) \leq \nu_R(a, b)$$

and

$$\mu_R(c^{-1}, c^{-1}) \geq \mu_R(ac, bc),$$
$$\nu_R(c - 1, c^{-1}) \leq \nu_R(ac, bc).$$

Since R is intuitionistic right conformable,

$$\mu_R(a, b) = \mu_R(acc^{-1}, bcc^{-1}) \geq \mu_R(ac, bc) \geq \mu_R(a, b)$$

and

$$\nu_R(a, b) = \nu_R(acc^{-1}, bcc^{-1}) \leq \nu_R(ac, bc) \leq \nu_R(a, b).$$

Thus $\mu_R(a, b) \geq \mu_R(ac, bc)$ and $\nu_R(a, b) \leq \nu_R(ac, bc)$. Hence $R(ac, bc) = R(a, b)$.

The proofs of (2) and (3) are similar. □

Corollary 5.1. (1) If R is an intuitionistic fuzzy G-congruences on a group K, then $R(cad, cbd) = R(ad, bd) = R(ca, cb) = R(a, b)$, whenever $a \neq b$, $c, d \in K$.

(2) If R is an intuitionistic fuzzy G-reflexive and symmetric relation on a group K, which is both intuitionistic fuzzy right and left conformable, then $R(a^{-1}, b^{-1}) = R(a, b)$, whenever $a \neq b \in K$.

(3) If R is an intuitionistic fuzzy G-congruences on a group K, then $R(a^{-1}, b^{-1}) = R(a, b)$, whenever $a \neq b \in K$.

Proof. (1) By Proposition 4.9(1), R is both intuitionistic fuzzy right and left conformable. Hence, by Proposition 5.1(3), we get the result.

(2) Let $a \neq b \in K$. Then, by Proposition 5.1(3), $R(a, b) = R(b^{-1}aa^{-1}, b^{-1}ba^{-1}) = R(b^{-1}, a^{-1}) = R(a^{-1}, b^{-1})$.

(3) It follows from Proposition 4.9(1) and Corollary 5.1(2). □

Example 5.2. Consider the intuitionistic fuzzy relation R given in Example 4.4. Then R is intuitionistic fuzzy G-reflexive and right conformable on K with $R(ab, ab) \neq R(a, a)$.

Definition 5.3[9]. Let $(S, \cdot)$ be a groupoid and let $A \in$ IFS(S). Then A is called an *intuitionistic fuzzy subgroupoid* (in short, *IFGP*) of S if for any $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

We will denote the set of all $IFGP_S$ of a groupoid S as IFGP(S). Then it is clear that $0_\sim$ and $1_\sim \in$ IFGP(S) .

Definition 5.4[10]. Let K be a group and let $A \in$ IFGP(K). Then A is called an *intuitionistic fuzzy subgroup* (in short, *IFG*) of K if $A(x^{-1}) \geq A(x)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in K$.

We will denote the set of all IFG_S of K as IFG(K).

Result 5.A[10, Proposition 2.6]. Let K be a group and let $A \in$ IFG(K). Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e)$, $\nu_A(x) \geq \nu_A(e)$ for each $x \in K$, where e is the identity element of K.

Definition 5.5[10]. Let K be a group and let $A \in$ IFG(K). Then A is said to be *normal* if $A(xy) = A(yx)$ for any $x, y \in K$.

We will denote the family of all intuitionistic fuzzy normal subgroups of a group K as IFNG(K). In particular, we will denote the set $\{N \in IFNG(K) : N(e) = (1, 0)\}$ as IFN(K).

Definition 5.6[15]. An IFER R on a groupoid S is called an:

(1) *intuitionistic fuzzy left congruence* (in short, $IFLC$)

if it is intuitionistic fuzzy left compatible.

(2) *intuitionistic fuzzy right congruence*(in short, $IFRC$) if it is intuitionistic fuzzy right compatible.

(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)].

Result 5.B[16, Lemma 5.6]. Let K be a group and let $A \in$ IFN(K). We define the complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : K \times K \to I \times I$ as follows: For each $(a,b) \in K \times K$,

$$R_A(a,b) = A(ab^{-1}).$$

Then $R_A \in$ IFC(K).

The following is the modification of Result 5.B.

Lemma 5.7. Let K be a group and let A be an intuitionistic fuzzy nonempty normal subgroup of K. We define a complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : K \times K \to I \times I$ as follows: For any $a, b \in K$,

$$R_A(a,b) = A(ab^{-1}).$$

Then $R_A \in IFC_G(K)$.

