Stability of nonlinear differential system by Lyapunov method

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Abstract We abtain some stability results for a very general differential system using the method of cone valued vector Lyapunov functions and conversely some sufficient conditions for existence of such vector Lyapunov functions.

Key Words : local lipschitz conditions, quasimonotone, cone valued vector Lyapunov function, $\phi(t)$ -stalility

1. Preliminaries and Definitions

Lyapunov second methods are now well established subjects as the most powerful techniques of analysis for the stability and qualitative properties of nonlinear differential equations $x' = f(t,x),\ x(t_0) = x_0 \in R^N$.

One of the original Lyapunov theorems is as follows:

Lyapunov Theorems. For $x' = f(t,x)$, assume that there exists a function $V: R_+ \times S_\rho \rightarrow R_+$ such that

(i) V is C^1-function and positive definite,

(ii) V is decresent,

(iii) $\frac{d}{dt} V(t,x) = V_t(t,x) + V_x \cdot f(t,x) \le -a(\| x \|)$

for $t \ge 0,\ x \in S_\rho$, where $S_\rho = \{x \in R^N \| x \| < \rho\}$ for $\rho > 0$, $a(r)$ is strictly increasing function with $a(0) = 0$.

Then the trivial solution $x(t) \equiv 0$ is uniformly asymptotically stable.

The advantage of the method is that it does not require the knowledge of solutions to analysis the stability of the equations. However in practical sense, how to find suitable Laypunov functions V for given equations are the most difficult questions. Hence weakening the conditions (i), (ii), and (iii), and enlarging the class of Lyapunov functions are basic trends in Lyapunov stability theory [2, 3, 4, 5, 6, 11].

In the unified comparison frameworks, Ladde [7] analysed the stability of comparison differential equations by using vector Lyapunov function methods.

Lakeshmikantham and Leela [9] initiated the cone valued Lyapunov function methods to avoid the quasimonotonicity assumption of comparison equations. They obtained various useful differential inequalities with cone-valued Lyapunov functions, Akpan and Akinyele [1] extended and generalized the results of [7,8] to the ϕ_0-stability of the compari -son differential equations by using the cone-valued Lyapunov functions.

Here we generalize, in some sense, the results of [1] to the $\phi(t)$-stabilities of comparison equations below.

Let R^n denote the n-dimensional Euclidean

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space with any equivalent norm $\| \cdot \|$, and scalar product$(,)$.

$R_+ = [0,\infty)$. $C[R_+ \times R^n, R^n]$ denotes the space of continuous functions from $R_+ \times R^n$ into R^n.

Definition 1.1 ([11]). A proper subset K of R^n is called a cone if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K+K \subset K$; (iii) $K=\overline{K}$; (iv) $K^\circ \neq \varnothing$; (v) $K \cap (K), = =\{0\}$ where $\overline{K}$ and K° denote the closure and interior of K, respectively and ∂K denotes the boundary of K. The order relation on R^n induced by the cone K is defined as follow:

For $x, y \in R^n$, $x \leq_k y$ iff $x-y \in K$, and $x \leq_{k^\circ} y$ iff $y-x \in K^\circ$.

Definition 1.2 ([11]). The set $K^* = \{\phi \in R^n : (\phi, x) \geq 0,$ for all $x \in K\}$ is called the *adjoint cone* of K if K^* itself satisfies Definition 1.1.

Note that $x \in \partial K$ if and only if $(\phi, x)=0$ for some $\phi \in K^*$, where $K_0 = K - \{0\}$.

Consider the differential equation

$$x' = f(t,x),\ x(t_0)=x_0,\ t_0 \geq 0 \qquad (1)$$

where $f \in C[R_+ \times R^N, R^N]$ and $f(t,0)=0$ for all $t \geq 0$. Let $S_\rho = \{x \in R^N : \| x \| < \rho\}$, $\rho > 0$. Let $K \subset R^n$ be a cone in R^n, $n \leq N$. For $V \in C[R_+ \times S_\rho, K]$ at $(t,x) \in R_+ \times S_\rho$, let

$$D^+ V(t,x) = \lim_{h \to 0^+} \frac{1}{h} [V(t+h, x+hf(t,x)) - V(t,x)]$$

be a Dini derivative of V along the solution curves of the equations (1).

