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Moment-Based Density Approximation Algorithm for Symmetric Distributions

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Abstract

Given the moments of a symmetric random variable, its density and distribution functions can be accurately approximated by making use of the algorithm proposed in this paper. This algorithm is specially designed for approximating symmetric distributions and comprises of four phases. This approach is essentially based on the transformation of variable technique and moment-based density approximants expressed in terms of the product of an appropriate initial approximant and a polynomial adjustment. Probabilistic quantities such as percentage points and percentiles can also be accurately determined from approximation of the corresponding distribution functions. This algorithm is not only conceptually simple but also easy to implement. As illustrated by the first two numerical examples, the density functions so obtained are in good agreement with the exact values. Moreover, the proposed approximation algorithm can provide the more accurate quantities than direct approximation as shown in the last example.

Keywords: Approximation algorithm; density approximation; moments; percentage points; symmetric distributions; transformation of variables.

1. Introduction

Symmetric distributions have been given much attention not only in the statistical literature but also in many scientific fields. For instance, test statistics for symmetry such as the Cramer von Mises test were developed and a family of symmetric unimodal distributions on the circle was considered by Rothman and Woodroofe (1972) and Jones and Pewsey (2005). In connection with Tukey's symmetric Lambda distributions, the k^{th} moment of the proposed order statistics was explicitly derived for approximating its expectation in Joiner and Rosenblatt (1971); as can be seen in this case, it is possible to

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determine the moments of various statistical quantities, whereas their exact density and distribution functions are often analytically intractable or difficult to obtain in closed forms.

Several types of moment- or cumulant-based approximations to the distributions of various random quantities of interest have been proposed in the statistical literature. Gram-Charlier series and Cornish-Fisher expansions as well as saddlepoint type approximations have been extensively discussed over the last several decades in connection with this problem. These techniques were introduced for approximating density functions for finite sums of independent and identically distributed random variables. While the moment-generating functions or characteristic functions of convolutions can be easily obtained, this is not the case for their density functions. It should be noted that these methods are very useful for approximating the density functions of test statistics since under the null hypothesis, many test criteria can be expressed in terms of a convolution. A Gram-Charlier approximation might be a good option for approximating a density function when the normal approximation does not provide enough accuracy. The saddlepoint approximation method, which was pioneered by Daniels (1954), has been much investigated. It was shown to provide excellent approximations in many statistical applications. The most significant advantage of the saddlepoint method is that the approximants so obtained are usually quite accurate in the tail areas of the target density. It ought to be pointed out that those existing methods might have difficulties to provide accurate approximations to the examples used in this paper since they have unusual features.

The distribution of a symmetric random variable can be approximated from its moments either directly on its entire support or via a transformation. The latter approach is recommended when the distribution exhibits a sharp peak or trough at the point of symmetry. This paper introduces a density approximation algorithm that is based on the transformation of variables techniques and a density approximant proposed by Ha and Provost (2007). This approach makes use of the fact that, once translated by λ, the point of symmetry, the distribution being approximated and the corresponding distribution defined on the positive half-line share the same even moments.

The required notation is introduced in Section 2 and the density approximation algorithm specially designed for the symmetric distributions on the basis of a matching-moment technique is described in Section 3. Initial approximations having beta and gamma densities are being considered. In Section 4, the proposed methodology is applied to three artificial examples wherein the accuracy measures are computed to illustrate the advantage of the proposed approximation algorithm. Certain computational aspects and concluding remarks are discussed in Section 5.

2. Notation

The following notation will be used in this paper.

λ : the point of symmetry,

Z : a continuous random variable with symmetric distribution about the point of symmetry λ,

$f_Z(\cdot)$ and $\mu_Z(h)$: the symmetric density function of Z and the h^{th} raw moment of the random variable Z, respectively,

X : the corresponding random variable centered about zero, that is, $X = Z - \lambda$,

$f_X(\cdot)$ and $\mu_X(h)$: the density function and the h^{th} raw moment of X, respectively,

R : a random variable having a *bona fide* density function, which is defined on the positive part of X,

$f_R(\cdot)$ and $\mu_R(h)$: the density function and the h^{th} raw moment of R, respectively,

T : random variable denoting a square of R, that is, $T = R^2$,

$f_T(\cdot)$ and $\mu_T(h)$: the density function and the h^{th} raw moment of T, respectively,

$\psi(\cdot)$: an initial approximation to a density function,

$m(h)$: the h^{th} raw moment of an initial density approximant,

$f_{Z_d}(z)$, $f_{X_d}(x)$, $f_{R_d}(r)$ and $f_{T_d}(t)$: density approximants with polynomial adjustments of degree d for Z, X, R and T, respectively,

$\mathcal{I}_S(x)$: the indicator function with respect to the set S, which is equal to 1 when $x \in S$ and 0 otherwise.