Proof. Let $a \neq b \in K$. Then clearly $\mu_{R_A}(a,a) = \mu_A(aa^{-1}) = \mu_A(e) > 0$ and $\nu_{R_A}(a,a) = \nu_A(aa^{-1}) = \nu_A(e) < 1$. On the other hand,

$$\begin{aligned}\delta_1(R_A) &= \textstyle\bigwedge_{t\in K} \mu_{R_A}(t,t) = \mu_A(e) \\ &\geq \mu_A(ab^{-1}) = \mu_{R_A}(a,b)\end{aligned}$$

and

$$\begin{aligned}\delta_2(R_A) &= \textstyle\bigvee_{t\in K} \nu_{R_A}(t,t) = \nu_A(e) \\ &\leq \nu_A(ab^{-1}) = \nu_{R_A}(a,b).\end{aligned}$$

Thus R_A is intuitionistic fuzzy G-reflexive. By Result 5.B, R_A is intuitionistic fuzzy symmetric and transitive. Also, by Result 5.B, R_A is intuitionistic fuzzy compatible. Hence $R_A \in IFC_G(K)$. □

Result 5.C[15, Proposition 2.18]. Let K be a group and let $R \in$ IFC(K). We define the complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : K \to I \times I$ as follows: For each $a \in K$,

$$A_R(a) = R(a,e) = R_e(a).$$

Then $A_R \in$ IFN(K).

The following is the modification of Result 5.C.

Lemma 5.8. Let K be a group and let $R \in IFC_G(K)$. We define a complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : K \to I \times I$ as follows: For each $x \in K$,

$$A_R(x) = R(x,e).$$

Then A_R is an intuitionistic fuzzy nonempty normal subgroup of K.

Proof. Let $x, y \in K$. Since R is intuitionistic fuzzy compatible,

$$\begin{aligned}\mu_{A_R}(xy) &= \mu_R(xy,e) = \mu_R(xy,ee) \\ &\geq \mu_R(x,e) \wedge \mu_R(y,e) \\ &= \mu_{A_R}(x) \wedge \mu_{A_R}(y)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_R}(xy) &= \nu_R(xy,e) = \nu_R(xy,ee) \\ &\leq \nu_R(x,e) \vee \nu_R(y,e) \\ &= \nu_{A_R}(x) \vee \nu_{A_R}(y).\end{aligned}$$

Thus, $A_R \in IFGP(K)$. Let $x \neq e$. Then, by Corollary 5.1(3),

$$A_R(x^{-1}) = R(x^{-1},e) = R(x,e) = A_R(x).$$

So $A_R \in IFG(K)$. Now let $xy \neq e$. Then, by Corollary 5.1(1),

$$\begin{aligned}A_R(xy) &= R(xy,e) = R(x^{-1}xyx, x^{-1}ex) \\ &= R(yx,e) = A_R(yx).\end{aligned}$$

Thus $A_R \in IFNG(K)$. Moreover, $\mu_{A_R}(e) = \mu_R(e,e) > 0$ and $\nu_{A_R}(e) = \nu_R(e,e) < 1$. So $A_R \neq 0_{\sim}$. Hence A_R is an intuitionistic fuzzy nonempty normal subgroup of K. □

Let K be any group. We define a relation $\sim$ on $IFC_G(K)$ as follows: For any $P, Q \in IFC_G(K)$, $P \sim Q$ if and only if $P(x,y) = Q(x,y)$ for all $x \neq y \in K$ and $P(e,e) = Q(e,e)$. Then clearly, $\sim$ is an equivalence relation on $IFC_G(K)$, which partitions $IFC_G(K)$ into disjoint equivalence classes $[R]$, $R \in IFC_G(K)$. Let $IFC_G(K)/\sim$ be the family of all these equivalence classes and let $IFN^*(K)$ denote the set of all intuitionistic fuzzy nonempty normal subgroups of K.

Proposition 5.9. Let K be a group. Then, under the product $[P][Q] = [P \cap Q]$, $P, Q \in IFC_G(K)$, $IFC_G(K)/\sim$ is a commutative monoid of idempotents.

Proof. We note that, by Proposition 4.5, $IFC_G(K)$ is closed under the formation of finite intersections. For any $P, P_1, Q, Q_1 \in IFC_G(K)$, suppose $P \sim P_1$ and $Q \sim Q_1$. Let $x \neq y \in K$. Then

$$\begin{aligned}\mu_{P\cap Q}(x,y) &= \mu_P(x,y) \wedge \mu_Q(x,y) \\ &= \mu_{P_1}(x,y) \wedge \mu_{Q_1}(x,y) \\ &= \mu_{P_1\cap Q_1}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\cap Q}(x,y) &= \nu_P(x,y) \vee \nu_Q(x,y) \\ &= \nu_{P_1}(x,y) \vee \nu_{Q_1}(x,y) \\ &= \nu_{P_1\cap Q_1}(x,y).\end{aligned}$$

Thus $(P \cap Q)(x,y) = (P_1 \cap Q_1)(x,y)$. Also, $(P \cap Q)(e,e) = (P_1 \cap Q_1)(e,e)$. Thus $P \cap Q \sim P_1 \cap Q_1$. So the product on $IFC_G(K)/\sim$ is well-defined. We can easily see that $(IFC_G(K)/\sim, \cdot)$ forms a commutative semigroup of idempotents. Now we define a complex mapping $\epsilon = (\mu_\epsilon, \nu_\epsilon) : K \times K \to I \times I$ as follows: For any $x, y \in K$, $\epsilon(x,y) = (1,0)$. Then clearly $\epsilon \in IFC_G(K)$. Moreover,

$$[R][\epsilon] = [R] = [\epsilon][R] \text{ for each } R \in IFC_G(K).$$

Hence $[\epsilon]$ is the identity element of $IFC_G(K)/\sim$. This completes the proof. □

Proposition 5.10. Let K be any group. Then $(IFN^*(K), \cap)$ forms a commutative monoid of idempotents.