Consider a comparison differentia equation

$$u' = g(t,u),\ u(t_0)=u_0,\ t_0 \geq 0 \qquad (2)$$

where $g \in C[R_+ \times K, R^n]$, $g(t,0)=0$ for all $t \geq 0$ and K is a cone in R^n.

Let $S(\rho) = \{u \in K : \| u \| < \rho\}$, $\rho > 0$. for $v \in C[R_+ \times S(p), K]$ $(t,u) \in R_+ \times S(p)$ let $D^+ v(t,u)$ $= \lim_{h \to 0^+} \frac{1}{h} [v(t+h, u+hg(t,u)) - v(t,u)]$ be a Dini derivative of v along solution curves of the equation (2).

Definition 1.3 ([11]). A function $g : D \to R^n$, $D \subset R^n$ is said to be *quasimonotone* nondecreasing relative to the cone K when it satisfies that if $x, y \in D$ with $x \leq_K y$ and $(\phi_0, y-x) = 0$ for some $\phi_0 \in K_{0*}$, then $(\phi, g(y)-g(x)) \geq 0$.

Definition 1.4 ([8,10]). The trivial solution $x=0$ of (1) is (S_1) *equistable* if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $\| x_0 \| < \delta$ implies $\| x(t, t_0, x_0) \| < \varepsilon$, for all $t \geq t_0$.

Other stability notions ($S_2 \sim S_8$) can be similarly defined [8,10].

Now we give cone-valued $\phi(t)$-stability definitions of the trival solution of (2).

Let $\phi : [0,\infty] \to K^*$ be a cone-valued function.

Definition 1.5 ([12]). The trivial solution $u=0$ of (2) is

(S_1^*) $\phi(t)$-equistable if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon$, for all $t \geq t_0$ where $r(t)$ is a maximal solution of (2);

(S_2^*) *uniformly* $\phi(t)$-stable if the δ in (S_1^*) is independent of t_0;

Other $\phi(t)$-stability notions ($S_3^* \sim S_8^*$) can be similarly defined [12].

Lemma 1.6 ([9,11]). Assume that (i) $V \in C[R_+ \times S_\rho, K]$, $V(t,x)$ satisfies a local Lipschitz condition in x relative to the cone and for $(t,x) \in R_+ * S_\rho$, $D^+ V(t,x) \leq_K g(t, V(t,x))$;

(ii)$g \in C[R^+ \times K, R^n]$ and g(t,u) is quasimonotone in u with respect to K for each $t \in R_+$.

If $r(t,t_0,u_0)$ is a maximal solution of (2) relative to K and $x(t,t_0,u_0)$ is any solution of (1) with $V(t_0,x_0) \leq_K u_0$, then on the common interval of existence, we have $V(t,x(t,t_0,x_0)) \leq_K r(t,t_0,u_0)$.

2. Main Resuts

In this section, we investigate sufficient conditions for $\phi(t)$-stability of the trivial solution u=0 of the comparison system(2). We also investigate the corresponding stability concepts of the trivial solution x=0 of (1) using differential inequalities with the method of cone-valued Lyapu -nov vector function.

Theorem 2.1. Assume that

(i) $V \in C[R_+ \times S_p, K]$, $V(t,x)$ is locally Lipschitzian in x relative to K and for $(t,x) \in R_+ \times S_\rho$, $D^+ V(t,x) \leq_K 0$,

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$,

(iii) $\phi(t) \in K_0^*$ is a bounded continuous function on $[0, \propto]$ and $a([(\phi(t),r(t)]) \leq (\phi(t),v(t,u(t))$, $t \geq t_0 \geq 0$ for some function $a \in K$.

