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# On the Effects of Plotting Positions to the Probability Weighted Moments Method for the Generalized Logistic Distribution*

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Abstract

Five plotting positions are applied to the computation of probability weighted moments (PWM) on the parameters of the generalized logistic distribution. Over a range of parameter values with some finite sample sizes, the effects of five plotting positions are investigated via Monte Carlo simulation studies. Our simulation results indicate that the Landwehr plotting position frequently tends to document smaller biases than others in the location and scale parameter estimations. On the other hand, the Weibull plotting position often tends to cause larger biases than others. The plotting position $(i - 0.35)/n$ seems to report smaller root mean square errors (RMSE) than other plotting positions in the negative shape parameter estimation under small samples. In comparison to the maximum likelihood (ML) method under the small sample, the PWM do not seem to be better than the ML estimators in the location and scale parameter estimations documenting larger RMSE. However, the PWM outperform the ML estimators in the shape parameter estimation when its magnitude is near zero. Sensitivity of right tail quantile estimation regarding five plotting positions is also examined, but superiority or inferiority of any plotting position is not observed.

*Keywords*: Generalized logistic distribution; plotting position; probability weighted moments; quantile; shape parameter.

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## 1. Introduction

The maximum likelihood (ML) method has been usually regarded as the best estimation technique on the parameters of several extreme value distributions (Jenkinson, 1955; Prescott and Walden, 1980). However, its limitations were reported in many articles. Landwehr *et al.* (1979a) and Hosking *et al.* (1985) claimed the decreasing accuracy problems of the ML method under small or moderate sample sizes for the Gumbel distribution and Generalized Extreme Value (GEV) distribution, respectively. Similar results for the small sample performance of the ML method in the Generalized Pareto (GP) distribution were also documented in Hosking and Wallis (1987). Furthermore, the ML method sometimes does not provide its local maximum of log likelihood (Hosking *et al,* 1985). In particular, when the ML method is applied to the multi-parameter estimation, it often leads to such failures. In that case, iteration based approach may be requested, which is computationally complicate. As an alternative, the probability weighted moments (PWM) method was considered, which is based on relatively simple algorithms.

The PWM method on the parameters of some extreme distributions was introduced by Greenwood *et al.* (1979). Its superiority in the small sample performances over the ML method was studied in numerous papers (Landwehr *et al.*, 1979a; Hosking *et al.*, 1985; Hosking and Wallis, 1987). Hosking (1990) studied the PWM based L-moments approach, and applied it to various distributions including the generalized logistic (GLO) distribution. He claimed that this method is advantageous over conventional moment methods due to the robustness to outliers. Moreover, he argued that the PWM method tends to be reasonably efficient compared with the ML method under certain conditions in cases of multi-parameter estimation.

The accuracy of the PWM method is known to be affected by the choice of the plotting positions. Landwehr *et al.* (1979a) proposed a plotting position leading to unbiased properties. Later, Landwehr *et al.* (1979b) recommended the plotting position, $(i-0.35)/n$ for the Wakeby distribution. This plotting position was also recommended for the estimation of the GEV and GP distributions by Hosking *et al.* (1985) and Hosking and Wallis (1987), respectively. Haktanir and Bozduman (1995) compared the effects of four plotting positions (Landwehr, $(i-0.35)/n$, Weibull, Cunnane) for three parameter log-normal, log-Pearson-3, GEV, and Wakeby distributions via simulation studies. They used the parameter estimates from the annual flood peaks series in the simulation. Specifically, using the quantile based relative errors, they compared box plot outputs. They concluded that the Landwehr plotting position was slightly better than others for the GEV and log-Pearson-3 distributions. For the log-normal distribution, the Landwehr, $(i-0.35)/n$, or the ML method was evenly good.

Recently, using the PWM method Gettinby *et al.* (2004, 2006) showed that the GLO model performed better fitness to many financial dynamics than other extreme models. Regardless of an increasing importance of the GLO distribution, the effects of plotting positions for the GLO distribution were not rigorously investigated in the literature yet. In this paper, five plotting positions are applied to the computation of the PWM on the parameters and right tail quantiles of the GLO distribution. Additional to the four plotting positions used in Haktanir and Bozduman (1995), a plotting position by Gettinby *et al.* (2006) is considered. Via Monte Carlo simulation studies, the effects of five plotting positions are evaluated over a range of parameter values. For more complete analysis, the PWM are additionally compared with the numerically computed ML estimators under the small sample situation.

Our simulation results indicate that the Landwehr plotting position frequently tends to document smaller biases than others in the location and scale parameter estimations. On the other hand, the Weibull plotting position often tends to cause larger biases than others. The plotting position $(i-0.35)/n$ seems to report smaller root mean square errors (RMSE) than other plotting positions in the negative shape parameter estimation under small samples. Compared with the ML method under a small sample, the PWM do not appear to provide better accuracy than the numerically computed ML estimators in the location and scale parameter estimations documenting larger RMSE. However, the PWM outperform the ML estimators in the shape parameter estimation when its magnitude is near zero. Sensitivity of right tail quantile estimation regarding five plotting positions is also examined, but superiority or inferiority of any plotting position is not observed.

The rest of the paper is organized as follows. In Section 2, the PWM method and the GLO distribution are briefly reviewed. Five plotting positions are also introduced. The accuracy of the PWM method for five plotting positions is examined via simulation studies in Section 3, where the right tail quantile estimation results are reported. Some concluding remarks are given in Section 4.

## 2. PWM Method and GLO Distribution

Hosking *et al.* (1985) specifically considered the $h$th PWM of a random variable $X$ with its marginal cumulative distribution function $F(X)$ as follows:

$$\beta_h = \mathrm{E}[X\{F(X)\}^h], \quad h = 0, 1, \ldots. \tag{2.1}$$

They suggested estimating (2.1) by ,

$$\hat{\beta}_h = \frac{1}{n}\sum_{i=1}^{n} p_{i,n}^h x_i,$$

where $p_{i,n}$ is a plotting position and $x_i$ is an ordered sample. According to them, some reasonable choices of a plotting position such as $p_{i,n} = (i-v)/n$, $0 < v < 1$, or $p_{i,n} = (i-v)/(n+1-2v)$, $-0.5 < v < 0.5$, produce consistent estimators for (2.1).

There have been many versions of plotting positions. The oldest and simple well-known one is $i/(n+1)$, which is referred to as a Weibull plotting position. A slightly modified one is $(i-0.4)/(n+0.2)$ by Cunnane (1978). Landwehr *et al.* (1979a) suggested an unbiased estimator for (2.1) using the following plotting position

$$p_{i,n}^h = [(i-1)(i-2)\cdots(i-h)]/[(n-1)(n-2)\cdots(n-h)].$$

Later, some plotting positions yielding better estimation results for the underlying distribution were pursued. As mentioned, a plotting position $(i-0.35)/n$ was shown to produce good estimation results for the Wakeby, GEV and GP distributions in Landwehr *et al.* (1979b), Hosking *et al.* (1985) and Hosking and Wallis (1987), respectively. Recently, Gettinby *et al.* (2006) estimated the GEV and GLO distributions using a plotting position $p_{i,n} = (i+\alpha)/(n+\omega)$, where $\alpha = \{\sqrt{n(n-1)} - (n+1)\}/2$ and $\omega = 1+2\alpha$. The effects of these five plotting positions (Weibull, Cunnane, Landwehr, $(i-0.35)/n$, and Gettinby) will be compared in the next section.