3. The Algorithm

The following approximation algorithm, which comprises of four phases, is specially designed for approximating symmetric distributions on the basis of their theoretical moments. Our aim is to obtain an approximate density function for a symmetric random variable on the basis of its moments.

Phase 1. **Transformation**

The *first* phase is the transformation of the random variable Z, which comprises of three steps. The *first* step is a transformation of the symmetric random variable and its moments about its point of symmetry. Letting λ be the point of symmetry of the random variable Z, the proposed methodology is applied to $X = Z - \lambda$, whose h^{th} moment is

$$\mu_X(h) = E(Z-\lambda)^h = \sum_{j=0}^{h}\binom{h}{j}(-\lambda)^j E(Z^{h-j}) = \sum_{j=0}^{h}\binom{h}{j}(-\lambda)^j \mu_Z(h-j). \quad (3.1)$$

The resulting density approximant is symmetric about zero, that is, $f_X(x) = f_X(-x)$. The *second* step consists in defining a random variable R whose support is on a subset of the positive half-line. Its density function $f_R(x)$ is such that

$$f_X(x) = \frac{1}{2} f_R(-x)\, \mathcal{I}_{(-\infty,0)}(x) + \frac{1}{2} f_R(x)\, \mathcal{I}_{(0,\infty)}(x) \tag{3.2}$$

and

$$f_R(r) = 2\, f_X(r)\, \mathcal{I}_{(0,\infty)}(r)\,. \tag{3.3}$$

We note that $\mu_R(2h) = \mu_X(2h)$, $h = 0, 1, \ldots$. Clearly, all the odd moments of X are equal to zero and no information is available on the odd moments of R. In the *final* step of the first phase, we let $T = R^2$ and $\mu_T(h)$ denote the h^{th} moment of T. Since the odd moments of R are not available from those of X, we approximate the density of T whose moments, which are denoted by $\mu_T(h)$ with $\mu_T(h) = \mu_X(2h)$, $h = 0, 1, \ldots$, are all available.

Phase 2. **Initial Approximation**

The *second* phase consists in obtaining an initial approximation to the density of T. As pointed out in Ha and Provost (2007), when selecting a density function as the initial approximation, it should be kept in mind that one may choose a certain density function to be asymptotically or approximately distributed. For example, if it is known that a certain random quantity has approximately or asymptotically a chi-squared distribution, for instance, quadratic forms in normal random variables, then a gamma density as an initial approximation would be appropriate. Initial approximants can also be determined from the information about support of the target density. For instance, such an initial approximant could be a uniform or beta density function when the support is compact, or a gamma or Weibull density function when the support is the positive half-line. The choice of an initial approximate density function can be made on the basis of the support of T and possibly on the behavior of its moments.

For example, when a beta density in the interval (0, 1) is suitable as an initial approximation, the baseline density, $\psi(x)$, is

$$\psi(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, t^{\alpha-1}(1-t)^{\beta-1}\, \mathcal{I}_{[0,1]}(t)\,, \tag{3.4}$$

$\mathcal{I}_A(t)$ denoting the indicator function, which is equal to 1 when $t \in A$ and 0 otherwise. The parameters α and β are estimated as follows:

$$\alpha = \frac{\mu_T(1)(\mu_T(1) - \mu_T(2))}{\mu_T(2) - \mu_T(1)^2} \quad \text{and} \quad \beta = \frac{(1-\mu_T(1))(\alpha+1)}{\mu_T(1)} \tag{3.5}$$

(see for instance Johnson *et al.*, 1995, Section 25). The j^{th} moment of this beta distribution is given by

$$m(j) = \frac{\Gamma(\alpha+\beta)\, \Gamma(\alpha+j)}{\Gamma(\alpha)\, \Gamma(\alpha+j+\beta)} = \frac{\prod_{k=0}^{j-1}(k+\alpha)}{\prod_{k=0}^{j-1}(k+\alpha+\beta)}\,, \qquad j = 1, 2, \ldots. \tag{3.6}$$