Then the trivial solution $u=0$ of (2) is $\phi(t)$ -eqistable.

Proof) Let ϵ>0 be arbitarily given and let M =sup{ $\| \phi(t) \| \, |t \geq 0$}.

Since $a^{-1}(Ma(\eta))$ is continuous and $a^{-1}(Ma(0))$=0, there ϵ_1>0 such that $a^{-1}(Ma(\eta)) \leq \epsilon$ for $0 \leq \eta \leq \epsilon_1$. Since v(t,0)=0 and v(t,u) is continuous in u, given a(ϵ_1)>0, $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t, a(\epsilon_1))$ such that $\| u_0 \| < \delta_1$ implies $\| v(t_0,u_0) \| < a(\epsilon_1)$. Now for the bounded continuous function $\phi(t) \in K_0$*, $(\phi(t_0),u_0) \leq \| \phi(t_0) \| \, \| u_0 \| \leq \| \phi(t_0) \| \delta_1$ implies $(\phi(t),v(t_0,u_0) < \| \phi(t) \| a(\epsilon_1)$. Put $\delta = \| \phi(t_0) \| \delta_1$.Then$(\phi(t_0),u_0) < \delta$ implies $(\phi(t),v(t_0,u_0) \leq \| \phi(t) \| \, \| (v(t_0,u_0) \| < Ma(\epsilon_1)$. Let u(t) be any solution of (2) such that $(\phi(t_0),u_0)) < \delta$ implies $a([(\phi(t),r(t)]) \leq (\phi(t),v(t,u(t)) \leq (\phi(t),v(t_0,u(t_0)) < Ma(\epsilon_1)$.

Hence $(\phi(t),r(t)) \leq a^{-1}(Ma(\epsilon_1)) \leq \epsilon$ which completes the proof.

Theorem 2.2 ([12]). Assume that

(i) $V \in C[R_+ \times S_p, K]$, $V(t,x)$ is locally Lipschitzian in x relative to K and for $(t,x) \in R_+ \times S_\rho$, $D^+V(t,x) \leq_K g(t, V(t,x))$,

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$,

(iii) there exist $a,b \in K$ such that for some $\phi(t) \in K_{0^*}$, for each $x \in S_P$, $b(\| x \|) \leq (\phi(t), V(t,x)) \leq a(\| x \|)$ $t \geq t_0 \geq 0$

Then the trivial solution $x=0$ of (1) has the corresponding one of the stability ($S_1 \sim S_8$) properties if the trivial solution $u=0$ of (2) has each one of the $\phi(t)$-stability ($S_1^* \sim S_8^*$) properties in Definition 1.5.

Theorem 2.3 Assume that

(i) $V \in C[R_+ \times S_p, K]$, $V(t,x)$ is positive definite and locally Lipschitzian in x relativeto K and for $(t,x) \in R_+ \times S_\rho$, where A(t)>0 is continuous function, $A(t)D^+ V(t,x) + V(t,x) D^+ A(t) \leq_K g(t, V(t,x)A(t))$,

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$,

(iii) there exist $b \in K$ such that for some

$\phi(t) \in K_{0*}$, for each $x \in S_P$,
$b(\| x \|) \leq (\phi(t), V(t,x)) \leq a(\| x \|)$,
$t \geq t_0 \geq 0$

Then if the trivial solution $u=0$ of (2) is $\phi(t)$-eqistable(*uniformly* $\phi(t)$-stable), the trivial solution $x=0$ of (1) is stable (uniformly stable).