Gettinby *et al.* (2004, 2006) claimed that the GLO distribution provided better fitness to the extremes in indices of share return series of many countries than other extreme distributions. The marginal cumulative distribution function of the GLO distribution is as follows:

$$F(x) = 1/\left[1+\left\{1-\frac{\xi}{\sigma}(x-\mu)\right\}^{1/\xi}\right], \qquad \xi \neq 0, \tag{2.2}$$

where $x$ is bounded as $\mu+\sigma/\xi \leq x < \infty$ under $\xi < 0$ and as $-\infty < x \leq \mu+\sigma/\xi$ under $\xi > 0$. If $\xi = 0$, an ordinary logistic distribution is obtained. Three parameters, $\mu$, $\sigma$ and $\xi$ stand for the location, scale, and shape parameters, respectively. Note that $\sigma$ should be a positive value.

The L-moments or the PWM method for the parameter estimation of the GLO distribution was rigorously investigated by Hosking (1990). The L-moments are expected values of some linear combinations of ordered statistics, which can be applied to estimating the measures of location, scale, skewness, and etc. For example, first three L-moments $m_1, m_2$ and $m_3$ indicate the location, scale and skewness measures of a distribution, respectively. They have linear relationships with the PWM as follows: $m_1 = \beta_0$, $m_2 = 2\beta_1 - \beta_0$, $m_3 = 6\beta_2 - 6\beta_1 + \beta_0$. The first three L-moments are estimated by $\hat{m}_1, \hat{m}_2$ and $\hat{m}_3$ which are obtained by replacing $\beta_h$ with $\hat{\beta}_h$. According to Hosking (1990), three parameters in the GLO distribution are sequentially estimated as follows: $\hat{\xi} = -\hat{m}_3/\hat{m}_2$, $\hat{\sigma} = \hat{m}_2/\{\Gamma(1+\hat{\xi})\Gamma(1-\hat{\xi})\}$, $\hat{\mu} = \hat{m}_1 + (\hat{m}_2 - \hat{\sigma})/\hat{\xi}$. Besides the effects of the plotting positions in the parameter estimation of the GLO distribution, we are also interested in the quantile estimation cases. The $p$-quantile $x(p)$ of the GLO distribution can be computed from the following inverse distribution function of (2.2) (Hosking, 1990)

$$x(p) = \mu + \sigma[1 - \{(1-p)/p\}^{\xi}]/\xi]. \tag{2.3}$$

Using the methods above, the effects of aforementioned five plotting positions to the parameter and quantile estimations of the GLO distribution are investigated in the following section.

## 3. Simulation Results

According to Hosking *et al.* (1985) and Hosking and Wallis (1987), the PWM methods on the parameters of the GEV and GP distributions are appropriate only under some specific range of shape parameters in the sense of both theory and practice. In particular, their approximation efficiency is known to be appropriate when their shape parameters are ranged from $-0.2$ to $0.2$. For the GLO distribution, to the best of our knowledge, those conditions have not been rigorously studied yet. However, according to our pilot simulation experiments, the significantly decreasing accuracy seemed to be observed as the shape parameter was farther away from zero. Due to these preliminary results, we restricted the shape parameter space into the range from $-0.6$ to $0.6$.

As it would be explained, some plotting positions tended to have different patterns based on the sign of the shape parameter in our main simulation studies. Because of this, we considered positive and negative signs of shape parameters separately when we examine the scale and location parameter estimations. In particular, the shape parameter values are fixed into $0.2$ and $-0.2$ in those cases. Since these values are near zero, we can attenuate the effect of the shape parameter itself and may be able to examine the pure effects of the parameter of our interest. For same purposes, the scale and location parameters are fixed into $0.2$ and $0$, respectively when other parameters are examined. The scale parameter was investigated over the range from $0.2$ to $5$, which are positive values. The location parameter was examined under the range from $-10$ to $10$, which are realistic. One referee kindly recommended reconsidering these spaces for the scale and location parameters in the main simulation studies.

Five plotting positions (Weibull, Cunnane, Landwehr, $(i-0.35)/n$, and Gettinby) were considered. For each plotting position, we considered sample sizes 20, 50, 100 and 500. For each set of plotting position, parameters, and sample size, we generated the GLO distribution 1,000 times, and evaluated the PWM estimation performance. The bias and RMSE (in the parenthesis) for the PWM based shape parameter estimates are reported in Table 3.1. Tables 3.2 and 3.3 include the scale parameter estimation results in cases of positive and negative shape parameters, respectively. The location parameter estimates with positive and negative shape parameters are reported in Tables 3.4 and 3.5, respectively. In the tables, the M1, M2, M3, M4 and M5 represent $(i-0.35)/n$, Cunnane, Weibull, Landwehr, and Gettinby plotting positions, respectively. According to Tables 3.1–3.5, larger deviations were overall observed as the true parameter value was farther away from zero in the shape and scale parameters. Such patterns were not documented in the location parameter. Since the PWM estimators are sequentially computed from the shape parameter to the location parameter via the scale parameter, we investigated the shape parameter estimation first.

Table 3.1 indicates that M1 seems to provide smaller RMSE than other plotting positions under the small sample sizes like 20 and 50 when the shape parameter is negative. However, it is difficult to find any systematic superiority of M1 when the sample size is larger than 50. The other interesting pattern we need to pay attention to is on M3. This plotting position seems to provide larger biases than others when the shape parameter is negative over all the sample sizes from 20 to 500. Such pattern is also observed when the sample size is 500 under the positive shape parameter. Except for these two plotting positions, we could not find any particular pattern of other plotting positions over the

