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# SOME COMPUTATIONS AND EXTREMAL PROPERTIES OF OPERATORS

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**Abstract**

In [6] some computaion of spectral measures induced by normal operators $T^{*n}T^n$ was introduced. In this note we improve some computations by using spectral measures, which are related to extremal vectors. Also, we discuss the extremal value properties and apply our spectral measure equations to moment sequences which are induced by weighted shifts.

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## 1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. And also the study of invariant subspaces is very important to study of structure of Hilbert space operators. In these sequel studies, Per Enflo introduced a new technique involving some "extremal vectors" for producing invariant subspaces for certain quasinilpotent operators in $\mathcal{L}(\mathcal{H})$ (cf. [1]). One modified those techniques and expanded in [4] to produce better invariant subspace theorems for some quasinilpotent operators, and continued to explore the limits of Enflo's technique for producing invariant and hyperinvariant subspaces in [5]. In [7], the spectral techniques were contributed to solve the hyperinvariant subspace problem of subnormal operators.

In this paper we study improved some computations in the techniques introduced in [6]. In Section 2, we discuss some spectral equations which are related to extremal vectors. In Section 3, we study the extremal value properties. Finally, in Section 4 we apply our spectral measure equations to moment sequences which are induced by weighted shifts.

Throughout this paper, $\mathbb{N}$ is the set of natural numbers.

## 2. Some constructions

Let $N$ be a normal operator in $\mathcal{L}(\mathcal{H})$. Then there exists a corresponding spectral measure $E$ on $\sigma(N)$ such that for each Borel function $f$ on $\sigma(N)$, $f(N) = \int f(\lambda)\ dE$ exists, in particular, if $f = \chi_\Delta$ is a characteristic function on a Borel subset $\Delta \subset \sigma(N)$, then $f(N) = E(\Delta)$ and for $f \in C(\sigma(N))$ there exist polynomials $p_n(\lambda)$ with $\|f - p_n\|_\infty \to 0$ such that $f(N) = \lim \int p_n(\lambda)\ dE$, where $C(\sigma(N))$ is the Banach space of all continuous functions on $\sigma(N)$.

By the spectral mapping theorem (cf. [2]), there exists a corresponding spectral measure $E^{(n,p)}$ to $(T^{*n}T^n)^p$ such that

$$(T^{*n}T^n)^p = \int_{[0,||T^n||^{2p}]} \lambda dE^{(n,p)} \qquad (n \in \mathbb{N})$$

and

$$T^{*n}T^n = \int_{[0,||T^n||^2]} \lambda dE^{(n)} \qquad (n \in \mathbb{N}).$$

For brevity we write $E_\lambda^{(n,p)} = E^{(n,p)}([0,\lambda])$ and $E_{\lambda^-}^{(n,p)} = E^{(n,p)}([0,\lambda))$ (see [5] for this notation). First we discuss some fundamental properties of these spectral measures.

**Proposition 2.1.** *Under the above notation we have the following statements:*

(i) $E_\lambda^{(n,p)} = E_{\lambda^{1/p}}^{(n)}$ *for all* $\lambda \in [0,\infty)$,

(ii) $E_{\lambda^p}^{(n,p)} = E_\lambda^{(n)}$ *for all* $\lambda \in [0,\infty)$.

*Proof.* (i) First we claim that $E^{(n,p)}([0,\lambda]) = E^{(n)}([0,\lambda^{1/p}])$. Observe that

$$\begin{aligned}\int_{[0,\|T^n\|^{2p}]} \lambda dE^{(n,p)} &= (T^{*n}T^n)^p = \int_{[0,\|T^n\|^2]} \lambda^p dE^{(n)} \\ &= \int_{[0,\|T^n\|^2]} \varphi(\lambda)\, dE^{(n)} \qquad (\text{where } \varphi(\lambda) = \lambda^p) \\ &= \int_{\varphi[0,\|T^n\|^2]} \lambda dE^{(n)} \circ \varphi^{-1} \\ &= \int_{[0,\|T^n\|^{2p}]} \lambda dE^{(n)} \circ \varphi^{-1}.\end{aligned}$$

Hence

$$\int_{[0,\|T^n\|^{2p}]} \lambda dE^{(n,p)} = \int_{[0,\|T^n\|^{2p}]} \lambda dE^{(n)} \circ \varphi^{-1}.$$

Continuing this process, we have that

$$\int_{[0,\|T^n\|^{2p}]} \lambda^m dE^{(n,p)} = \int_{[0,\|T^n\|^{2p}]} \lambda^m dE^{(n)} \circ \varphi^{-1} \text{ for all } m \in \mathbb{N}.$$

