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Component Importance for Continuum Structure Functions with Underlying Binary Structures*

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Abstract

A continuum structure function (CSF) is a non-decreasing mapping from the unit hypercube to the unit interval. A B-type CSF, defined in the text, is a CSF whose behaviour is modeled by its underlying binary structures. As the measure of importance of a system component for a B-type CSF, the structural and reliability importance of a component at a system level $\alpha (0 < \alpha < 1)$ are defined and their properties are deduced.

Keywords: Continuum structures, reliability importance; structural importance.

1. Introduction

A structure function in reliability theory models the operation of a complex system by relating the states $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of the components $C = \{1, 2, \ldots, n\}$ of a system to that of the system itself. The classic theory of structure functions, as introduced by Birnbaum *et al.* (1961), assumes that each component can be in one of only two states, operating or failed and similarly that the system whose status is determined solely by the statuses of these components can itself be in only one of these two states. Consequently, the structure function is a binary function of binary variables. A coherent *binary structure function* (BSF) is a mapping $\phi : \{0,1\}^n \to \{0,1\}$ which is non-decreasing in each argument and for which no component is irrelevant. In many applications, however, the system and its components are capable of assuming a range of levels of performance, varying from perfect functioning to complete failure, and in these situations the dichotomous model is an oversimplification. In order to describe more adequately the performance of such systems and components, set-theoretic and axiomatic approaches have been adopted by a number of authors to introduce a variety of classes of multistate structures which permit the states of the components and the system itself to

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take an arbitrary finite number of values. Extending the domain and range from $\{0,1\}$ to $\{0,1,\ldots,M\}$, Barlow and Wu (1978) propose a class of multistate structure functions having a one-to-one correspondence between the BSF and the multistate structure function (MSF), and Natvig (1982) suggests a natural generalization of this class of MSF by permitting the underlying BSF ϕ to vary. A *continuum structure function* (CSF), introduced by Baxter (1984), is a mapping $\gamma : \Delta \to [0,1]$ which is non-decreasing in each argument and which satisfies $\gamma(\mathbf{0}) = 0$ and $\gamma(\mathbf{1}) = 1$, where Δ denotes the unit hypercube $[0,1]^n$ and $\boldsymbol{\alpha}$ denotes $(\alpha,\alpha,\ldots,\alpha) \in \Delta$. In the spirit of Natvig's suggestion, the following class of CSFs is proposed.

Definition 1.1 (Baxter, 1986). *A CSF γ is said to be a Natvig CSF if there exists a class of coherent BSF $\{\phi_\alpha, 0 < \alpha \le 1\}$ such that*

$$\gamma(\boldsymbol{x}) \ge \alpha \text{ iff } \phi_\alpha(\boldsymbol{I}_\alpha(\boldsymbol{x})) = 1 \ (\boldsymbol{x} \in \Delta, 0 < \alpha \le 1),$$

where $\boldsymbol{I}_\alpha(\boldsymbol{x}) = (I_\alpha(x_1),\ldots,I_\alpha(x_n))$ and $I_\alpha(x_i)$ is the indicator of $\{x_i \ge \alpha\}$, $i = 1,2,\ldots,n$.

Let γ be a Natvig CSF. The class of underlying BSFs $\{\phi_\alpha, 0 < \alpha \le 1\}$ is finite, since there are only finitely many BSFs of n components. Further, $\{\phi_\alpha, 0 < \alpha \le 1\}$ cannot be chosen arbitrarily. For γ to be a CSF, ϕ_α is to be left-continuous in α for fixed $\boldsymbol{x}$ and $\phi_\beta(\boldsymbol{y}) \ge \phi_\alpha(\boldsymbol{y})$ ($\beta < \alpha, \boldsymbol{y} \in \{0,1\}^n$). For various suggestions and the axiomatic characterizations of such classes, see Baxter and Lee (1989a), Borges and Rodrigues (1983), Griffith (1997), Kim and Baxter (1987), and Lee (2003).

In this paper, we consider a new class of CSFs, called B-type CSFs, which includes the class of Natvig CSFs. For a B-type CSF, a relevant component at a certain state is allowed to become irrelevant at some other states of the system or vice versa. In section 2, the formal definition of B-type CSFs is presented and some properties of such CSFs are deduced. In Section 3, we present the definitions of component importance and derive their properties. An example is also discussed for illustrative purpose.