Table 3.1: The bias and root mean square errors (in the parenthesis) of the PWM based shape parameter estimates over a range of −0.6 to 0.6 regarding five plotting positions for the GLO distribution are reported. The location and scale parameters are fixed into 0 and 0.2, respectively. Considered sample sizes are 20, 50, 100, and 500. M1, M2, M3, M4 and M5 represent $(i - 0.35)/n$, Cunnane, Weibull, Landwehr and Gettinby plotting positions, respectively. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| | | $\xi$ | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | −0.6 | −0.4 | −0.2 | 0.2 | 0.4 | 0.6 |
| 20 | $M_1$ | 0.1271 | 0.0532 | −0.0047 | −0.0841 | −0.1396 | −0.2126 |
| | | (0.2031) | (0.1504) | (0.1240) | (0.1537) | (0.1999) | (0.2677) |
| | $M_2$ | 0.1801 | 0.1023 | 0.0393 | −0.0380 | −0.0956 | −0.1861 |
| | | (0.2384) | (0.1753) | (0.1249) | (0.1275) | (0.1694) | (0.2441) |
| | $M_3$ | 0.2230 | 0.1343 | 0.0623 | −0.0599 | −0.1346 | −0.2284 |
| | | (0.2667) | (0.1923) | (0.1275) | (0.1293) | (0.1927) | (0.2714) |
| | $M_4$ | 0.1274 | 0.0602 | 0.0281 | −0.0241 | −0.0610 | −0.1153 |
| | | (0.2160) | (0.1719) | (0.1405) | (0.1392) | (0.1754) | (0.2121) |
| | $M_5$ | 0.1782 | 0.0894 | 0.0380 | −0.0360 | −0.0921 | −0.1711 |
| | | (0.2390) | (0.1683) | (0.1305) | (0.1248) | (0.1769) | (0.2351) |
| 50 | $M_1$ | 0.0859 | 0.0338 | 0.0063 | −0.0290 | −0.0618 | −0.1211 |
| | | (0.1517) | (0.1114) | (0.0839) | (0.0922) | (0.1270) | (0.1792) |
| | $M_2$ | 0.0938 | 0.0454 | 0.0209 | −0.0183 | −0.0498 | −0.0989 |
| | | (0.1565) | (0.1231) | (0.0900) | (0.0886) | (0.1209) | (0.1584) |
| | $M_3$ | 0.1230 | 0.0643 | 0.0326 | −0.0264 | −0.0677 | −0.1254 |
| | | (0.1753) | (0.1200) | (0.0899) | (0.0874) | (0.1247) | (0.1758) |
| | $M_4$ | 0.0794 | 0.0314 | 0.0065 | −0.0160 | −0.0284 | −0.0698 |
| | | (0.1569) | (0.1182) | (0.0908) | (0.0889) | (0.1229) | (0.1519) |
| | $M_5$ | 0.0996 | 0.0504 | 0.0162 | −0.0098 | −0.0420 | −0.1033 |
| | | (0.1628) | (0.1232) | (0.0856) | (0.0841) | (0.1203) | (0.1642) |
| 100 | $M_1$ | 0.0553 | 0.0178 | 0.0003 | −0.0159 | −0.0335 | −0.0726 |
| | | (0.1209) | (0.0924) | (0.0655) | (0.0644) | (0.0936) | (0.1306) |
| | $M_2$ | 0.0666 | 0.0290 | 0.0088 | −0.0090 | −0.0262 | −0.0686 |
| | | (0.1269) | (0.0960) | (0.0620) | (0.0642) | (0.0925) | (0.1280) |
| | $M_3$ | 0.0727 | 0.0371 | 0.0118 | −0.0134 | −0.0315 | −0.0804 |
| | | (0.1281) | (0.0920) | (0.0630) | (0.0624) | (0.0971) | (0.1324) |
| | $M_4$ | 0.0559 | 0.0190 | 0.0061 | −0.0060 | −0.0171 | −0.0599 |
| | | (0.1242) | (0.0939) | (0.0673) | (0.0656) | (0.0892) | (0.1220) |
| | $M_5$ | 0.0727 | 0.0305 | 0.0083 | −0.0060 | −0.0267 | −0.0641 |
| | | (0.1285) | (0.0896) | (0.0657) | (0.0633) | (0.0908) | (0.1297) |
| 500 | $M_1$ | 0.0214 | 0.0036 | −0.0005 | −0.0023 | −0.0081 | −0.0259 |
| | | (0.0739) | (0.0491) | (0.0293) | (0.0296) | (0.0476) | (0.0748) |
| | $M_2$ | 0.0260 | 0.0067 | 0.0017 | −0.0024 | −0.0077 | −0.0221 |
| | | (0.0774) | (0.0471) | (0.0299) | (0.0295) | (0.0473) | (0.0756) |
| | $M_3$ | 0.0340 | 0.0096 | 0.0039 | −0.0033 | −0.0094 | −0.0288 |
| | | (0.0742) | (0.0468) | (0.0297) | (0.0293) | (0.0495) | (0.0736) |
| | $M_4$ | 0.0193 | 0.0057 | 0.0008 | −0.0001 | −0.0062 | −0.0233 |
| | | (0.0780) | (0.0459) | (0.0304) | (0.0304) | (0.0472) | (0.0735) |
| | $M_5$ | 0.0251 | 0.0082 | 0.0017 | −0.0031 | −0.0061 | −0.0236 |
| | | (0.0746) | (0.0479) | (0.0299) | (0.0281) | (0.0508) | (0.0750) |

Table 3.2: The bias and root mean square errors (in the parenthesis) of the PWM based scale parameter estimates over a range of 0.2 to 5 regarding five plotting positions for the GLO distribution are reported. The location and shape parameters are fixed into 0 and 0.2, respectively. Considered sample sizes are 50, 100, and 500. M1, M2, M3, M4 and M5 represent $(i-0.35)/n$, Cunnane, Weibull, Landwehr and Gettinby plotting positions, respectively. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| | | $\sigma$ | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | 0.2 | 0.4 | 0.6 | 1 | 3 | 5 |
| 20 | $M_1$ | −0.0070 | −0.0158 | −0.0243 | −0.0478 | −0.1649 | −0.2409 |
| | | (0.0427) | (0.0845) | (0.1279) | (0.2181) | (0.6438) | (1.1051) |
| | $M_2$ | −0.0128 | −0.0289 | −0.0447 | −0.0827 | −0.1859 | −0.3549 |
| | | (0.0428) | (0.0835) | (0.1295) | (0.2076) | (0.6442) | (1.0591) |
| | $M_3$ | −0.0188 | −0.0376 | −0.0545 | −0.0960 | −0.2745 | −0.4262 |
| | | (0.0443) | (0.0901) | (0.1308) | (0.2169) | (0.6390) | (1.0581) |
| | $M_4$ | −0.0057 | −0.0120 | −0.0185 | −0.0363 | −0.0697 | −0.0933 |
| | | (0.0416) | (0.0837) | (0.1197) | (0.2087) | (0.6210) | (1.0922) |
| | $M_5$ | −0.0100 | −0.0267 | −0.0397 | −0.0637 | −0.1923 | −0.2434 |
| | | (0.0417) | (0.0851) | (0.1190) | (0.2040) | (0.6471) | (1.0219) |
| 50 | $M_1$ | −0.0033 | −0.0085 | −0.0088 | −0.0231 | −0.0709 | −0.0849 |
| | | (0.0268) | (0.0537) | (0.0749) | (0.1269) | (0.4011) | (0.6546) |
| | $M_2$ | −0.0057 | −0.0111 | −0.0143 | −0.0299 | −0.0831 | −0.1537 |
| | | (0.0263) | (0.0534) | (0.0786) | (0.1328) | (0.3758) | (0.6608) |
| | $M_3$ | −0.0088 | −0.0169 | −0.0205 | −0.0304 | −0.1347 | −0.1577 |
| | | (0.0273) | (0.0538) | (0.0793) | (0.1320) | (0.3994) | (0.6632) |
| | $M_4$ | −0.0037 | −0.0046 | −0.0059 | −0.0173 | −0.0478 | −0.0729 |
| | | (0.0259) | (0.0519) | (0.0797) | (0.1245) | (0.3798) | (0.6512) |
| | $M_5$ | −0.0036 | −0.0105 | −0.0159 | −0.0244 | −0.0778 | −0.1470 |
| | | (0.0257) | (0.0520) | (0.0768) | (0.1286) | (0.3850) | (0.6436) |
| 100 | $M_1$ | −0.0017 | −0.0040 | −0.0078 | −0.0091 | −0.0181 | −0.0687 |
| | | (0.0182) | (0.0371) | (0.0523) | (0.0912) | (0.2612) | (0.4677) |
| | $M_2$ | −0.0031 | −0.0073 | −0.0098 | −0.0198 | −0.0523 | −0.0795 |
| | | (0.0178) | (0.0359) | (0.0527) | (0.0909) | (0.2756) | (0.4623) |
| | $M_3$ | −0.0047 | −0.0070 | −0.0122 | −0.0182 | −0.0502 | −0.1033 |
| | | (0.0191) | (0.0371) | (0.0544) | (0.0947) | (0.2798) | (0.4742) |
| | $M_4$ | −0.0006 | −0.0016 | −0.0018 | −0.0072 | −0.0247 | −0.0126 |
| | | (0.0185) | (0.0374) | (0.0533) | (0.0897) | (0.2654) | (0.4452) |
| | $M_5$ | −0.0026 | −0.0057 | −0.0075 | −0.0141 | −0.0452 | −0.0670 |
| | | (0.0186) | (0.0370) | (0.0559) | (0.0954) | (0.2741) | (0.4817) |
| 500 | $M_1$ | −0.0006 | −0.0015 | −0.0008 | −0.0013 | −0.0115 | −0.0006 |
| | | (0.0081) | (0.0160) | (0.0252) | (0.0405) | (0.1238) | (0.2076) |
| | $M_2$ | −0.0006 | −0.0019 | −0.0035 | −0.0036 | −0.0077 | −0.0196 |
| | | (0.0084) | (0.0160) | (0.0248) | (0.0406) | (0.1199) | (0.2058) |
| | $M_3$ | −0.0010 | −0.0029 | −0.0034 | −0.0036 | −0.0090 | −0.0138 |
| | | (0.0083) | (0.0168) | (0.0245) | (0.0418) | (0.1198) | (0.2082) |
| | $M_4$ | −0.0003 | 0.0000 | −0.0009 | −0.0022 | −0.0041 | −0.0005 |
| | | (0.0081) | (0.0165) | (0.0241) | (0.0406) | (0.1239) | (0.2058) |
| | $M_5$ | −0.0009 | −0.0016 | −0.0015 | −0.0034 | −0.0070 | −0.0205 |
| | | (0.0084) | (0.0162) | (0.0252) | (0.0417) | (0.1230) | (0.2135) |