When the target density of T has an approximate or asymptotic gamma distribution, the initial approximant can be chosen to be a gamma density function with parameters γ and δ, that is,

$$\psi(t) = \frac{1}{\Gamma(\gamma)\,\delta^{\gamma}} t^{\gamma-1} e^{-t/\delta}\, \mathcal{I}_{(0,\infty)}(t) \tag{3.7}$$

whose h^{th} moment is given by

$$m(h) = \frac{\delta^h \Gamma(\gamma+h)}{\Gamma(\gamma)} = \delta^h \prod_{i=1}^{h} (\gamma + h - i)\ , \quad h = 0, 1, \ldots, \tag{3.8}$$

the parameters γ and δ being estimated as follows:

$$\gamma = \frac{\mu_T(1)^2}{\mu_T(2) - \mu_T(1)^2} \quad \text{and} \quad \delta = \frac{\mu_T(2)}{\mu_T(1)} - \mu_T(1) \tag{3.9}$$

(see for example Johnson *et al*, 1994, Section 17).

Phase 3. **Polynomial Adjustment**

In this phase, we determine a polynomial adjustment such that the first d moments of the approximant coincide with those of T. As shown in Provost (2005), the coefficients ξ_i of the d^{th} degree polynomial adjustment, $\sum_{i=0}^{d} \xi_i t^i$, satisfy the following equation. It could be useful to provide the main equation which yields this system,

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{d-1} \\ \xi_d \end{bmatrix} = \begin{bmatrix} m(0) & m(1) & \cdots & m(d) \\ m(1) & m(2) & \cdots & m(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ m(d-1) & m(d) & \cdots & m(2d-1) \\ m(d) & m(d+1) & \cdots & m(2d) \end{bmatrix}^{-1} \begin{bmatrix} \mu_T(0) \\ \mu_T(1) \\ \vdots \\ \mu_T(d-1) \\ \mu_T(d) \end{bmatrix}. \tag{3.10}$$

Thus, the density approximant denoted by $f_{T_d}(t)$ can be expressed as the product of an initial approximation, $\psi(x)$, which is in fact a density function whose parameters are estimated by matching the moments of the target density to those of the initial approximant, and a polynomial adjustment, $\sum_{i=0}^{d} \xi_i t^i$, whose coefficients ξ_i satisfy Equation (3.10), that is,

$$f_{T_d}(t) = \psi(t) \sum_{i=0}^{d} \xi_i t^i\,. \tag{3.11}$$

It should be mentioned that the choice of the initial approximant plays a key role for approximating the density function of T. If the convergence rate of the tails of the initial approximant agrees with those of the target density, the approximation by making use of the product of an initial approximant and a polynomial adjustment is more likely to provide accuracy to the target density than the approximation on the basis of the initial approximant chosen without consideration about the tail convergence rate.

Phase 4. **Inverse Transformation for obtaining a Density Approximant to Z**

In this final phase, we wish to obtain a density approximant for Z the original symmetric random variable on the basis of our density approximant of T. This phase comprises of three steps as was the case for the inverse of the transformation obtained in the first phase. First, we apply the inverse of the square transformation to obtain a density approximant for R, denoted by $f_{R_d}(r)$. This yields

$$f_{R_d}(r) = 2r f_{T_d}(r^2) = 2r\psi(r^2) \sum_{i=0}^{d} \xi_i r^{2i} . \tag{3.12}$$

Secondly, we symmetrize and normalize the approximate density of R to obtain a symmetric density approximant for X denoted by $f_{X_d}(x)$. That is, we let

$$f_{X_d}(x) = \frac{1}{2} f_{R_d}(-x)\, \mathcal{I}_{(-\infty,0)}(x) + \frac{1}{2} f_{R_d}(x)\, \mathcal{I}_{(0,\infty)}(x) . \tag{3.13}$$

Finally, we translate the distribution of X so that it is centered at the location parameter λ in order to obtain the following d^{th} degree density approximant for Z:

$$f_{Z_d}(z) = f_{X_d}(z+\lambda) . \tag{3.14}$$

It should be noted that the cumulative distribution function of X can simply be approximated by integration of the proposed density approximant. The resulting quantiles are usually quite accurate.

4. Numerical Examples

The proposed approximation algorithm is applied to three symmetric distributions in this section. A symmetrized mixture of uniform and beta distributions is considered as an example of application on a compact support. The distribution of a symmetrized mixture of exponential and generalized beta distributions, whose support is the entire real line, will also be considered. A symmetrized mixture of two equally weighted Beta distributions is also considered to show how the proposed approximation algorithm improves accuracy.