Proof. We can easily prove that $(IFN^*(K), \cap)$ forms a commutative semigroups of idempotents. It is clear that $1_\sim \in IFN^*(K)$. Moreover, $1_\sim \cap A = A = A \cap 1_\sim$ for each $A \in IFN^*(K)$. So $1_\sim$ is the identity element of $IFN^*(K)$. □

Theorem 5.11. Let K be any group. Then $IFC_G(K)$ is isomorphic to $IFN^*(K)$.

Proof. Consider the mapping $\Psi : IFC_G(K) \to IFN^*(K)$ defined by $\Psi([R]) = A_R$ for each $R \in IFC_G(K)$. Then we can easily see that Ψ is well-defined. Let $A \in IFN^*(K)$. Then, by Lemma 5.7, $R_A \in IFC_G(K)$. Let $x \in K$. Then $A_{R_A}(x) = R_A(x, e) = A(xe^{-1}) = A(x)$. Thus $A_{R_A} = A$. So $\Psi([R_A]) = A$. Hence Ψ is surjective. For any $P, Q \in IFC_G(K)$, suppose $\Psi([P]) = \Psi([Q])$. Then $A_P = A_Q$. Thus $P(x, e) = Q(x, e)$ for each $x \in K$. Let $x \neq y \in K$. Then, by Corollary 5.1(1),

$$P(x,y) = P(xy^{-1}, e) = Q(xy^{-1}, e) = Q(x, y).$$

Thus $P \sim Q$, i.e., $[P] = [Q]$. So Ψ is injective. Now let $P, Q \in IFC_G(K)$ and let $x \in K$. Then

$$\begin{aligned}\mu_{A_{P\cap Q}}(x) &= \mu_{P\cap Q}(x,e)\\ &= \mu_P(x,e) \wedge \mu_Q(x,e)\\ &= \mu_{A_P}(x) \wedge \mu_{A_Q}(x)\\ &= \mu_{A_P\cap A_Q}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_{P\cap Q}}(x) &= \nu_{P\cap Q}(x,e)\\ &= \nu_P(x,e) \vee \nu_Q(x,e)\\ &= \nu_{A_P}(x) \vee \nu_{A_Q}(x)\\ &= \nu_{A_P\cap A_Q}(x).\end{aligned}$$

Thus $A_{P\cap Q} = A_P \cap A_Q$. So

$$\begin{aligned}\Psi([P][Q]) &= \Psi([P\cap Q])\\ &= A_{P\cap Q}\\ &= A_P \cap A_Q\\ &= \Psi([P]) \cap \Psi([Q]).\end{aligned}$$

Moreover, $\Psi([\epsilon]) = A_\epsilon = 1_\sim$. Hence Ψ is a monoid homomorphism. Therefore $IFC_G(K)/\sim$ and $IFN^*(K)$ are isomorphic under Ψ. □

Proposition 5.12. Let K be a group. If $P, Q \in IFC_G(K)$ such that $\delta_1(P) = \delta_1(Q)$ and $\delta_2(P) = \delta_2(Q)$, then $P \circ Q = Q \circ P$ and $P \circ Q \in IFC_G(K)$ such that $\delta_1(P \circ Q) = \delta_1(P) = \delta_1(Q)$ and $\delta_2(P \circ Q) = \delta_2(P) = \delta_2(Q)$.

Proof. Let $x \neq y \in K$. Then

$$\begin{aligned}&\mu_{Q\circ P}(x,y)\\ &= \textstyle\bigvee_{t\in K}[\mu_P(x,t) \wedge \mu_Q(t,y)]\\ &= (\textstyle\bigvee_{x\neq t\neq y}[\mu_P(x,t) \wedge \mu_Q(t,y)])\\ &\quad \vee(\mu_P(x,x) \wedge \mu_Q(x,y)) \vee (\mu_P(x,y) \wedge \mu_Q(y,y))\\ &= (\textstyle\bigvee_{x\neq t\neq y} \mu_P(yt^{-1}x, y) \wedge \mu_Q(x, yt^{-1}x))\\ &\quad \vee(\mu_Q(x,y) \wedge \mu_P(y,y)) \vee (\mu_Q(x,x) \wedge \mu_P(x,y))\\ &\qquad \text{(By Corollary 5.1(1))}\\ &= (\textstyle\bigvee_{x\neq t\neq y}[\mu_P(yt^{-1}x, y) \wedge \mu_Q(x, yt^{-1}x)])\\ &\quad \vee(\mu_P(x,x) \wedge \mu_Q(x,y)) \vee (\mu_P(x,y) \wedge \mu_Q(y,y))\\ &\qquad \text{(By Corollary 5.1(1))} \qquad (5.1)\end{aligned}$$

and

$$\begin{aligned}&\nu_{Q\circ P}(x,y)\\ &= \textstyle\bigwedge_{t\in K}[\nu_P(x,t) \vee \nu_Q(t,y)]\\ &= (\textstyle\bigwedge_{x\neq t\neq y}[\nu_P(x,t) \vee \nu_Q(t,y)]) \wedge (\nu_P(x,x)\\ &\quad \vee \nu_Q(x,y)) \wedge (\nu_P(x,y) \vee \nu_Q(y,y))\\ &= (\textstyle\bigwedge_{x\neq t\neq y}[\nu_P(yt^{-1}x, y) \vee \nu_Q(x, yt^{-1}x)]) \wedge (\nu_P(x,x)\\ &\quad \vee \nu_Q(x,y)) \wedge (\nu_P(x,y) \vee \nu_Q(y,y)). \qquad (5.2)\end{aligned}$$