Table 3.3: The bias and root mean square errors (in the parenthesis) of the PWM based scale parameter estimates over a range of 0.2 to 5 regarding five plotting positions for the GLO distribution are reported. The location and shape parameters are fixed into 0 and −0.2, respectively. Considered sample sizes are 20, 50, 100 and 500. M1, M2, M3, M4 and M5 represent $(i-0.35)/n$, Cunnane, Weibull, Landwehr and Gettinby plotting positions, respectively. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| | | $\sigma$ | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | 0.2 | 0.4 | 0.6 | 1 | 3 | 5 |
| 20 | $M_1$ | −0.0180 | −0.0338 | −0.0485 | −0.0843 | −0.2392 | −0.3589 |
| | | (0.0438) | (0.0838) | (0.1262) | (0.2009) | (0.6424) | (1.0301) |
| | $M_2$ | −0.0131 | −0.0260 | −0.0393 | −0.0628 | −0.2003 | −0.3007 |
| | | (0.0416) | (0.0853) | (0.1267) | (0.2167) | (0.6371) | (1.0369) |
| | $M_3$ | −0.0194 | −0.0372 | −0.0573 | −0.0875 | −0.3004 | −0.4535 |
| | | (0.0451) | (0.0858) | (0.1339) | (0.2276) | (0.6835) | (1.0804) |
| | $M_4$ | −0.0070 | −0.0101 | −0.0178 | −0.0346 | −0.0783 | −0.1297 |
| | | (0.0409) | (0.0846) | (0.1248) | (0.2076) | (0.6351) | (1.0290) |
| | $M_5$ | −0.0110 | −0.0246 | −0.0423 | −0.0610 | −0.2150 | −0.3187 |
| | | (0.0423) | (0.0839) | (0.1268) | (0.2100) | (0.6224) | (0.9720) |
| 50 | $M_1$ | −0.0079 | −0.0160 | −0.0163 | −0.0425 | −0.1224 | −0.1665 |
| | | (0.0272) | (0.0520) | (0.0753) | (0.1298) | (0.4092) | (0.6503) |
| | $M_2$ | −0.0062 | −0.0108 | −0.0198 | −0.0250 | −0.0749 | −0.1007 |
| | | (0.0250) | (0.0541) | (0.0792) | (0.1311) | (0.4058) | (0.6754) |
| | $M_3$ | −0.0072 | −0.0154 | −0.0257 | −0.0368 | −0.1113 | −0.1896 |
| | | (0.0255) | (0.0532) | (0.0815) | (0.1317) | (0.3915) | (0.6618) |
| | $M_4$ | −0.0028 | −0.0032 | −0.0074 | −0.0174 | −0.0193 | −0.0567 |
| | | (0.0262) | (0.0516) | (0.0779) | (0.1244) | (0.3969) | (0.6517) |
| | $M_5$ | −0.0056 | −0.0084 | −0.0146 | −0.0251 | −0.0668 | −0.1106 |
| | | (0.0262) | (0.0505) | (0.0771) | (0.1335) | (0.3814) | (0.6477) |
| 100 | $M_1$ | −0.0040 | −0.0090 | −0.0110 | −0.0248 | −0.0610 | −0.0858 |
| | | (0.0186) | (0.0376) | (0.0538) | (0.0902) | (0.2673) | (0.4707) |
| | $M_2$ | −0.0030 | −0.0060 | −0.0091 | −0.0157 | −0.0417 | −0.0490 |
| | | (0.0181) | (0.0374) | (0.0555) | (0.0933) | (0.2989) | (0.4558) |
| | $M_3$ | −0.0040 | −0.0096 | −0.0115 | −0.0220 | −0.0491 | −0.1197 |
| | | (0.0193) | (0.0376) | (0.0546) | (0.0911) | (0.2904) | (0.4509) |
| | $M_4$ | −0.0013 | −0.0023 | −0.0030 | −0.0111 | −0.0103 | −0.0223 |
| | | (0.0178) | (0.0370) | (0.0532) | (0.0904) | (0.2769) | (0.4528) |
| | $M_5$ | −0.0037 | −0.0061 | −0.0056 | −0.0152 | −0.0503 | −0.0639 |
| | | (0.0181) | (0.0375) | (0.0553) | (0.0913) | (0.2769) | (0.4585) |
| 500 | $M_1$ | −0.0008 | −0.0012 | −0.0021 | −0.0027 | −0.0053 | −0.0187 |
| | | (0.0082) | (0.0159) | (0.0243) | (0.0437) | (0.1256) | (0.2023) |
| | $M_2$ | −0.0005 | −0.0018 | −0.0019 | −0.0037 | −0.0117 | −0.0211 |
| | | (0.0083) | (0.0167) | (0.0246) | (0.0416) | (0.1260) | (0.2087) |
| | $M_3$ | −0.0010 | −0.0014 | −0.0023 | −0.0041 | −0.0157 | −0.0207 |
| | | (0.0081) | (0.0158) | (0.0246) | (0.0400) | (0.1246) | (0.2129) |
| | $M_4$ | −0.0002 | −0.0011 | 0.0003 | −0.0024 | −0.0041 | −0.0144 |
| | | (0.0083) | (0.0159) | (0.0252) | (0.0408) | (0.1268) | (0.2044) |
| | $M_5$ | −0.0005 | −0.0006 | −0.0011 | −0.0032 | −0.0084 | −0.0109 |
| | | (0.0084) | (0.0166) | (0.0250) | (0.0407) | (0.1298) | (0.2078) |