2. The Class of B-type Continuum Structures

Definition 2.1 *Let $\{(\phi_\alpha, C_\alpha)\}$ be a collection of pairs such that*

(i) $\phi_\alpha : \{0,1\}^{C_\alpha} \to \{0,1\}$ is a coherent BSF, the components of which are the elements of C_α for each $\alpha \in (0,1]$,

(ii) $\cup_\alpha C_\alpha = C$,

(iii) For each $\boldsymbol{x} \in \Delta, \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{x})) \ge \phi_\beta(\boldsymbol{I}_\beta^{C_\beta}(\boldsymbol{x}))$ whenever $\alpha < \beta$ $(0 < \alpha, \beta \le 1)$.

The function $\gamma : \Delta \to [0,1]$ is said to be a *B-type CSF* if it satisfies the condition

$$\gamma(\boldsymbol{x}) \ge \alpha \text{ iff } \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{x})) = 1 \ (0 < \alpha \le 1) \text{ for all } \boldsymbol{x} \in \Delta.$$

Proposition 2.1 A B-type CSF is continuous.

Proof: Note that $\boldsymbol{I}_\alpha(\boldsymbol{x}) = \boldsymbol{I}_\alpha(\alpha \boldsymbol{I}_\alpha(\boldsymbol{x}))$ and $\boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{x}) = \boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{0}^{C-C_\alpha}\boldsymbol{x}^{C_\alpha})$ for all $\boldsymbol{x} \in \Delta$ and $\alpha (0 < \alpha < 1)$. Now, for a B-type CSF γ,

$$\begin{aligned}\gamma(\boldsymbol{x}) \geq \alpha &\Rightarrow \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{x})) = 1\\ &\Rightarrow \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(\boldsymbol{0}^{C-C_\alpha}, \boldsymbol{x}^{C_\alpha})) = 1\\ &\Rightarrow \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(\alpha \boldsymbol{I}_\alpha(\boldsymbol{0}^{C-C_\alpha}, \boldsymbol{x}^{C_\alpha}))) = 1\\ &\Rightarrow \gamma(\alpha \boldsymbol{I}_\alpha(\boldsymbol{0}^{C-C_\alpha}, \boldsymbol{x}^{C_\alpha})) \geq \alpha\\ &\Rightarrow \gamma(\alpha \boldsymbol{I}_\alpha(\boldsymbol{x})) \geq \alpha \text{ since } \boldsymbol{x} \geq (\boldsymbol{0}^{C-C_\alpha}, \boldsymbol{x}^{C_\alpha}) \text{ and } \gamma \text{ is non-decreasing.}\end{aligned}$$

□

Hence, by Theorem 2 of Lee (2003), γ is continuous, completing the proof.

Proposition 2.2 *Let γ be a B-type CSF. Then, $i \in C_\alpha$ if and only if there exists a state vector $\boldsymbol{x}$ such that $\gamma(\alpha_i, \boldsymbol{x}) \geq \alpha$ whereas $\gamma((\alpha - \epsilon)_i, \boldsymbol{x}) < \alpha$ for every $\epsilon > 0$.*

Proof: ('If') Choose $i \in C$ and suppose that $\boldsymbol{x}$ be a vector such that $\gamma(\alpha_i, \boldsymbol{x}) \geq \alpha$ whereas $\gamma((\alpha - \epsilon)_i, \boldsymbol{x}) < \alpha$ for every $\epsilon > 0$. Then, since $\gamma(\alpha_i, \boldsymbol{x}) \geq \alpha$, we have $\phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(1_i, \boldsymbol{x})) = 1$. Further, If $i \notin C_\alpha$, then $\phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(0_i, \boldsymbol{x})) = \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(1_i, \boldsymbol{x}))$ so that $\gamma(0_i, \boldsymbol{x}) \geq \alpha$, contradicting the assumption, and hence $i \in C_\alpha$ as required.

('Only if') Choose $i \in C_\alpha$. Since ϕ_α is coherent, there exists a binary vector $\boldsymbol{y}$ such that $\phi_\alpha(1_i, \boldsymbol{y}^{C_\alpha - \{i\}}) (= 1) > \phi_\alpha(0_i, \boldsymbol{y}^{C_\alpha - \{i\}}) (= 0)$, which in turn implies that $\gamma(\alpha_i, \alpha \boldsymbol{y}) \geq \alpha > \gamma((\alpha - \epsilon)_i, \alpha \boldsymbol{y})$ for every $\epsilon > 0$, as claimed. □

In a structure function, a component is considered to be relevant to the system if changing the state of the component somehow changes the system state. Thus, C_α is the set of component which are relevant to γ at level α. We note that a component which is relevant to the system at level α needs not be relevant at different levels. A CSF γ is said to be *weakly coherent* (Baxter, 1986), if $\inf_{i \in C} \sup_{\boldsymbol{x} \in \Delta} \{\gamma(1_i, \boldsymbol{x}) - \gamma(0_i, \boldsymbol{x})\} > 0$, i.e., every component is relevant to the system.