Proof) Suppose that the trivial solution $u=0$ of (2) is $\phi(t)$-eqistable. Let $0<\epsilon<\rho$ be arbitrarily given and $t_0 \in R_+$. Then there exists $\delta=\delta(t_0,\epsilon)>0$ such that $(\phi(t_0),u_0)<\delta$ implies $(\phi(t),r(t))<b(\epsilon)$ for all $t \geq t_0$ where r(t) be a maximal solution of (2) relative to K. For given $x_0=x(t_0) \in S_\rho$, we can take $u_0=u(t_0)$ in K such that $a(\| x(t_0) \|)=(\phi(t_0), V(t_0)A(t_0))$ and $V(t_0, x(t_0)A(t_0)) \leq_K u_0$.
Note that if $x(t,t_0,x_0)$ is any solution of (1) such that $V(t_0,x(t_0)A(t_0)) \leq_K u_0$, then $V(t,x(t)A(t)) \leq_K r(t)$.
From (iii), we may assume that $V(t,0)=0$. Suppose $u_0 \in K^0$ and $(\phi(t_0),u_0A(t_0)<\delta$. Since $V(t,x)A(t)$ is continuous in x, there exist $\delta^*(u_0)>0$ such that $V(t_0,x(t_0)A(t_0)) \leq_K u_0$ for any $\| x \| <\delta^*$.
Now choose $\delta_1>0$ such that $a(\delta_1) \leq \delta$ and $\delta_1 \leq \delta^*$. Then the inequalities $\| x(t_0) \| <\delta_1$ and $a(\| x(t_0) \|)<\delta$ hold simultaneously. Since $b(\| x(t) \|)< b(\epsilon) \leq (\phi(t), V(t,x(t)A(t))) \leq (\phi(t),r(t))$ for all $t \geq t_0$, $\| x(t;t_0,x_0) \| <\epsilon$ whenever $\| x(t_0) \| <\delta_1$. Hence the trivial solution x=0 of (1) is eqistable.

Let $H=\{a \in C[R_+,R_+] \mid a(t)$ is strict -ly increasing in t and $a(0)=0\}$.

Theorem 2.4 Assume that

(i) $f \in C[R_+ \times S_\rho^*, R^N]$, $f(t,0)=0$, and $f(t,x)$ satisfies a Lipschitz condition in x such that $\| f(t,x)-f(t,y) \|_{G_N} \leq_K L_1(t)$ $\times \| x-y \|_{K_N}, (t,x),(t,y) \in R_+ \times S_P^*$ with $\theta>0, t \geq 0, L_1(s)ds \leq N\theta$ for some constant $N>0$.

(ii) The solution $x(t,0,x_0)$ of (1) satis -fies that any $x_0 \in S_p^*$,
$\beta_1(\| x_0 \|) \leq \| x(t,0,x_0) \| \leq \beta_2(\| x_0 \|)$, for some $\beta_1,\beta_2 \in$ K (3)

(iii) $g \in C[R_+ \times K, R^n], g(t,0)=0$, and $g(t,u)$ satisfies a Lipschitz condition in u such that $\| g(t,u)-g(t,v) \|_{K_n}$ $\leq_K L_2(t) \| u-v \|_{K_n}, (t,u),(t,v) \in R_+ \times K$.

(iv) The solution $u(t,0,u_0)$ of (2) satisfies that
$\gamma_1[(\phi(t),u_0)]$
$\leq (\phi(t),u(t,0,u_0)) \leq \gamma_2[(\phi(t),u_0)], t \geq 0$ (4)
for some $\phi(t) \in K_0^*$, some $\gamma_1,\gamma_2 \in$ K.

Then there exists a cone-valued function V with the properties
(a) $D^+V(t,x) \leq_K g(t,V(t,x))$.
(b) $b(\| x \|) \leq (\phi(t), V(t,x)) \leq a(\| x \|)$, for some $a,b \in$ K

Proof) From (i)and (iii), the existence and uniqueness of solutions of (1) and (2) as well as their continuous dependence on the initial values are followed.

Let $x(t,0,x_0), u(t,0,u_0)$ be the solutions of (1) and (2) passing through the points $(0,x_0)$ and $(0,u_0)$ satisfying (3)and (4), respectively.

