

CONVERGENCE OF APPROXIMATE SOLUTIONS TO SCALAR CONSERVATION LAWS BY DEGENERATE DIFFUSION

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ABSTRACT. In this paper, we show the convergence of approximate solutions to the convective porous media equation using methodology developed in [8]. First, we obtain the approximate transport equation for the given convective porous media equation. Then using the averaging lemma, we obtain the convergence.

1. Introduction

For scalar multi-dimensional conservation laws

$$(1.1) \quad \begin{aligned} \partial_t u + \sum_{j=1}^d \partial_{x_j} F_j(u) &= 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}^+, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

there are available two equivalent notions for weak solutions: the Kruzhkov entropy solution [9], stating that u satisfies the entropy inequalities

$$(1.2) \quad \partial_t \eta(u) + \operatorname{div} q(u) \leq 0 \quad \text{in } \mathcal{D}',$$

for any entropy pair $\eta - q$ with η convex, and the kinetic formulation of Lions-Perthame-Tadmor [10]. Both concepts lead to uniqueness, stability theorems and error estimates for approximate entropy solutions [9, 16].

Consider the following approximation of (1.1) obtained by adding a porous media operator to the right hand side of (1.1):

$$(1.3) \quad \begin{aligned} \partial_t u + \sum_{j=1}^d \partial_{x_j} F_j(u) &= \varepsilon \sum_{j=1}^d \partial_{x_j x_j} (|u|^{m-1} u), \quad x \in \mathbb{R}^d, t \geq 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

where $m > 1$ and $a_j = F'_j$.

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This equation (1.3) and its variants model degenerate diffusion-convection motions of ideal fluids and arise in a wide variety of important applications, including two phase flows in porous media (see, for example, [3] and the references cited therein) and sedimentation-consolidation process (see, for example, [1] and the reference cited therein). A well-posedness theory for (1.3) and its variants is relatively well understood; see [24], [15], [2].

In particular, in [15], the existence theory of travelling waves solution to (1.3) has been carried out revealing the advantage of using the degenerate diffusive approximation (1.3) with respect to the usual one. Indeed, the latter method provides an approximating solutions which, in some situations, coincides with the exact solution of (1.1) outside a compact set (in the space variable, for fixed time), while the perturbation effects of the usual viscosity always yield to undesired modifications of far fields (for numerical applications in this direction, see [12]).

In this paper, our main object is to show that the solutions u^ε of (1.3) converge in the strong topology to the entropy solution of (1.1). Convergence of (1.3) has been established in the 1-dimensional case by Marcati [11] using the compensated compactness. The vanishing viscosity limit was studied by several authors in the special case of a linear (and therefore non-degenerate) diffusion term which corresponds to $m = 1$; this activity started with Oleinik's [14] and Kruzhkov's [9] works. For different kind of viscosity coefficients, see [12], [13]. In this paper, for multi-dimensional case, we prove the convergence of the approximate solutions u^ε to the entropy solution of (1.1) using different methodology developed in [8].

This methodology was developed to understand the compactness of approximate solutions $\{u^\varepsilon\}$ bounded in some L^p -norm ($p > 1$) which satisfy the entropy dissipation measure in the sense

$$(1.4) \quad \partial_t \eta(u^\varepsilon) + \operatorname{div} q(u^\varepsilon) \quad \text{is precompact in } H_{loc,x,t}^{-1}.$$

The compactness of u^ε which satisfy (1.4) has been proved in one-space dimension in both the L^∞ and L^p stability settings by Tartar [22] and Schonbek [20] (see [19] for a simplified proof using singular entropies and [23] for an analysis of the compensated compactness bracket in multi-d). For multi-dimensional conservation laws, convergence of approximate solutions is usually deduced by using a framework of DiPerna [4] and Szepessy [21]. The argument hinges on showing that a Young-measure solution (with certain regularity in time) that satisfies (1.2) for all convex η and is a Dirac mass at $t = 0$ is in fact a regular weak solution. It yields compactness for bounded families of approximate entropy solutions, *i.e.* approximate solutions $\{u^\varepsilon\}$ that satisfy the dissipation structure

$$(1.5) \quad \partial_t \eta(u^\varepsilon) + \operatorname{div} q(u^\varepsilon) \leq \mathcal{P}^\varepsilon(u^\varepsilon)$$

with $\mathcal{P}^\varepsilon(u^\varepsilon) \rightarrow 0$ in \mathcal{D}' as $\varepsilon \rightarrow 0$.