Proposition 2.3 *A B-type CSF is weakly coherent.*

Proof: Suppose that γ is a B-type CSF. Choose $i \in C$. Then, since $\cup_\alpha C_\alpha = C$, there exists an $\alpha > 0$ such that $i \in C_\alpha$. By Proposition 2.2, there exists a vector $\boldsymbol{x}$ such that $\gamma(\alpha_i, \boldsymbol{x}) \geq \alpha$ whereas $\gamma((\alpha - \epsilon)_i, \boldsymbol{x}) < \alpha$ for every $\epsilon > 0$. Then, $\gamma(0_i, \boldsymbol{x}) < \alpha \leq \gamma(1_i, \boldsymbol{x})$ so that $\sup_{x \in \Delta} \{\gamma(1_i, \boldsymbol{x}) - \gamma(0_i, \boldsymbol{x})\} > 0$. Since i is arbitrary, γ is weakly coherent as claimed. □

3. Component Importance on B-type CSFs

For a BSF ϕ, Birnbaum (1969) defines the reliability importance of component i as

$$I^{\phi}(i) = P\left\{\phi(1_i, \boldsymbol{Y}) - \phi(0_i, \boldsymbol{Y}) = 1\right\},$$

where $\boldsymbol{Y}$ is a binary random state vector of components. Various authors have proposed extensions of this concept to the multistate case (*e.g.* Barlow and Wu, 1978; Griffith, 1980; Natvig, 1982; Block and Savits, 1982) and on continua (*e.g.* Kim and Baxter, 1987; Baxter and Lee, 1989b). With a B-type CSF, we may regard part of the unit interval, say $[0, \alpha)$ $(0 < \alpha \leq 1)$, as corresponding to the failure states of the system and the components and to regard $[\alpha, 1]$ as the operating states. When this binary approach is applied, a natural generalization of $I^{\phi}(i)$ would be as follows.

Definition 3.1 *The reliability importance $R_i(\alpha)$ of component i at level $\alpha \in (0,1]$ for the B-type CSF γ is defined as*

$$R_i(\alpha) = P\{\phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(1_i, \boldsymbol{X})) > \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(0_i, \boldsymbol{X}))\},$$

where $\boldsymbol{X}$ is a random state vector.

We note that $R_i(\alpha)$ depends on level α. When P is replaced with μ, the Lebesgue measure on $\boldsymbol{R}^n$, $R_i(\alpha)$ can be interpreted as the *structural importance* which is the relative proportion of state vectors at which component i is relevant to the system at level α. It can be easily seen that $\mu\{\boldsymbol{x} \in \Delta | \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(1_i, \boldsymbol{x})) > \phi_\alpha(\boldsymbol{I}_\alpha^{C_\alpha}(0_i, \boldsymbol{x}))\} = 0$ if and only if i is not relevant to the system at level α and, hence, that the following Proposition holds.

Proposition 3.1 *Let γ be a B-type CSF and let $X_1, X_2, \ldots, X_n$ are independent, absolutely continuous random variables. Then, for all $\alpha \in (0,1]$ and each $i \in C$, $R_i(\alpha) > 0$ if and only if $i \in C_\alpha$.*

It is of interest to determine when one component is uniformly more important than another, i.e., when $R_i(\alpha) \geq R_j(\alpha)$ for all α.

Definition 3.2 *Let γ be a B-type CSF. We say that component i is connected in series (parallel) to the remainder of the components, if, for all $\alpha \in (0,1], \gamma(\boldsymbol{x}) \geq \alpha$ only if (if) $x_i \geq \alpha$.*

It is easily seen that if the random variable X has support $[0,1]$, then $0 < P(X \geq \alpha) < 1$ for all $\alpha \in (0,1)$. Let γ be a B-type CSF and suppose that $X_1, X_2, \ldots, X_n$ are independent, absolutely continuous random variables, each with support the unit interval. Then, we may write

$$R_i(\alpha) = P_X(U_\alpha \cap D_{i\alpha})/P_X(D_{i\alpha}) - P_X(U_\alpha \cap D_{i\alpha}^c)/P_X(D_{i\alpha}^c),$$

where $U_\alpha = \{\boldsymbol{x} \in \Delta | \gamma(\boldsymbol{x}) \geq \alpha\}$ and $D_{i\alpha} = \{\boldsymbol{x} \in \Delta | x_i \geq \alpha\}$. Kim and Baxter (1987) suggest a component importance measure for general CSFs as conditional probability with the key vector $\boldsymbol{\delta}_\alpha$, which is the intersection of the boundary of U_α and the diagonal of Δ. When restricted to B-type CSFs, their importance measure coincides with $R_i(\alpha)$ expressed as above under the same condition. Noticing $\delta_\alpha = \alpha$ for all $\alpha \in (0,1)$ in a B-type CSF, the proof of the following Proposition is similar to those of Theorem 4.1 and Theorem 4.2 in Baxter and Lee (1989b), and hence omitted.