Since $\delta_1(P) = \delta_1(Q)$, and P and Q are intuitionistic fuzzy G-reflexive,

$$\begin{aligned}\mu_P(x,x) \wedge \mu_Q(x,y) &= \mu_Q(x,y)\\ &= \mu_Q(x,y) \wedge \mu_P(y,y)\end{aligned}$$

and

$$\begin{aligned}\mu_P(x,y) \wedge \mu_Q(y,y) &= \mu_P(x,y)\\ &= \mu_Q(x,x) \wedge \mu_P(x,y).\end{aligned}$$

Since $\delta_2(P) = \delta_2(Q)$, and P and Q are intuitionistic fuzzy G-reflexive,

$$\begin{aligned}\nu_P(x,x) \vee \nu_Q(x,y) &= \nu_Q(x,y)\\ &= \nu_Q(x,y) \vee \nu_P(y,y)\end{aligned}$$

and

$$\begin{aligned}\nu_P(x,y) \vee \nu_Q(y,y) &= \nu_P(x,y)\\ &= \nu_Q(x,x) \vee \nu_P(x,y).\end{aligned}$$

Thus, by (5.1) and (5.2),

$$\begin{aligned}&\mu_{Q\circ P}(x,y)\\ &= (\textstyle\bigvee_{x\neq t\neq y}[\mu_P(yt^{-1}x, y) \wedge \mu_Q(x, yt^{-1}x)])\\ &\quad \vee(\mu_Q(x,y) \wedge \mu_P(y,y)) \vee (\mu_Q(x,x) \wedge \mu_P(x,y))\\ &= \textstyle\bigvee_{t\in K}[\mu_Q(x,t) \wedge \mu_P(t,y)]\\ &= \mu_{P\circ Q}(x,y)\end{aligned}$$

and

$$\begin{aligned}&\nu_{Q\circ P}(x,y)\\ &= (\textstyle\bigwedge_{x\neq t\neq y}[\nu_P(yt^{-1}x, y) \vee \nu_Q(x, yt^{-1}x)])\\ &\quad \wedge(\nu_Q(x,y) \vee \nu_P(y,y)) \wedge (\nu_Q(x,x) \vee \nu_P(x,y))\\ &= \textstyle\bigwedge_{t\in K}[\nu_Q(x,t) \vee \nu_P(t,y)] = \nu_{P\circ Q}(x,y).\end{aligned}$$

Hence $P \circ Q = Q \circ P$. Moreover, by Proposition 4.6, $P \circ Q \in IFC_G(K)$ such that $\delta_1(P \circ Q) = \delta_1(P) = \delta_1(Q)$ and $\delta_2(P \circ Q) = \delta_2(P) = \delta_2(Q)$. This completes the proof. □

6. The (λ, μ)-partition of $IFC_G(K)$

Definition 6.1[9]. Let $(X, \cdot)$ be a groupoid and let $A, B \in IFS(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$, is defined as follows: For each $x \in X$,

$$(A \circ B)(x) = \begin{cases} (\bigvee_{yz=x}[\mu_A(y) \wedge \mu_B(z)], \\ \quad \bigwedge_{yz=x}[\nu_A(y) \vee \nu_B(z)]), \\ (0,1) \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Result 6.A[9, Proposition 2.3]. Let $(X, \cdot)$ be a groupoid. If "$\cdot$" is a associative [resp. commutative], then so is "$\circ$" in IFS(X).

Result 6.B[10, Proposition 2.4]. Let A be an IFG of a group G. Then $A \circ A = A$.

Result 6.C[10, Proposition 3.2]. Let K be a group, let $A \in$ IFS(K) and let $B \in$ IFNG(K). Then $A \circ B = B \circ A$.

Result 6.D[10, Proposition 3.4]. Let K be a group and let $A, B \in$ IFNG(K). Then $A \circ B \in$ IFNG(K).

For a group K and for each $(\lambda, \mu) \in (0,1] \times [0,1)$ with $\lambda + \mu \leq 1$, let

$$IFC_{G,(\lambda,\mu)}(K) = \{R \in IFC_G(K) : \delta_1(R) = \lambda \text{ and } \delta_2(R) = \mu\}.$$

We define a relation $\sim$ on $IFC_{G,(\lambda,\mu)}(K)$ as follows: For any $P, Q \in IFC_{G,(\lambda,\mu)}(K)$,

$P \sim Q$ if and only if $P(x,y) = Q(x,y)$,whenever $x \neq y \in K$, and $P(e,e) = Q(e,e)$.