Table 3.4: The bias and root mean square errors (in the parenthesis) of the PWM based location parameter estimates over a range of −10 to 10 regarding five plotting positions for the GLO distribution are reported. The scale and shape parameters are fixed into 0.2. Considered sample sizes are 20, 50, 100 and 500. M1, M2, M3, M4 and M5 represent $(i-0.35)/n$, Cunnane, Weibull, Landwehr and Gettinby plotting positions, respectively. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| $n$ | | $\mu$ = −10 | −5 | −1 | −0.2 | 0.2 | 1 | 5 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 20 | $M_1$ | −0.0431 | −0.0357 | −0.0300 | −0.0245 | −0.0241 | −0.0241 | −0.0198 | −0.0106 |
| | | (0.0882) | (0.0884) | (0.0877) | (0.0854) | (0.0845) | (0.0869) | (0.0818) | (0.0812) |
| | $M_2$ | −0.1771 | −0.0998 | −0.0282 | −0.0175 | −0.0084 | 0.0057 | 0.0645 | 0.1125 |
| | | (0.1965) | (0.1298) | (0.0852) | (0.0852) | (0.0859) | (0.0822) | (0.0991) | (0.1339) |
| | $M_3$ | −0.1689 | −0.2412 | −0.0928 | −0.0390 | −0.0052 | 0.0476 | 0.0948 | 0.0066 |
| | | (0.2210) | (0.2730) | (0.1274) | (0.1007) | (0.0845) | (0.0974) | (0.1167) | (0.0746) |
| | $M_4$ | 0.0025 | −0.0035 | 0.0013 | 0.0026 | −0.0014 | −0.0042 | −0.0061 | −0.0036 |
| | | (0.0790) | (0.0798) | (0.0816) | (0.0804) | (0.0837) | (0.0813) | (0.0854) | (0.0832) |
| | $M_5$ | −0.0111 | −0.0109 | −0.0107 | −0.0094 | −0.0099 | −0.0134 | −0.0119 | −0.0082 |
| | | (0.0812) | (0.0811) | (0.0798) | (0.0798) | (0.0810) | (0.0842) | (0.0793) | (0.0792) |
| 50 | $M_1$ | −0.0142 | −0.0131 | −0.0118 | −0.0107 | −0.0109 | −0.0096 | −0.0089 | −0.0082 |
| | | (0.0548) | (0.0526) | (0.0534) | (0.0526) | (0.0531) | (0.0531) | (0.0516) | (0.0519) |
| | $M_2$ | −0.0700 | −0.0388 | −0.0117 | −0.0089 | −0.0045 | 0.0017 | 0.0258 | 0.0543 |
| | | (0.0874) | (0.0645) | (0.0536) | (0.0532) | (0.0508) | (0.0508) | (0.0581) | (0.0729) |
| | $M_3$ | −0.2534 | −0.1659 | −0.0385 | −0.0170 | −0.0071 | 0.0201 | 0.1114 | 0.1301 |
| | | (0.2619) | (0.1746) | (0.0661) | (0.0560) | (0.0510) | (0.0544) | (0.1212) | (0.1373) |
| | $M_4$ | −0.0016 | 0.0004 | 0.0010 | −0.0027 | −0.0038 | −0.0005 | −0.0014 | −0.0024 |
| | | (0.0526) | (0.0517) | (0.0501) | (0.0498) | (0.0523) | (0.0507) | (0.0501) | (0.0505) |
| | $M_5$ | −0.0047 | −0.0055 | −0.0039 | −0.0057 | −0.0024 | −0.0056 | −0.0070 | −0.0054 |
| | | (0.0513) | (0.0525) | (0.0500) | (0.0499) | (0.0497) | (0.0530) | (0.0514) | (0.0506) |
| 100 | $M_1$ | −0.0066 | −0.0053 | −0.0088 | −0.0074 | −0.0050 | −0.0065 | −0.0054 | −0.0037 |
| | | (0.0367) | (0.0359) | (0.0364) | (0.0372) | (0.0360) | (0.0364) | (0.0372) | (0.0365) |
| | $M_2$ | −0.0349 | −0.0188 | −0.0054 | −0.0021 | −0.0008 | −0.0005 | 0.0126 | 0.0260 |
| | | (0.0504) | (0.0415) | (0.0366) | (0.0378) | (0.0353) | (0.0350) | (0.0386) | (0.0451) |
| | $M_3$ | −0.1590 | −0.0808 | −0.0217 | −0.0089 | −0.0022 | 0.0100 | 0.0649 | 0.1153 |
| | | (0.1632) | (0.0891) | (0.0430) | (0.0382) | (0.0371) | (0.0384) | (0.0739) | (0.1198) |
| | $M_4$ | 0.0007 | 0.0008 | −0.0005 | −0.0001 | −0.0028 | −0.0017 | −0.0001 | 0.0003 |
| | | (0.0367) | (0.0367) | (0.0371) | (0.0360) | (0.0357) | (0.0362) | (0.0366) | (0.0363) |
| | $M_5$ | −0.0023 | −0.0030 | −0.0050 | −0.0045 | −0.0030 | −0.0019 | −0.0034 | −0.0013 |
| | | (0.0372) | (0.0374) | (0.0359) | (0.0354) | (0.0361) | (0.0356) | (0.0365) | (0.0356) |
| 500 | $M_1$ | −0.0017 | −0.0021 | −0.0004 | −0.0012 | −0.0020 | −0.0009 | −0.0009 | −0.0003 |
| | | (0.0157) | (0.0160) | (0.0167) | (0.0156) | (0.0162) | (0.0163) | (0.0163) | (0.0167) |
| | $M_2$ | −0.0074 | −0.0035 | −0.0021 | −0.0005 | −0.0014 | 0.0001 | 0.0021 | 0.0058 |
| | | (0.0173) | (0.0167) | (0.0160) | (0.0163) | (0.0160) | (0.0159) | (0.0165) | (0.0173) |
| | $M_3$ | −0.0322 | −0.0160 | −0.0046 | −0.0009 | −0.0009 | 0.0024 | 0.0141 | 0.0294 |
| | | (0.0362) | (0.0232) | (0.0163) | (0.0168) | (0.0156) | (0.0161) | (0.0214) | (0.0339) |
| | $M_4$ | 0.0005 | −0.0010 | 0.0002 | −0.0010 | 0.0002 | 0.0002 | 0.0000 | −0.0005 |
| | | (0.0156) | (0.0161) | (0.0165) | (0.0161) | (0.0165) | (0.0163) | (0.0158) | (0.0165) |
| | $M_5$ | −0.0006 | −0.0006 | 0.0002 | −0.0006 | 0.0002 | 0.0003 | −0.0012 | −0.0011 |
| | | (0.0158) | (0.0156) | (0.0155) | (0.0156) | (0.0155) | (0.0162) | (0.0161) | (0.0155) |