4.1. Symmetrized mixture of uniform and beta distributions

Consider the symmetrized mixture of a uniform random variable on $(0, 0.5)$ and a beta random variable with parameters 2 and 6, whose support is the interval $(0, 1)$. The point of symmetry is zero in this case. We note that this is not a simple density function

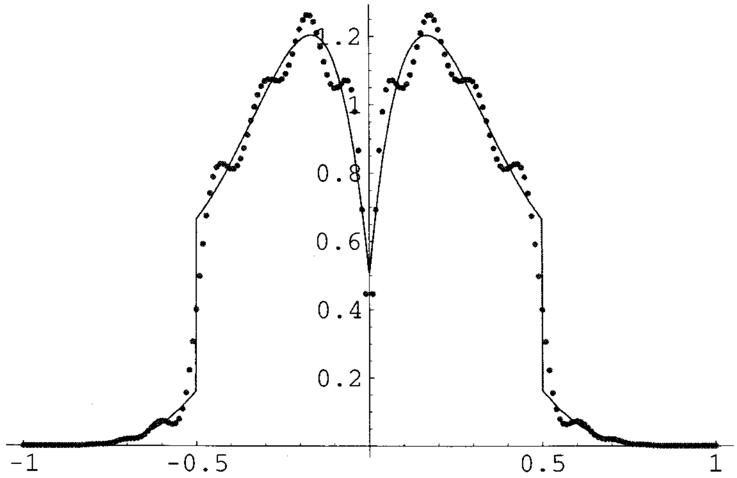

Figure 4.1: Density approximant for a certain mixture of uniform and beta distributions (dotted) and exact density

to approximate since it involves distributions having different supports and the resulting density function is not differentiable at the points of $-1/2$, 0 and $1/2$. The twentieth degree beta-polynomial approximant to the density of T, as determined from the proposed algorithm, is plotted and superimposed on the exact density in Figure 4.1. As can be seen from Figure 4.1, the proposed approach produces a density approximant that is quite accurate on the entire support including neiborhoods of the point of symmetry and the points of non differentiability.

4.2. Symmetrized mixture of exponential and generalized beta distributions

We now consider the symmetrized mixture of an exponential random variable with parameter 4 and a beta random variable with parameters $(4, 2)$ which is defined on the interval $[1/2, 2]$ with respective weights of $4/5$ and $1/5$. We approximated this density by means of a twelfth-degree gamma-polynomial approximant, making use of the four-phase approach. Figure 4.2 shows the exact and approximate densities of T, while the exact and approximate densities of X are plotted in Figure 4.3.

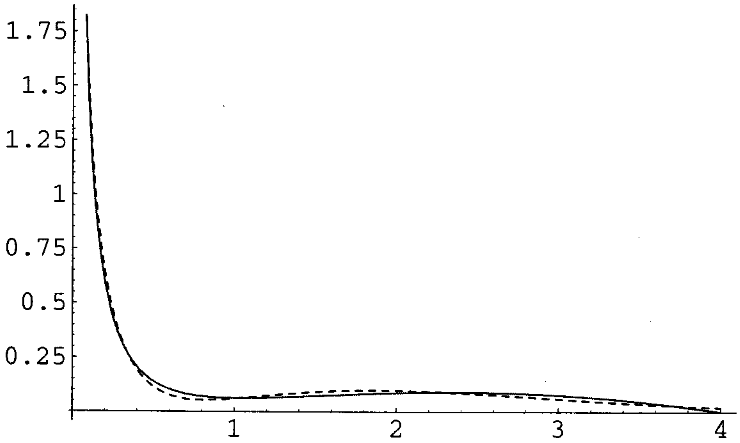

Figure 4.2: Exact and approximate (dashed) densities for a certain mixture of exponential and beta distributions