Then we can easily prove that $\sim$ is an equivalence relation on $IFC_{G,(\lambda,\mu)}(K)$. For each $R \in IFC_{G,(\lambda,\mu)}(K)$, let $[R]_{(\lambda,\mu)}$ be the equivalence class in $IFC_{G,(\lambda,\mu)}(K)$ containing R. Let $IFC_{G,(\lambda,\mu)}(K)/\sim$ be the family of all these equivalence classes. If $0 < \lambda < s \leq 1$ and $0 \leq t < \mu < 1$ such that $\lambda + \mu \leq 1$ and $s + t \leq 1$, then $IFC_{G,(\lambda,\mu)}(K) \cap IFC_{G,(s,t)}(K) = \emptyset$ and $IFC_{G,(\lambda,\mu)}(K)/\sim \cap IFC_{G,(s,t)}(K)/\sim = \emptyset$. Finally, let $IFNG_{(\lambda,\mu)}(K) = \{A \in IFNG(K) : \mu_A(x) \leq \lambda \leq \mu_A(e)$ and $\nu_A(e) \leq \mu \leq \nu_A(x)$, $e \neq x \in K\}$ if $K \neq (e)$ and let $IFNG_{(\lambda,\mu)}(K) = \{(e)_{(\lambda,\mu)}\}$ if $K = (e)$.

Proposition 6.2. Let K be a group and let $(\lambda, \mu) \in (0,1] \times [0,1)$ with $\lambda + \mu \leq 1$. Then $(IFC_{G,(\lambda,\mu)}(K), \circ)$ is a commutative semigroup of idempotents.

Proof. By Proposition 5.12, $(IFC_{G,(\lambda,\mu)}(K), \circ)$ is a commutative groupoid. Moreover, by Result 6.A it is clear that $\circ$ is associative. On the other hand, by Proposition 3.2(2), each member of $IFC_{G,(\lambda,\mu)}(K)$ is an idempotent. Hence $(IFC_{G,(\lambda,\mu)}(K), \circ)$ is a commutative semigroup of idempotents. □

Lemma 6.3. Let K be a group and let $(\lambda, \mu) \in (0,1] \times [0,1)$ with $\lambda + \mu \leq 1$. We define a binary relation $*$ on $IFC_{G,(\lambda,\mu)}(K)/\sim$ as follows: For any $P, Q \in IFC_{G,(\lambda,\mu)}(K)$,

$$[P]_{(\lambda,\mu)} * [Q]_{(\lambda,\mu)} = [P \circ Q]_{(\lambda,\mu)}.$$

Then $(IFC_{G,(\lambda,\mu)}(K)/\sim, *)$ is a commutative monoid of idempotents.

Proof. Suppose $K = (e)$. Then $IFC_{G,(\lambda,\mu)}(K)/\sim = \{[R]_{(\lambda,\mu)} = \{R\}\}$, where $R(e,e) = (\lambda, \mu)$. Thus the lemma is trivially true in this case. Suppose $K \neq (e)$. We are obliged to prove that $*$ is well-defined. Let $P, P_1, Q, Q_1 \in IFC_{G,(\lambda,\mu)}(K)$ such that $P \sim P_1$ and $Q \sim Q_1$. Let $x \neq y \in K$. Then

$$\begin{aligned}
&\mu_{Q \circ P}(x,y) \\
&= \textstyle\bigvee_{t \in K}[\mu_P(x,t) \wedge \mu_Q(t,y)] \\
&= (\textstyle\bigvee_{x \neq t \neq y}[\mu_P(x,t) \wedge \mu_Q(t,y)]) \\
&\quad \vee(\mu_P(x,x) \wedge \mu_Q(x,y)) \vee (\mu_P(x,y) \wedge \mu_Q(y,y)) \\
&= (\textstyle\bigvee_{x \neq t \neq y}[\mu_{P_1}(x,t) \wedge \mu_{Q_1}(t,y)]) \\
&\quad \vee\mu_Q(x,y) \vee \mu_P(x,y) \\
&= (\textstyle\bigvee_{x \neq t \neq y}[\mu_{P_1}(x,t) \wedge \mu_{Q_1}(t,y)]) \\
&\quad \vee(\mu_{P_1}(x,x) \wedge \mu_{Q_1}(x,y)) \vee (\mu_{P_1}(x,y) \wedge \mu_{Q_1}(y,y)) \\
&= \textstyle\bigvee_{t \in K}[\mu_{P_1}(x,t) \wedge \mu_{Q_1}(t,y)] = \mu_{Q_1 \circ P_1}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{Q \circ P}(x,y) \\
&= \textstyle\bigwedge_{t \in K}[\nu_P(x,t) \vee \nu_Q(t,y)] \\
&= (\textstyle\bigwedge_{x \neq t \neq y}[\nu_P(x,t) \vee \nu_Q(t,y)]) \\
&\quad \wedge(\nu_P(x,x) \vee \nu_Q(x,y)) \wedge (\nu_P(x,y) \vee \nu_Q(y,y)) \\
&= (\textstyle\bigwedge_{x \neq t \neq y}[\nu_{P_1}(x,t) \vee \nu_{Q_1}(t,y)]) \\
&\quad \wedge\nu_Q(x,y) \wedge \nu_P(x,y) \\
&= (\textstyle\bigwedge_{x \neq t \neq y}[\nu_{P_1}(x,t) \vee \nu_{Q_1}(t,y)]) \\
&\quad \wedge(\nu_{P_1}(x,x) \vee \nu_{Q_1}(x,y)) \wedge (\nu_{P_1}(x,y) \vee \nu_{Q_1}(y,y)) \\
&= \textstyle\bigwedge_{t \in K}[\nu_{P_1}(x,t) \vee \nu_{Q_1}(t,y)] = \nu_{Q_1 \circ P_1}(x,y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mu_{Q \circ P}(e,e) &= \mu_P(e,e) \wedge \mu_Q(e,e) \\
&= \mu_{P_1}(e,e) \wedge \mu_{Q_1}(e,e) \\
&= \mu_{Q_1 \circ P_1}(e,e)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{Q \circ P}(e,e) &= \nu_P(e,e) \vee \nu_Q(e,e) \\
&= \nu_{P_1}(e,e) \vee \nu_{Q_1}(e,e) \\
&= \nu_{Q_1 \circ P_1}(e,e).
\end{aligned}$$