Then obviously we have

$$\int_{[0,\|T^n\|^{2p}]} p(\lambda) dE^{(n,p)} = \int_{[0,\|T^n\|^{2p}]} p(\lambda) dE^{(n)} \circ \varphi^{-1} \text{ for any polynomial } p,$$

which implies that $E^{(n,p)} = E^{(n)} \circ \varphi^{-1}$. Also we have that

$$E_\lambda^{(n,p)} = E^{(n,p)}([0,\lambda]) = E^{(n)} \circ \varphi^{-1}([0,\lambda]) = E^{(n)}([0,\lambda^{1/p}]) = E_{\lambda^{1/p}}^{(n)}.$$

(ii) This follows obviously from (i). ■

The following lemma is a well-known property.

**Lemma 2.2 [2, p.266].** *Let $N$ be a normal operator in $\mathcal{L}(\mathcal{H})$ and let $E$ be the corresponding spectral measure to $N$. Then $\lambda \in \sigma_p(N)$ if and only if $E(\{\lambda\}) \neq 0$.*

**Proposition 2.3.** *Let $x_0 \in \mathcal{H}$ be a unit vector. Under the above notation we have the following statements:*
(i) *The functions $\lambda \to E_\lambda^{(n,p)}$ and $\lambda \to E_{\lambda-}^{(n,p)}$ are monotone increasing,*
(ii) *$\lambda \to \langle E_\lambda^{(n,p)} x_0, x_0\rangle$ is continuous from the right,*

(iii) $\lambda \to \langle E^{(n,p)}_{\lambda^-} x_0, x_0\rangle$ *is continuous from the left.*

*In particular, the functions in* (ii) *and* (iii) *are continuous at* $\lambda_0$ *if* $\lambda_0 \notin \sigma_p((T^{*n}T^n)^p)$.

*Proof.* (i) If $\lambda_1 < \lambda_2$, then

$$\begin{aligned} E^{(n,p)}_{\lambda_2} &= E^{(n,p)}([0,\lambda_2]) = E^{(n,p)}([0,\lambda_1] \cup (\lambda_1,\lambda_2]) \\ &= E^{(n,p)}([0,\lambda_1]) + E^{(n,p)}((\lambda_1,\lambda_2]) \geq E^{(n,p)}([0,\lambda_1]) = E^{(n,p)}_{\lambda_1}. \end{aligned}$$

(ii) Let $\lambda_k \to \lambda_0^+ \ (k\to\infty)$. Then

$$\begin{aligned} \langle E^{(n,p)}_{\lambda_k} x_0, x_0\rangle - \langle E^{(n,p)}_{\lambda_0} x_0, x_0\rangle &= \left\langle (E^{(n,p)}([0,\lambda_k]) - E^{(n,p)}([0,\lambda_0]))x_0, x_0\right\rangle \\ &= \left\langle E^{(n,p)}((\lambda_0,\lambda_k])\,x_0, x_0\right\rangle \leq \left\| E^{(n,p)}((\lambda_0,\lambda_k]) \right\|. \end{aligned}$$

Since

$$\lim_{k\to\infty} \left\| E^{(n,p)}((\lambda_0,\lambda_k]) \right\| = \left\| E^{(n,p)}(\cap_{k=1}^{\infty}(\lambda_0,\lambda_k]) \right\| = \left\| E^{(n,p)}(\varnothing) \right\| = 0,$$

we have that

$$\langle E^{(n,p)}_{\lambda_k} x_0, x_0\rangle \to \langle E^{(n,p)}_{\lambda_0} x_0, x_0\rangle.$$

We will show the left continuity at $\lambda_0 \notin \sigma_p((T^{*n}T^n)^p)$. To do so, let $\lambda_k \to \lambda_0^- \ (k\to\infty)$. By the similar method, we have that

$$\lim_{k\to\infty} \left\| E^{(n,p)}((\lambda_k,\lambda_0]) \right\| = \left\| E^{(n,p)}(\cap_{k=1}^{\infty}(\lambda_k,\lambda_0]) \right\| = \left\| E^{(n,p)}(\{\lambda_0\}) \right\|.$$

By Lemma 2.2, $\lim_{k\to\infty} \left\| E^{(n,p)}((\lambda_k,\lambda_0]) \right\| = 0$. Hence

$$\langle E^{(n,p)}_{\lambda_k} x_0, x_0\rangle \to \langle E^{(n,p)}_{\lambda_0} x_0, x_0\rangle.$$

(iii) Similar to (ii). ■

## 3. Extremal value property

In this section we discuss extremal values of $T \in \mathcal{L}(\mathcal{H})$.