Table 3.5: The bias and root mean square errors (in the parenthesis) of the PWM based location parameter estimates over a range of −10 to 10 regarding five plotting positions for the GLO distribution are reported. The scale and shape parameters are fixed into 0.2 and −0.2, respectively. Considered sample sizes are 20, 50, 100 and 500. M1, M2, M3, M4 and M5 represent $(i-0.35)/n$, Cunnane, Weibull, Landwehr and Gettinby plotting positions, respectively. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| $n$ | | $\mu$: −10 | −5 | −1 | −0.2 | 0.2 | 1 | 5 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 20 | $M_1$ | n/a | −0.0056 | −0.0044 | −0.0054 | −0.0072 | −0.0038 | 0.0080 | 0.0078 |
| | | n/a | (0.0788) | (0.0809) | (0.0818) | (0.0763) | (0.0784) | (0.0847) | (0.0788) |
| | $M_2$ | −0.1107 | −0.0640 | −0.0040 | 0.0114 | 0.0188 | 0.0266 | 0.1037 | 0.1749 |
| | | (0.1336) | (0.1004) | (0.0838) | (0.0838) | (0.0809) | (0.0842) | (0.1331) | (0.1950) |
| | $M_3$ | −0.0040 | −0.0954 | −0.0533 | 0.0087 | 0.0323 | 0.0980 | 0.2356 | 0.1620 |
| | | (0.0731) | (0.1177) | (0.0976) | (0.0854) | (0.0951) | (0.1301) | (0.2695) | (0.2198) |
| | $M_4$ | 0.0025 | 0.0006 | 0.0032 | 0.0013 | −0.0004 | −0.0017 | 0.0023 | 0.0015 |
| | | (0.0794) | (0.0833) | (0.0825) | (0.0826) | (0.0789) | (0.0798) | (0.0812) | (0.0838) |
| | $M_5$ | 0.0080 | 0.0115 | 0.0152 | 0.0151 | 0.0131 | 0.0100 | 0.0105 | 0.0146 |
| | | (0.0814) | (0.0833) | (0.0846) | (0.0812) | (0.0816) | (0.0814) | (0.0821) | (0.0808) |
| 50 | $M_1$ | −0.0016 | −0.0023 | −0.0021 | −0.0021 | 0.0016 | −0.0003 | 0.0010 | 0.0007 |
| | | (0.0513) | (0.0510) | (0.0482) | (0.0511) | (0.0504) | (0.0501) | (0.0512) | (0.0486) |
| | $M_2$ | −0.0555 | −0.0259 | −0.0003 | 0.0060 | 0.0071 | 0.0126 | 0.0419 | 0.0717 |
| | | (0.0751) | (0.0568) | (0.0506) | (0.0500) | (0.0517) | (0.0520) | (0.0674) | (0.0885) |
| | $M_3$ | −0.1329 | −0.1112 | −0.0207 | 0.0006 | 0.0168 | 0.0374 | 0.1626 | 0.2553 |
| | | (0.1403) | (0.1208) | (0.0555) | (0.0512) | (0.0540) | (0.0657) | (0.1714) | (0.2639) |
| | $M_4$ | 0.0000 | 0.0007 | −0.0004 | 0.0018 | 0.0023 | 0.0019 | 0.0011 | 0.0029 |
| | | (0.0516) | (0.0509) | (0.0494) | (0.0513) | (0.0509) | (0.0518) | (0.0523) | (0.0507) |
| | $M_5$ | 0.0075 | 0.0055 | 0.0038 | 0.0060 | 0.0058 | 0.0016 | 0.0074 | 0.0044 |
| | | (0.0489) | (0.0496) | (0.0530) | (0.0518) | (0.0507) | (0.0513) | (0.0515) | (0.0497) |
| 100 | $M_1$ | −0.0001 | −0.0007 | −0.0011 | −0.0014 | 0.0007 | −0.0020 | −0.0014 | −0.0002 |
| | | (0.0333) | (0.0351) | (0.0362) | (0.0348) | (0.0344) | (0.0370) | (0.0338) | (0.0362) |
| | $M_2$ | −0.0264 | −0.0130 | 0.0001 | 0.0026 | 0.0054 | 0.0057 | 0.0192 | 0.0362 |
| | | (0.0431) | (0.0381) | (0.0369) | (0.0356) | (0.0366) | (0.0343) | (0.0405) | (0.0524) |
| | $M_3$ | −0.1143 | −0.0634 | −0.0090 | 0.0019 | 0.0085 | 0.0215 | 0.0846 | 0.1602 |
| | | (0.1189) | (0.0732) | (0.0365) | (0.0352) | (0.0375) | (0.0423) | (0.0927) | (0.1643) |
| | $M_4$ | 0.0000 | −0.0001 | 0.0012 | 0.0033 | 0.0016 | 0.0001 | 0.0003 | 0.0005 |
| | | (0.0368) | (0.0340) | (0.0367) | (0.0357) | (0.0360) | (0.0340) | (0.0361) | (0.0366) |
| | $M_5$ | 0.0012 | 0.0022 | 0.0029 | 0.0014 | 0.0034 | 0.0012 | 0.0008 | 0.0032 |
| | | (0.0352) | (0.0352) | (0.0362) | (0.0365) | (0.0366) | (0.0369) | (0.0356) | (0.0354) |
| 500 | $M_1$ | −0.0004 | 0.0002 | −0.0005 | 0.0009 | 0.0001 | 0.0001 | −0.0002 | 0.0000 |
| | | (0.0158) | (0.0151) | (0.0157) | (0.0164) | (0.0164) | (0.0156) | (0.0160) | (0.0162) |
| | $M_2$ | −0.0058 | −0.0028 | 0.0009 | 0.0000 | 0.0017 | 0.0010 | 0.0040 | 0.0064 |
| | | (0.0168) | (0.0165) | (0.0162) | (0.0159) | (0.0166) | (0.0154) | (0.0164) | (0.0174) |
| | $M_3$ | −0.0290 | −0.0147 | −0.0022 | 0.0006 | 0.0028 | 0.0042 | 0.0166 | 0.0326 |
| | | (0.0329) | (0.0214) | (0.0164) | (0.0165) | (0.0160) | (0.0167) | (0.0234) | (0.0367) |
| | $M_4$ | 0.0002 | 0.0003 | −0.0003 | −0.0002 | 0.0000 | −0.0004 | 0.0006 | 0.0004 |
| | | (0.0155) | (0.0158) | (0.0159) | (0.0158) | (0.0161) | (0.0161) | (0.0160) | (0.0161) |
| | $M_5$ | 0.0013 | 0.0011 | 0.0011 | −0.0001 | −0.0003 | 0.0004 | 0.0005 | −0.0001 |
| | | (0.0163) | (0.0161) | (0.0156) | (0.0163) | (0.0157) | (0.0163) | (0.0161) | (0.0159) |