Proposition 3.2 *Let γ be a B-type CSF and suppose that $X_1, X_2, \ldots, X_n$ are independent, absolutely continuous random variables. Each with support the unit interval. If component i is connected in series (parallel) to the remainder of components and if $X_i \leq^{st} (\geq^{st}) X_j$, then $R_i(\alpha) \geq R_j(\alpha)$ for all $a \in (0,1)$, $j \neq i$.*

Proof: Omitted. □

In case of binary structures we have the following corollary.

Corollary 3.1 *Let ϕ be a BSF and suppose that $Y_1, Y_2, \ldots, Y_n$ are independent binary random variables such that $P\{Y_i = 1\} > 0$ for all $i \in C$. If component i is connected in series (parallel) to the remainder of components and if $Y \leq^{st} (\geq^{st}) Y_j$, then $I^\phi(i) \geq I^\phi(j)$.*

A stronger result may be obtained for modules of a B-type CSF, as we now show.

Definition 3.3 *A pair (A, χ) is said to be a module of a B-type CSF γ, if (A, χ) is a module of (C_α, ϕ_α) for every $\alpha \in (0,1]$.*

Proposition 3.3 *For a B-type CSF γ, let (A, χ) be a module of γ and suppose that $X_1, X_2, \ldots, X_n$ are independent, absolutely continuous random variables, each with support the unit interval. If component i is connected in series (parallel) to the remainder of components in the module, and if $X_i \leq^{st} (\geq^{st}) X_j$, then $R_i(\alpha) \geq R_j(\alpha)$ for all $\alpha \in (0,1)$ and all $j \in A, j \neq i$.*

Proof: Choose $\alpha \in (0,1)$ and component $j \in A$, $j \neq i$. Suppose that ϕ_α is the underlying BSF of γ at level α. We see that, by Birnbaum (1969), $I^{\phi_\alpha}(i) = I^\chi(j) I^{\phi_\alpha}(\chi)$ where $I^\chi(j)$ is the reliability importance of component i in the structure χ and $I^{\phi_\alpha}(\chi)$ is the reliability importance of the module regarded as a component in the structure ϕ_α. Define the binary random variable $Y_j = I_{\{X_j \geq \alpha\}}$, and observe that if $X_i \leq^{st} (\geq^{st}) X_j$ then $Y_i \leq^{st} (\geq^{st}) Y_j$. Then, it is easily seen that $R_j(\alpha) = I^{\phi_\alpha}(j)$ and hence $R_j(\alpha) = I^\chi(j) I^{\phi_\alpha}(\chi)$. Further, since $I^{\phi_\alpha}(\chi) \geq 0$, it suffices to show that $I^\chi(i) \geq I^\chi(j)$. Now, since component i is connected in series (parallel) to the remainder of components in the module and since $Y_i \leq^{st} (\geq^{st}) Y_j$, we have $I^\chi(i) \geq I^\chi(j)$ by Corollary 3.1. Since α and $j \in A$ are arbitrary, the result follows. □

Example 3.1 *Let* $\phi : \{0,1\}^6 \to \{0,1\}$ *be a coherent BSF defined by*

$$\phi(\boldsymbol{x}) = (x_1 \vee x_2) \wedge x_3 \wedge [x_4 \vee (x_5 \wedge x_6)]$$

and let γ *be a B-type CSF such that* $\phi_\alpha = \phi$ *for all* $\alpha \in (0,1]$. *If* $X_1 \geq^{st} X_2 \geq^{st} X_3$, *then* $R_3(\alpha) \geq R_1(\alpha) \geq R_2(\alpha)$ *for all* $\alpha \in (0,1)$. *If* $X_4 \geq^{st} X_6 \geq^{st} X_5 \geq^{st} X_3$, *then* $R_3(\alpha) \geq R_4(\alpha) \geq R_5(\alpha) \geq R_6(\alpha)$ *for all* $\alpha \in (0,1]$.

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