**Definition 3.1.** For $\theta$ with $0 < \theta < 1$ and $x_0 \in \mathcal{H}$ with $||x_0|| = 1$, we define

$$\lambda_n(\theta, x_0, p) = \inf\{\lambda \in [0, ||T^n||^{2p}] : ||E^{(n,p)}_{\lambda} x_0|| \geq \theta\}.$$

In this case $\lambda_{n,p} := \lambda_n(\theta, x_0, p)$ is called an *extremal value* of $T$ with respect to $(\theta, x_0, p)$.

Let $A$ be a positive operator and let $Ax = 0$. Some computation shows that $A^p x = 0$ for any $p > 0$ .

And for $p > 0$, obviously we have that $T^{*n}T^n$ is one to one if and only if $(T^{*n}T^n)^p$ is one to one.

**Proposition 3.2 (Boundedness Property).** *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\lambda_n(\theta, x_0, p) \leq |\sigma(T)|^{2pn} \quad \textit{for all} \quad n \in \mathbb{N},$$

*where $|\sigma(T)|$ is the spectral radius of $T$.*

*Proof.* Since $\lambda_{n,p} \leq \|T^n\|^{2p}$, we have $(\lambda_{n,p})^{\frac{1}{n}} \leq \|T^n\|^{\frac{2p}{n}}$. Hence

$$(\lambda_{n,p})^{\frac{1}{n}} \leq \limsup_{n\to\infty} (\lambda_{n,p})^{\frac{1}{n}} \leq (\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}})^{2p} = |\sigma(T)|^{2p}.$$

Hence the proof is complete. ■

The following is the main theorem of this note.

**Theorem 3.3 (Extremal and injective property).** *Let $T \in \mathcal{L}(\mathcal{H})$. Then $T$ is one to one if and only if $\lambda_n(\theta, x_0, p) > 0$ for any $x_0$ in $\mathcal{H}$ with $||x_0|| = 1$, any $(n,p) \in \mathbb{N} \times (0, \infty)$ and any $\theta \in (0,1)$.*

*Proof.* $(\Rightarrow)$ Assume that $T$ is one to one. To the contrary, we suppose that there exist $x_0$ in $\mathcal{H}$ with $||x_0|| = 1$, $(n,p) \in \mathbb{N} \times (0, \infty)$ and $\theta \in (0,1)$ such that $\lambda_{n,p} = 0$. Then by the definition of infimum of $\lambda_n(\theta, x_0, p)$, there exists a sequence $\{\lambda_{n,p}^{(k)}\}_{k=1}^{\infty}$ such that $\lambda_{n,p}^{(1)} \geq \lambda_{n,p}^{(2)} \geq \cdots \longrightarrow \lambda_{n,p}$ $(= 0)$. Then we have

$$\begin{aligned} 0 &< \theta \leq \lim_{k\to\infty} \left\|E^{(n,p)}([0, \lambda_{n,p}^{(k)}])x_0\right\| \leq \lim_{k\to\infty} \left\|E^{(n,p)}([0, \lambda_{n,p}^{(k)}])\right\| \|x_0\| \\ &= \lim_{k\to\infty} \left\|E^{(n,p)}([0, \lambda_{n,p}^{(k)}])\right\| = \left\|E^{(n,p)}(\cap_{k=1}^{\infty}([0, \lambda_{n,p}^{(k)}])\right\| \\ &= \left\|E^{(n,p)}(\{0\})\right\|. \end{aligned}$$

To show $E^{(n,p)}(\{0\}) = 0$, we will prove that $0 \notin \sigma_p((T^{*n}T^n)^p)$, (i.e. $(T^{*n}T^n)^p$ is one to one). Since $T$ is one to one, $T^n$ is one to one. Suppose $T^{*n}T^n x = 0$. Then $\langle T^{*n}T^n x, x\rangle = 0$. So $\|T^n x\| = 0$ and so $T^n x = 0$. Hence $x = 0$. Thus $T^{*n}T^n$ is one to one. By the above remark, so $(T^{*n}T^n)^p$ is one to one.

Therefore by Lemma 2.2 $E^{(n,p)}(\{0\}) = 0$. This contradiction proves this implication.