considered parameter values. In Table 3.2, M4 seems to outperform other plotting positions in the sense of smaller biases over all the sample sizes evenly. As we have observed in Table 3.1, M3 tends to document larger biases than other plotting positions under the sample sizes 20 and 50. Similar patterns are observed in Table 3.3. M3 tends to document larger biases than others under a small sample like 20. M3 also reports larger RMSE than others under the small sample. A very interesting pattern in Table 3.3 is on M4. M4 provided smallest biases over all the sample sizes. In Table 3.4, we frequently observe that M4 provides smaller biases than others under small samples. Similar pattern happens under sample size 20 in Table 3.5. M3 tends to frequently provide larger biases and RMSE than other plotting positions over all the samples in both Tables 3.4 and 3.5.

For more complete analysis, the referees suggested comparing the PWM with the ML estimators under the small sample situation. Under the large sample case, the ML estimators are usually expected to be better than the PWM method. Meanwhile, the PWM are expected to be of some advantages over the ML estimators under the small samples. To the best of our knowledge, the ML algorithm on the parameters of the GLO distribution has not been rigorously considered yet in the literature. Furthermore, the ML estimation procedure might require a computationally complicated iteration based algorithm due to the multi parameters of the GLO distribution. Regarding this, one referee kindly proposed making a numerical computation of the ML estimator for one parameter of our interest with others fixed. Following the referee's suggestion, we first built the negative log likelihood function $-L$ of the GLO distribution as follows:

$$-L = n \log \sigma + (1-\xi)\sum_{i=1}^{n} y_i + 2\sum_{i=1}^{n} \log(1+e^{-y_i}),$$

where $y_i = -\xi^{-1}\log\{1-\xi(x_i-\mu)/\sigma\}$. By numerically minimizing the negative log likelihood function with respect to the parameter of our interest, the ML estimator was obtained. Total four cases of simulation studies were considered here. Case 1 is for the ML estimators of the shape parameter ($\xi$) values from $-0.6$ to 0.6 with $(\mu, \sigma) = (0, 0.2)$. Cases 2 and 3 are for the scale parameter ($\sigma$) values over a range from 0.2 to 5 with $(\mu, \xi) = (0, 0.2)$ and $(0, -0.2)$, respectively. Cases 4 and 5 are for the location parameter ($\mu$) values ranged from $-5$ to 5 with $(\sigma, \xi) = (0.2, 0.2)$ and $(0.2, -0.2)$, respectively. For each set of parameters, the GLO distributions were generated 1,000 times and the ML estimators were evaluated via mean bias and RMSE. The considered sample size was 20. Some results are in Table 3.6. According to the results for Cases 4 and 5 in Table 3.6 and the outcomes for the sample sizes 20 in Tables 3.4 and 3.5, the ML estimators appeared to provide smaller RMSE than any of the aforementioned PWM. This implies that the ML estimators for the location parameter seemed to be better than the PWM in the sense of minimum RMSE under the small sample. In the aspect of the minimum mean bias criterion, some PWM methods performed better than the ML estimators. Similar patterns were observed in Cases 2 and 3. Documented RMSE of the ML estimators regarding the scale parameter estimation were smaller than those of all the PWM for

Table 3.6: The bias and root mean square errors (in the parenthesis) of the maximum likelihood (ML) estimates over several ranges of shape, scale, location parameters for the GLO distribution are reported. Case 1 is for the ML estimators for several shape parameter values ($\xi$) with $(\mu, \sigma) = (0, 0.2)$. Cases 2 and 3 are for the ML estimators for several scale parameter values ($\sigma$) with $(\mu, \xi) = (0, 0.2)$ and $(0, -0.2)$, respectively. Cases 4 and 5 are for the ML estimators for several location parameter values ($\mu$) with $(\sigma, \xi) = (0.2, 0.2)$ and $(0.2, -0.2)$, respectively. Considered sample size is 20. For each set of parameters, the GLO distributions are generated 1,000 times.

| | | | $\xi$ | | | |
|---|---|---|---|---|---|---|
| Case 1 | −0.6 | −0.4 | −0.2 | 0.2 | 0.4 | 0.6 |
| bias | −0.0795 | −0.0817 | −0.0999 | 0.0328 | 0.0682 | 0.0692 |
| rmse | (0.1620) | (0.2253) | (0.3448) | (0.2410) | (0.2043) | (0.1729) |
| | | | $\sigma$ | | | |
| Case 2 | 0.2 | 0.4 | 0.6 | 1 | 3 | 5 |
| bias | −0.0045 | −0.0058 | −0.0132 | −0.0114 | −0.0256 | −0.0729 |
| rmse | (0.0340) | (0.0686) | (0.1022) | (0.1683) | (0.5115) | (0.8562) |
| | | | $\sigma$ | | | |
| Case 3 | 0.2 | 0.4 | 0.6 | 1 | 3 | 5 |
| bias | −0.0054 | −0.0036 | −0.0066 | −0.0055 | −0.0814 | −0.0438 |
| rmse | (0.0340 ) | (0.0647) | (0.1009) | (0.1698) | (0.5207) | (0.8330) |
| | | | $\mu$ | | | |
| Case 4 | −5 | −1 | −0.2 | 0.2 | 1 | 5 |
| bias | −0.0013 | −0.0013 | −0.0020 | −0.0034 | −0.0053 | −0.0091 |
| rmse | (0.0669) | (0.0669) | (0.0702) | (0.0700) | (0.0661) | (0.0701) |
| | | | $\mu$ | | | |
| Case 5 | −5 | −1 | −0.2 | 0.2 | 1 | 5 |
| bias | 0.0060 | 0.0044 | 0.0071 | 0.0071 | 0.0071 | 0.0039 |
| rmse | (0.0696) | (0.0679) | (0.0709) | (0.0695) | (0.0710) | (0.0695) |

a sample size 20 in Tables 3.2 and 3.3. Meanwhile, the ML estimators for the shape parameter in Case 1 tended to be outperformed by all the PWM for a sample size 20 in Table 3.1 when its magnitude was near zero: $(-0.4, -0.2, 0.2, 0.4)$ in the sense of the minimum RMSE. In particular, the degree of inaccuracy of the ML estimators was stronger as the shape parameter was closer to zero: $(-0.2, 0.2)$. On the other hand, the ML method was better than any of the PWM method when the shape parameter was away from zero: $(-0.6, 0.6)$. Our results indicated that the accuracy of the PWM was reduced as the magnitude of the shape parameter value was going farther away from zero. These results on the shape parameter estimation are similar with those in GEV and GP distributions. Note that the accuracy of the PWM based shape parameter estimator for the GEV and GP is known to get worse as the shape parameter value goes farther away from zero.