Thus $Q \circ P \sim Q_1 \circ P_1$. So $*$ is well-defined. Moreover, by Proposition 6.2, we can see that $(IFC_{G,(\lambda,\mu)}(K)/\sim, *)$ is a commutative semigroup of idempotents.

Now we define a complex mapping $E : K \times K \to I \times I$ as follows: For any $x, y \in K$,

$$E(x,y) = \begin{cases} (1,0) & \text{if } x = y = e, \\ (\lambda, \mu) & \text{if } x = y \neq e, \\ (0,1) & \text{if } x \neq y. \end{cases}$$

Then we can routinely prove that $E \in IFC_{G,(\lambda,\mu)}(K)$ and that $E \circ R = R \circ E \sim R$ for each $R \in IFC_{G,(\lambda,\mu)}(K)$. Thus

$$\begin{aligned}
[E]_{(\lambda,\mu)} * [R]_{(\lambda,\mu)} &= [E \circ R]_{(\lambda,\mu)} \\
&= [R]_{(\lambda,\mu)} \\
&= [R \circ E]_{(\lambda,\mu)} \\
&= [R]_{(\lambda,\mu)} * [E]_{(\lambda,\mu)}.
\end{aligned}$$

So $[E]_{(\lambda,\mu)}$ is the identity element of $IFC_{G,(\lambda,\mu)}(K)/\sim$. This completes the proof. □

Proposition 6.4. Let K be a group and let $(\lambda,\mu)\in(0,1]\times[0,1)$ with $\lambda+\mu\leq 1$. Then $(IFNG_{(\lambda,\mu)},\circ)$ is a commutative monoid of idempotents.

Proof. Let $A,B\in IFNG_{(\lambda,\mu)}(K)$. Then, by Result 6.D, $A\circ B\in IFNG(K)$. Let $x\neq e\in K$. Since $A,B\in IFNG_{(\lambda,\mu)}(K)$,

$$\mu_A(x)\leq\lambda\leq\mu_A(e),\quad \nu_A(e)\leq\mu\leq\nu_A(x),$$

and

$$\mu_B(x)\leq\lambda\leq\mu_B(e),\quad \nu_B(e)\leq\mu\leq\nu_B(x).$$

Then $\mu_A(xt^{-1})\wedge\mu_B(t)\leq\lambda$ and $\nu_A(xt^{-1})\vee\nu_B(t)\geq\mu$ for each $t\in K$. Thus

$$\mu_{A\circ B}(x)=\bigvee_{t\in K}[\mu_A(xt^{-1})\wedge\mu_B(t)]\leq\lambda$$

and

$$\nu_{A\circ B}(x)=\bigwedge_{t\in K}[\nu_A(xt^{-1})\vee\nu_B(t)]\geq\mu.$$

Moreover, $\mu_{A\circ B}(e)=\mu_A(e)\wedge\mu_B(e)\geq\lambda$ and $\nu_{A\circ B}(e)=\nu_A(e)\vee\nu_B(e)\leq\mu$. So $A\circ B\in IFNG_{(\lambda,\mu)}(K)$. On the other hand, by Results 6.C and 6.B, $A\circ B=B\circ A$ and $A\circ A=A$. Furthermore, by Result 6.A, $\circ$ is associative. Hence $(IFNG_{(\lambda,\mu)}(K),\circ)$ is a commutative semigroup of idempotents. Finally, consider the intuitionistic fuzzy point $e_{(1,0)}$ of K. Then clearly, $e_{(1,0)}$ is the identity element of $IFNG_{(\lambda,\mu)}(K)$. This completes the proof. □

Proposition 6.5. Let P and Q be intuitionistic fuzzy G-reflexive relations on a group K such that $\delta_1(P)=\delta_1(Q)$ and $\delta_2(P)=\delta_2(Q)$. If P is intuitionistic fuzzy right conformable, then $A_{Q\circ P}=A_P\circ A_Q$.