($\Leftarrow$) Suppose that $T$ is not one to one. Then obviously, $(T^{*n}T^n)^p$ is not one to one by the above remark. Hence $0 \in \sigma_p((T^{*n}T^n)^p)$. So $E^{(n,p)}(\{0\}) \neq 0$ for all natural number $n$ and all $p > 0$. There exists a vector $x_0 \in \mathcal{H}$ with $\|x_0\| = 1$ such that $\left\|E^{(n,p)}(\{0\})x_0\right\| > \theta_0 > 0$. By the definition of extremal value, $\lambda_n(\theta_0, x_0, p) = 0$. This contradiction proves this implication. ■

## 4. Examples

Let $\alpha : \alpha_0, \alpha_1, \ldots$ be a weight sequence of positive real numbers. Let $W_\alpha$ be a hyponormal weighted shift with a weight sequence $\alpha$. Let$\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. We may consider $\mathcal{H}$ as $l^2(\mathbb{Z}_+)$, where $l_2 := l^2(\mathbb{Z}_+)$ is the set of square summable sequences in $\mathbb{C}$. Let $W_\alpha$ be a weighted shift with a weight sequence $\alpha$ and let $E^{(n,p)}$ be the spectral measure corresponding to $(W_\alpha^{*n}W_\alpha^n)^p$. Let $E_{e_i,e_j}^{(n,p)}$ be the scalar valued spectral measure on $[0, \|W_\alpha^n\|^{2p}]$ which is defined by $E_{e_i,e_j}^{(n,p)}(\triangle) = \left\langle E^{(n,p)}(\triangle)e_i, e_j\right\rangle$ for every Borel set $\triangle \in \mathcal{B}([0, \|W_\alpha^n\|^{2p}])$. Then, since $\left\langle\left(\int f(\lambda)dE\right)x, y\right\rangle = \int f(\lambda)dE_{x,y}$,

$$\int_{[0,\|W_\alpha^n\|^{2p}]} \lambda^m \, dE_{e_i,e_j}^{(n,p)} = \langle (W_\alpha^{*n}W_\alpha^n)^{mp} e_i, e_j \rangle.$$

**Example 4.1.** Let $W_\alpha$ be a weighted shift with a weight sequence $\alpha$. Since

$$(W_\alpha^{*n}W_\alpha^n)^{mp} = Diag\left\{\alpha_0^{2mp}\cdots\alpha_{n-1}^{2mp}, \alpha_1^{2mp}\cdots\alpha_n^{2mp}, \cdots, \alpha_i^{2mp}\cdots\alpha_{n+i-1}^{2mp}, \cdots\right\},$$

we have that for each $m \in N$,

$$\int_{[0,\|W_\alpha^n\|^{2p}]} \lambda^m \, dE_{e_i,e_j}^{(n,p)} = \langle (W_\alpha^{*n}W_\alpha^n)^{mp} e_i, e_j \rangle = \begin{cases} \alpha_i^{2mp}\cdots\alpha_{n+i-1}^{2mp} & (i = j) \\ 0 & (i \neq j) \end{cases}. \tag{4.1}$$

**Example 4.2.** Let $W_\alpha$ be a Bergmann shift (i.e. $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$). By Example 4.1., we have

$$\begin{aligned} \int_0^1 \lambda^m dE_{e_i,e_j}^{(n,p)}(\lambda) &= \alpha_i^{2mp}\alpha_{i+1}^{2mp}\cdots\alpha_{n+i-1}^{2mp} \\ &= \left(\frac{i+1}{i+2}\cdot\frac{i+2}{i+3}\cdots\frac{n+i}{n+i+1}\right)^{mp} \\ &= \left(\frac{i+1}{n+i+1}\right)^{mp}. \end{aligned}$$

Hence for each $m \in \mathbb{N}$,

$$\int_0^1 \lambda^m \, dE^{(n,p)}_{e_i,e_j}(\lambda) = \begin{cases} \left(\frac{i+1}{n+i+1}\right)^{mp} & (i=j) \\ 0 & (i \neq j) \end{cases} .$$

By Example 4.1., we have

$$\begin{aligned} \int_0^1 \lambda^m dE^{(n,p)}_{e_i,e_j}(\lambda) &= \alpha_i^{2mp}\alpha_{i+1}^{2mp}\cdots\alpha_{n+i-1}^{2mp} \\ &= \left(\frac{i+1}{i+2}\cdot\frac{i+2}{i+3}\cdots\frac{n+i}{n+i+1}\right)^{mp} \\ &= \left(\frac{i+1}{n+i+1}\right)^{mp}. \end{aligned}$$

**Example 4.3.** Let $W_\alpha$ be a unilateral shift of multiplicity one. Since $\|W_\alpha^n\|^{2p} = 1$, by Example 4.1 we have

$$\int_{[0,\|W_\alpha^n\|^{2p}]} \lambda^m \, dE^{(n,p)}_{e_i,e_j} = \int_0^1 \lambda^m dE^{(n,p)}_{e_i,e_j}(\lambda) = \delta_{ij},$$

where $\delta_{ij}$ is the usual notation.

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