We have compared five plotting position based PWM on the parameter estimation of the GLO distribution via the simulation studies. Additionally, the ML estimators are

Table 3.7: The bias and root mean square errors (in the parenthesis) of the PWM based 0.9, 0.99, 0.999 quantile estimates regarding five plotting positions for the GLO distribution with sample size 500 are reported. The shape parameter over a range of −0.4 to 0.4 is considered. The location and scale parameters are fixed as zero and 0.2, respectively. M1, M2, M3, M4 and M5 represent $(i - 0.35)/n$, Cunnane, Weibull, Landwehr, and Gettinby plotting positions, respectively. $x(p)$ indicates $p$-quantile. For each set of sample size, parameters, and plotting position, the GLO distributions are generated 1,000 times.

| $p$ | | $\xi$ = −0.4 | −0.2 | 0.2 | 0.4 |
|---|---|---|---|---|---|
| | $x(p)$ | 0.7041 | 0.5518 | 0.3556 | 0.2924 |
| | $M_1$ | −0.0036 | −0.0015 | −0.0002 | −0.0002 |
| | | (0.0653) | (0.0391) | (0.0153) | (0.0102) |
| | $M_2$ | 0.0001 | −0.0017 | −0.0003 | −0.0006 |
| | | (0.0671) | (0.0401) | (0.0155) | (0.0104) |
| 0.9 | $M_3$ | −0.0049 | −0.0018 | −0.0009 | −0.0003 |
| | | (0.0666) | (0.0384) | (0.0152) | (0.0101) |
| | $M_4$ | −0.0062 | −0.0021 | −0.0003 | 0.0001 |
| | | (0.0636) | (0.0398) | (0.0149) | (0.0104) |
| | $M_5$ | −0.0052 | −0.0032 | −0.0010 | −0.0003 |
| | | (0.0666) | (0.0400) | (0.0154) | (0.0106) |
| | $x(p)$ | 2.6421 | 1.5068 | 0.6011 | 0.4204 |
| | $M_1$ | −0.0082 | −0.0022 | 0.0035 | 0.0041 |
| | | (0.5022) | (0.1479) | (0.0314) | (0.0232) |
| | $M_2$ | 0.0070 | −0.0027 | 0.0012 | 0.0022 |
| | | (0.4846) | (0.1556) | (0.0299) | (0.0236) |
| 0.99 | $M_3$ | −0.0232 | −0.0034 | 0.0007 | 0.0032 |
| | | (0.4791) | (0.1508) | (0.0303) | (0.0236) |
| | $M_4$ | −0.0265 | −0.0051 | 0.0005 | 0.0033 |
| | | (0.4424) | (0.1535) | (0.0311) | (0.0233) |
| | $M_5$ | −0.0196 | −0.0040 | 0.0000 | 0.0037 |
| | | (0.4708) | (0.1562) | (0.0307) | (0.0239) |
| | $x(p)$ | 7.4213 | 2.9803 | 0.7488 | 0.4684 |
| | $M_1$ | 0.0980 | 0.0061 | 0.0085 | 0.0076 |
| | | (2.9415) | (0.4225) | (0.0540) | (0.0347) |
| | $M_2$ | 0.1327 | 0.0067 | 0.0038 | 0.0046 |
| | | (2.4676) | (0.4471) | (0.0505) | (0.0350) |
| 0.999 | $M_3$ | 0.0076 | 0.0056 | 0.0037 | 0.0063 |
| | | (2.3547) | (0.4397) | (0.0517) | (0.0352) |
| | $M_4$ | −0.0158 | −0.0012 | 0.0023 | 0.0061 |
| | | (2.1757) | (0.4403) | (0.0533) | (0.0348) |
| | $M_5$ | 0.0255 | 0.0089 | 0.0020 | 0.0071 |
| | | (2.2950) | (0.4540) | (0.0517) | (0.0357) |

numerically obtained, and are compared with the PWM. We have summarized the main important simulation results as follows:

1. Landwehr plotting position tended to provide more unbiased estimation results

than other plotting positions in the location and scale parameter estimations.

2. Weibull plotting position frequently tended to cause larger biases than other plotting positions under the location, scale, and shape parameter estimations over several sample sizes.

3. Plotting position $(i-0.35)/n$ seemed to provide smaller RMSE than other plotting positions in the negative shape parameter estimation under small samples like 20 and 50.

4. ML estimator tended to provide smaller RMSE than any of the PWM methods in the location and scale parameter estimations under a small sample 20. Meanwhile, the ML estimators were outperformed by all the PWM above in the shape parameter estimation when its magnitude was near zero: $(-0.4, -0.2, 0.2, 0.4)$.

Concluding this section, we compared the right tail quantile estimation performance for each five plotting position. Three right tail quantiles, $p = 0.9, 0.99$, and $0.999$ for a sample size 500 were considered. We restricted the location and scale parameters into zero and 0.2, respectively. Over a range of shape parameter from $-0.4$ to $0.4$, the quantiles above were estimated. Aforementioned five plotting positions were considered again. For each set of parameter and plotting position, we generated the GLO distribution 1,000 times. First, we estimated the parameters using the PWM method, and estimated the quantiles using the function (2.3). The bias and RMSE of the quantile estimates are reported in Table 3.7. Most of the plotting positions seemed to document evenly good accuracy under larger shape parameter value and smaller quantile value. On the other hand, they seemed to indicate evenly bad accuracy under smaller shape parameter value and more right tail extreme quantile value. Unlike the parameter estimation cases, we could not find a consistently superior or inferior plotting position in the quantile estimation in the sense of either minimum RMSE or minimum bias criteria. To make our analysis complete, we also considered similar experiments for a sample size 20. However, we do not report the simulation outputs here since neither superior or inferior plotting position was observed in the simulation studies.

## 4. Conclusion

In this paper, the effects of five plotting positions to the computation of the PWM on parameters and quantiles of the GLO distribution have been examined via Monte Carlo simulation studies. The Weibull, Cunnane, Landwehr, $(i-0.35)/n$, and Gettinby plotting positions were considered. We applied sample sizes 20, 50, 100, 500 and some appropriate range of parameter values. In the sense of minimum RMSE and minimum bias criteria, we investigated the accuracy of estimation. In the simulation studies, the Landwehr plotting position seemed to outperform other plotting positions in the sense of smaller biases in the scale and location parameter estimations. On the other hand, the Weibull plotting position frequently tended to perform badly in many parameter estimation under small or large samples documenting larger biases than other plotting positions. The plotting position $(i-0.35)/n$ seemed to be better than others in the sense of smaller RMSE in the negative shape parameter estimation under small samples. For more complete analysis, we have compared the PWM with the numerically computed ML estimators under a small sample. The ML estimators tended to have better accuracy than the PWM in the location and scale parameter estimation. However, the PWM seemed to be better than the ML estimators in the shape parameter estimation when the magnitude of the shape parameter is near zero. We also examined the sensitivity of right tail quantile estimation regarding five plotting positions. No superior or inferior plotting position was observed in the quantile estimation.

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