Let us choose a function $G(r)$ such that $G(0)=0, G'(0)=0, G(r)>0, G''(r)>0$ for $r>0$, and let $\alpha>1$,

$$G(r)=\int_0^r du \int_o^u G''(v)dv$$ and

$$G(\frac{r}{\alpha})=\int_0^{r/\alpha} du \int_0^u G''(v)dv,$$

we have, setting $u=w/\alpha$,

$$G(\frac{r}{\alpha}) = \frac{1}{\alpha}\int_0^r dw \int_o^{w/\alpha} G''(v)dv$$
$$< \frac{r}{\alpha}\int_0^r dw \int_0^w G''(v)dv = \frac{1}{\alpha}G(r).$$

Let w be any given point in K_0. Let $\sigma_w : S_p^* \to K$ be a function with values in the cone $K \subset R^n$, defined by for $x \in S_p^*$,

$$\sigma_w(x) = G(\| x(t+\delta,t,x) \|)(\frac{1+\alpha\delta}{1+\delta})w \qquad (5)$$

For $\delta = 0$, we have from (5) that $G(\| x \|)w \leq_K \sigma_w(x)$,

and $(\phi(t), G(\| x \| w) \leq (\phi(t),\sigma_w(x))$ for any $\phi(t) \in K_0^*$ and each $t \geq 0$. Suppose that $\eta_1 = \inf\{(\phi(t),w) : t \geq 0\} > 0$. Let $\beta_3(r) = \eta_1 G(r), r > 0$. Then $\beta_3(\| x \|) \leq (\phi(t),\sigma_w(x))$, for each $\epsilon > 0$, let $\delta = \beta_2^{-1}(\epsilon)$. From the estimate (3), $\| x_0 \| < \delta$ implies $\| x(t) \| < \beta_2(\| x_0 \|) < \beta_2(\delta) = \beta_2(\beta_2^{-1}(\epsilon)) = \epsilon, t \geq 0$.

Hence the solution $x = 0$ of (1) is uniformly stable. Thus by Theorem 5.4.3 in [14], $\| x(t+\delta,t,x) \| < c(\| x \|)$ $c \in K$. Therefore $G(\| x(t+\delta,t,x) \| 0) < G(c(\| x \|))$. Since $(1+\alpha\delta)/(1+\delta) < \alpha$, it follow that

$$(\phi(t),\sigma_w(w)) \leq (\phi(t), \alpha G(c(\| x \|))w)$$
$$\leq \eta_2 \alpha G(c(\| x \|).$$

Suppose that $\eta_2 = \sup\{(\phi(t),w) : t \geq 0\} < \infty$. Hence if $\beta_4(r) = \eta_2 \alpha G(c(r))$, then

$$\beta_3(\| x \|) \leq (\phi(t),\sigma_w(x)) \leq \beta_4(\| x \|) \qquad (6)$$

Define a cone-valued function $V(t,x)$ by $t \geq 0, x \in S_p^*$,

$$V(t,x) = u(t,0,\sigma_w(x(t,0,x))) \qquad (7)$$

where $u(t,0,u_0)$ are solutions of (2) passing through $(0,u_0)$. By hypotheses (i)and (iii) and the choice of $\sigma_w(x), V(t,x)$ is continuous in t and x.

For $h > 0$ sufficiently small,

$$V(t+h,x+hf(t,x)) - V(t,x)$$
$$\leq_K \beta(t) \| x + hf(t,x) - x(t+h,t,x) \|$$
$$\times e(t,x,h)$$
$$+ V(t+h,x(t,h,t,x)) - V(t,x),$$

where $\lim_{t\to\infty} \frac{1}{h} e(t,x,h) = 0$.