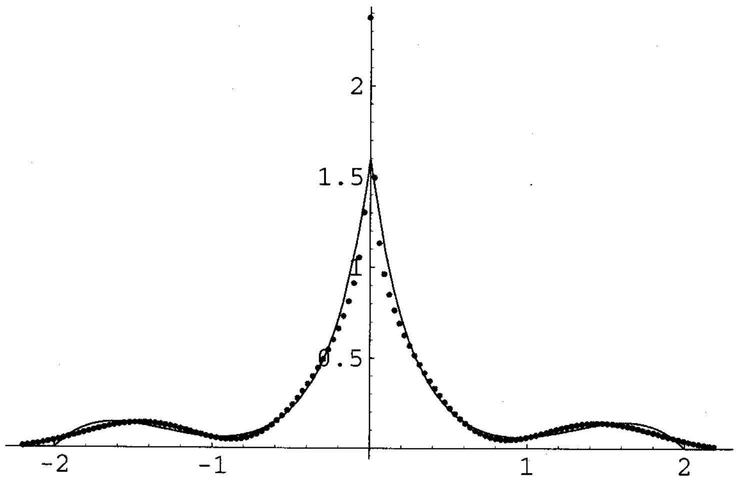

Figure 4.3: Exact and approximate (dotted) densities for X

Table 4.1: Error comparisons

Moments	*Quantities*	*Four-phase*	*Direct*
60	MSE	1.44189×10^{-10}	8.50741×10^{-10}
60	MaxSE	1.22329×10^{-9}	1.20263×10^{-8}
20	MSE	1.09936×10^{-8}	2.45313×10^{-7}
20	MaxSE	6.27226×10^{-8}	1.96224×10^{-6}

4.3. Symmetrized mixture of two equally weighted beta distributions

In order to give an indication of the improvement in accuracy that the proposed algorithm provides, we consider the symmetrized mixture of two equally weighted beta distributions with parameters $(3, 7)$ and $(8, 4)$, respectively. In this case, a uniform distribution (that is, a beta $(1, 1)$ distribution) and a sixtieth-degree (and a twentieth-degree) polynomial adjustment have been used as initial approximation to the density of R and polynomial adjustment, respectively. To illustrate the advantage gained by making use of the proposed approach, we compare it with the *direct* approximation of the density of X obtained from sixtieth and twentieth-degree Legendre polynomials which support is the interval $(-1, 1)$, which is described in Provost (2005). The direct and the proposed approaches are compared in Table 4.1 in terms of the following two quantities: the mean squared error (MSE) and the maximum squared error (MaxSE). It can be seen that in every case, the proposed approximation algorithm produces more accurate density approximants.

5. Concluding Remarks

The selection of the number of moments to be used for obtaining an appropriate approximant, which is equivalent to determining a suitable degree for the polynomial adjustment, can be determined by inspecting the density plots of approximants of successive degrees. For instance, when there are no observable differences between approximants of degrees δ and $\delta + 2$, then one can select $\delta + 1$ as the appropriate degree for the approximant. The symbolic computational package *Mathematica* was utilized for evaluating all the density approximants and suggests for plotting the graphs. The programming code in *Mathematica* is available from the author upon request. The proposed density approximation methodology is not only conceptually simple since it is essentially based on simple transformations and a moment-matching technique, but it also is easy to program. Three artificial numerical examples are proposed and approximated, which are required a relatively large number of moments since the distributions considered exhibited irregular features. Clearly, the more irregular the function to be approximated, the more moments will be required. Furthermore, as a future research, one may consider to investigate unartificial numerical examples in scientific fields which involve unusual symmetric distributions. Given the computational resources that are currently available, such approximants can quickly yield accurate percentage points, even when the calculations involve a large number of moments.

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References

Daniels, H. E. (1954). Saddlepoint approximations in statistics. *The Annals of Mathematical Statistics*, **25**, 631–650.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions, Vol 1*. 2nd ed., John Wiley & Sons, New York.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions, Vol 2*. 2nd ed., John Wiley & Sons, New York.

Joiner, B. L. and Rosenblatt, J. R. (1971). Some properties of the range in samples from Tukey's symmetric Lambda distributions. *Journal of the American Statistical Association*, **66**, 394–399.

Jones, M. C. and Pewsey, A. (2005). A family of symmetric distributions on the circle. *Journal of the American Statistical Association*, **100**, 1422–1428.

Ha, H-T. and Provost, S. B. (2007). A viable alternative to resorting to statistical tables. *Communication is Statistics: Simulation and Computation*, **36**, 1–17.

Provost, S. B. (2005). Moment-based density approximants. *The Mathematica Journal*, **9**, 727–756.

Rothman, E. D. and Woodroofe, M. (1972). A Cramer von-Mises type statistic for testing symmetry. *The Annals of Mathematical Statistics*, **43**, 2035–2038.

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