Proof. Let $x\in K$. Then

$$\begin{aligned}
&\mu_{A_P\circ A_Q}(x)\\
&=\bigvee_{t\in K}[\mu_{A_P}(xt^{-1})\wedge\mu_{A_Q}(t)]\\
&=\bigvee_{t\in K}[\mu_P(xt^{-1},e)\wedge\mu_Q(t,e)]\\
&=(\bigvee_{t\neq x}[\mu_P(xt^{-1},e)\wedge\mu_Q(t,e)])\\
&\quad\vee(\mu_P(e,e)\wedge\mu_Q(x,e))\\
&=\bigvee_{t\neq x}[\mu_P(x,t)\wedge\mu_Q(t,e)]\\
&\quad\vee(\mu_P(x,x)\wedge\mu_Q(x,e))\quad\text{(By Proposition 5.1(1))}\\
&=\bigvee_{t\in K}[\mu_P(x,t)\wedge\mu_Q(t,e))]\\
&=\mu_{Q\circ P}(x,e)\\
&=\mu_{A_{Q\circ P}}(x)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{A_P\circ A_Q}(x)\\
&=\bigwedge_{t\in K}[\nu_{A_P}(xt^{-1})\vee\nu_{A_Q}(t)]\\
&=\bigwedge_{t\in K}[\nu_P(xt^{-1},e)\vee\nu_Q(t,e)]\\
&=(\bigwedge_{t\neq x}[\nu_P(xt^{-1},e)\vee\nu_Q(t,e)])\\
&\quad\wedge(\mu_P(e,e)\vee\nu_Q(x,e))\\
&=(\bigwedge_{t\neq x}[\nu_P(x,t)\vee\nu_Q(t,e)])\\
&\quad\wedge(\nu_P(x,x)\vee\nu_Q(x,e))\\
&=\bigwedge_{t\in K}[\nu_P(x,t)\vee\nu_Q(t,e))]\\
&=\nu_{Q\circ P}(x,e)\\
&=\nu_{A_{Q\circ P}}(x).
\end{aligned}$$

Hence $A_{Q\circ P}=A_P\circ A_Q$. □

Corollary 6.5. Let K be a group and let $(\lambda,\mu)\in(0,1]\times[0,1)$ with $\lambda+\mu\leq 1$. If $P,Q\in IFC_{G,(\lambda,\mu)}(K)$, then $A_{P\circ Q}=A_P\circ A_Q\in IFNG_{(\lambda,\mu)}(K)$.

Proof. Let $P,Q\in IFC_{G,(\lambda,\mu)}(K)$. Then, by Proposition 5.12, $Q\circ P=P\circ Q\in IFC_{G,(\lambda,\mu)}(K)$. Thus, by Lemma 5.8, $A_{P\circ Q}\in IFNG_{(\lambda,\mu)}(K)$. Hence, by Propositions 5.12 and 6.5, $A_{P\circ Q}=A_P\circ A_Q$. □

Theorem 6.6. Let K be a group and let $(\lambda,\mu)\in(0,1]\times[0,1)$ with $\lambda+\mu\leq 1$. Then $(IFC_{G,(\lambda,\mu)}(K)/\sim,*)$ and $(IFNG_{(\lambda,\mu)}(K),\circ)$ are isomorphic.

Proof. Suppose $K=(e)$. Then $IFC_{G,(\lambda,\mu)}(K)/\sim$ and $IFNG_{(\lambda,\mu)}(K)$ are trivially isomorphic, since both are singletons. Suppose $K\neq(e)$. We define a mapping $\Psi:IFC_{G,(\lambda,\mu)}(K)\to IFNG_{(\lambda,\mu)}(K)$ by $\Psi([R]_{(\lambda,\mu)})=A_R$ for each $R\in IFC_{G,(\lambda,\mu)}(K)$. Then, as in the proof of Theorem 5.11, we can show that Ψ is a well-defined injection. Let $[E]_{(\lambda,\mu)}$ be the class which occurs in the proof of Lemma 6.3. Then clearly $[E]_{(\lambda,\mu)}$ is the identity element of $IFC_{G,(\lambda,\mu)}(K)$. Moreover, we can easily see that $\Psi([E]_{(\lambda,\mu)})=A_E=e_{(1,0)}$. Now let $P,Q\in IFC_{G,(\lambda,\mu)}(K)$. Then

$$\begin{aligned}
&\Psi([P]_{(\lambda,\mu)}*[Q]_{(\lambda,\mu)})\\
&=\Psi([P\circ Q]_{(\lambda,\mu)})\\
&=A_{P\circ Q}=A_P\circ A_Q\qquad\text{(By Corollary 6.5)}\\
&=\Psi([P]_{(\lambda,\mu)})\circ\Psi([Q]_{(\lambda,\mu)}).
\end{aligned}$$

So Ψ is a monoid homomorphism. Let $A\in IFNG_{(\lambda,\mu)}(K)$. Then, by Lemma 5.7, $R_A\in IFC_G(K)$. We define a complex mapping $P=(\mu_P,\nu_P):K\times K\to I\times I$ as follows: For any $x,y\in K$,

$$P(x,y)=\begin{cases}R_A(x,y) & \text{if}\quad x\neq y,\\ (\lambda,\mu) & \text{if}\quad x=y\neq e,\end{cases}$$

and

$$\mu_P(e,e)=\mu_{R_A}(e,e)=\mu_A(e)\geq\lambda,$$
$$\nu_P(e,e)=\nu_{R_A}(e,e)=\nu_A(e)\leq\mu.$$