Divide both sides by h and take lim as $h \to 0^+$ to obtain $D^+ V(t,x)$

$$\leq_K \lim_{h\to 0^+} [\frac{1}{h} V(t+h,x(t+h,t,x)) - V(t,x)]$$
$$= \lim_{h\to 0^+} \frac{1}{h}[u(t+h,0,\sigma_w(x(t+h,0,x)))$$
$$- u(t,0,\sigma_w(t,0,x)))]$$
$$= u'(t,0,\sigma_w(x(t,0,x))) = g(t, V(t,x)).$$

Now from (3), we can take $\beta_1, \beta_2 \in$ K which satisfy (3) and (8) simultaneously,

$$\beta_2^{-1}(\| x \|) \leq \| x(t,0,x) \| \leq \beta_1^{-1}(\| x \|)$$

(8). Since $(\phi(t), V(t,x)) = (\phi(t),u(t,0,\sigma_w(t,0,x)))$ from (4), (6), and (8) we have

$$(\phi(t), V(t,x) \geq \gamma_1(\phi(t),\sigma_w(x(t,0,x))))$$
$$\geq \gamma_1(\beta_3(\| x(t,0,x) \|))$$
$$\geq \gamma_1(\beta_3(\beta_2^{-1}[\| x \|)) \equiv b(\| x \|), b \in \mathrm{K}.$$

On the other hand

$$(\phi(t), V(t,x) \leq \gamma_2((\phi(t),\sigma_w(x(t,0,x))))$$
$$\leq \gamma_2(\beta_4(\| x \|))$$
$$\leq \gamma_2(\beta_4(\beta_1^{-1}(\| x \|), \quad a \in \mathrm{K}$$

This completes the proof of Theorem 2.4.

References

[1] Edet P. Akpan and Olysola Akinyele, *On the ϕ_0-Stability of Comparison Differential Systems* J. of Math. Anal. and Appl. 164 (1992) 307-324.

[2] F. Brauer, *Some refinements of Lyapunov's second method*, Canad. J. Math. (1965), 811-819.

[3] F. Brauer and J. A. Nohel, *Qulitative theory of ordinary differential equa -tions*, Benjamin, New York, 1969.

[4] F.M. Danan and S. Elaydi, *Lipschitz*

stability of nonlinear systems of differential equations, II, Lyapunov function, J. Math. Anal. Appl. 143(1989), 517-529.

[5] A. Halanay, *Differential equations: stability*, Oscillation, Time Lag. Academic Press, New York 1966.

[6] S. Heikkila, *On the quasimonotoni -city of linear differential systems*, Appl. Anal. 10 (1980), 121-126.

[7] G. S. Ladde, *Competitive processes and comparison differential systems*, Trans. Amer. Math. Soc. 221(1976), 391-402.

[8] V. Lakshmikantham and S. Leela, "*Differential and integral inequali -ties*, Thoery and Appl." Vol. I, Academic Press, New York, 1969.

[9] V. Lakshmikantham and S. Leela, *Cone-valued Lyapunov function*, J. Nonlinear Anal. Theory, Methods and Appl. Vol. 1 No. 3(1977), 215-222.

[10] V. Lakshmikantham, S. Leela, and A.A. Martynyuk, *Stability analysis of nonlinear system*, Marcel Dekker, Inc. 1989.

[11] V. Lakshmikantham, V. M. Matro -sov, and S. Sivasundaram, *Vector Lyapunov function and stability anal -ysis of nonlinear systems*, Klluwer Academic Publi -shers, 1991.

[12] Y. S. Oh and J. H. An, *On $\phi(t)$- stability for the comparison differen -tial systems*, J. of Korea Soc. of Math. Ed.(b):the pure and applied math. no.2(1997) pp.181-166.

[13] Y. S. Oh and J. H. An, *On uniform asymptotic stability of the nonlinear differential system*, J. Korea indu -strial ISS, Vol. 9, No. 4, 2004.

[14] M. R. M. Rao, *Ordinary differential equations*, Theory and Applications, East-West Press, New Delhi, 1980.

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