Then clearly $P\in IFR(K)$. Moreover, P is intuitionistic fuzzy G-reflexive and symmetric. Now let $x\neq y\in K$. Then

$$\begin{aligned}
&\mu_{P\circ P}(x,y)\\
&=\bigvee_{t\in K}[\mu_P(x,t)\wedge\mu_P(t,y)]\\
&=(\bigvee_{x\neq t\neq y}[\mu_P(x,t)\wedge\mu_P(t,y)])\vee\mu_P(x,y)\\
&=(\bigvee_{x\neq t\neq y}[\mu_{R_A}(x,t)\wedge\mu_{R_A}(t,y)])\vee\mu_{R_A}(x,y)\\
&=\mu_{R_A\circ R_A}(x,y)\\
&\leq\mu_{R_A}(x,y)\\
&=\mu_P(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{P\circ P}(x,y)\\
&=\bigwedge_{t\in K}[\nu_P(x,t)\vee\nu_P(t,y)]\\
&=(\bigwedge_{x\neq t\neq y}[\nu_P(x,t)\vee\nu_P(t,y)])\wedge\nu_P(x,y)\\
&=(\bigwedge_{x\neq t\neq y}[\nu_{R_A}(x,t)\vee\nu_{R_A}(t,y)])\wedge\nu_{R_A}(x,y)\\
&=\nu_{R_A\circ R_A}(x,y)
\end{aligned}$$

$\geq \nu_{R_A}(x,y)$
$= \nu_P(x,y).$
Thus $P \circ P \subset P$. So P is intuitionistic fuzzy transitive. Hence $P \in \mathrm{I}FE_G(K)$.

Now we show that P is intuitionistic fuzzy right conformable. For any $a, b, c \in K$. Suppose $\mu_P(c,c) \geq \mu_P(a,b)$ and $\nu_P(c,c) \leq \nu_P(a,b)$.

Case (i): Suppose $a \neq b$. Since R_A is intuitionistic fuzzy G-reflexive, $\mu_{R_A}(c,c) \geq \mu_{R_A}(a,b)$ and $\nu_{R_A}(c,c) \leq \nu_{R_A}(a,b)$. Also, by Proposition 4.9(1), R_A is intuitionistic fuzzy right conformable. Thus

$\mu_P(ac,bc) = \mu_{R_A}(ac,bc) \geq \mu_{R_A}(a,b) = \mu_P(a,b)$
and
$\nu_P(ac,bc) = \nu_{R_A}(ac,bc) \leq \nu_{R_A}(a,b) = \nu_P(a,b).$

Case (ii): Suppose $a = b$. If $c = e$, then $P(ac,bc) = P(a,b)$. If $c \neq e$, then

$\mu_P(ac,bc) = \mu_P(ac,ac) \geq \lambda = \mu_P(c,c) \geq \mu_P(a,b)$
and
$\nu_P(ac,bc) = \nu_P(ac,ac) \leq \mu = \nu_P(c,c) \leq \nu_P(a,b).$
So, in all, P is intuitionistic fuzzy conformable. By the similar arguments, we can see that P is intuitionistic fuzzy conformable. Hence, by Proposition 4.9(2), $P \in \mathrm{I}FC_{G,(\lambda,\mu)}(K)$. Let $x \in K$. Then

$A_P(x) = P(x,e) = R_A(x,e) = A(x).$
Thus $\Psi([P]_{(\lambda,\mu)}) = A_P = A$. So Ψ is surjective. Hence Ψ is a monoid isomorphism. Therefore $\mathrm{I}FC_{G,(\lambda,\mu)}(K)/\sim$ and $\mathrm{I}FNG_{(\lambda,\mu)}(K)$ are isomorphic monoids under Ψ. This completes the proof. □

Corollary 6.6. Let K be a group. Then the semigroup $(\mathrm{I}FC_{G,(1,0)}(K), \circ)$ is isomorphic to the semigroup $(\mathrm{I}FNG_{(1,0)}(K), \circ)$. In fact, $\mathrm{I}FC_{G,(1,0)}(K) = \mathrm{I}FC(K)$ and $\mathrm{I}FNG_{(1,0)}(K) = \mathrm{I}FN^*(K)$.

Proof. It is clear that $[R]_{(1,0)} = \{R\}$ for each $R \in \mathrm{I}FC_{G,(1,0)}(K)$. Then $(\mathrm{I}FC_{G,(1,0)}(K), \circ)$ can be identified with $(\mathrm{I}FC_{G,(1,0)}(K)/\sim, *)$. Hence, by Theorem 6.6, $(\mathrm{I}FC_{G,(1,0)}(K), \circ)$ is isomorphic to $(\mathrm{I}FNG_{(1,0)}(K), \circ)$ as semigroups. Moreover, $\mathrm{I}FC_{G,(1,0)}(K) = \mathrm{I}FC(K)$ and $\mathrm{I}FNG_{(1,0)}(K) = \mathrm{I}FN^*(K)$. □

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