An alternative compactness framework is proposed in Lions-Perthame-Tadmor [10] by means of the kinetic formulation and averaging lemmas (e.g. [17, 18]). The framework in [10] is developed for approximations that still satisfy (1.5). Nevertheless, as we will see, it can be easily adapted to apply to the structure (1.4).

This methodology successfully applied to the relaxation approximation of Jin-Xin type, diffusion-dispersion approximation, kinetic models of BGK type, and singularly perturbed higher order partial differential equations, see [6], [7], [8].

Main idea of this methodology is following: First turn the entropy production into a kinetic form using duality (see section 3) and it results to an approximate transport equation,

$$(1.6) \quad \partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon = \sum_{j=1}^d \partial_{x_j} (\bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon) + \partial_\xi m^\varepsilon,$$

for the function $\chi^\varepsilon = \mathbb{1}(u^\varepsilon(x, t), \xi)$, where

$$(1.7) \quad \mathbb{1}(u, \xi) = \begin{cases} \mathbb{1}_{0 < \xi < u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u < \xi < 0} & \text{if } u < 0 \end{cases}$$

is the usual Maxwellian associated with the kinetic formulation of scalar conservation laws, $\bar{g}_i^\varepsilon, g_i^\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and m^ε is uniformly bounded in positive measures. Convergence is then obtained via the averaging lemma in [18]. In the limit $\varepsilon \rightarrow 0$, $\chi^\varepsilon \rightarrow \mathbb{1}(u, \xi) =: \chi$ which satisfies

$$(1.8) \quad \partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi m \quad \text{in } D'_{x,t,\xi},$$

with m a positive bounded measure. We conclude that u an entropy solution.

From physical point of view, this result describes the behavior of the solution of the convective porous media equation in terms of the solution to the related conservation law, which the permeability of the medium tends to zero.

This paper is organized in this way: In section 2, we establish the finite propagation of speed and a priori estimates for approximate solutions. Then section 3 shows the convergence of approximate solutions to the unique entropy solution.

2. Finite propagation speed and a priori estimates

In this section we obtain the finite speed of propagation and a priori estimates for approximate solutions.

Proposition 2.1. *Let $m > 1, \varepsilon > 0$ be given. Let $F_j(u) \in C^1(\mathbb{R}), j = 1, \dots, d$. If $u_0^\varepsilon(x)$ is a continuous function with compact support, then there exists a function $\zeta^\varepsilon : \mathbb{R} \rightarrow [0, \infty]$ such that for all $t > 0$,*

$$\text{supp}(u^\varepsilon(\cdot, t)) \subset \{x : |x| \leq \zeta^\varepsilon(t)\}.$$

Proof. Let $T > 0$ and $u^\varepsilon = u$. Define v and $L[z]$ by

$$v = \frac{m}{m-1} |u|^{m-1}$$

and

$$L[z] = \partial_t z + \sum_{j=1}^d F'_j(u) \partial_{x_j} z - \varepsilon \sum_{j=1}^d (\partial_{x_j} z)^2 - \sum_{j=1}^d \varepsilon(m-1) z \partial_{x_j x_j} z.$$

Note that $L[v] = 0$ in the sense of distribution.

If we set

$$\hat{v}(x, t) = \phi(t) \left(1 - \frac{|x|^2}{\phi(t)}\right)^+, \quad x \in \mathbb{R}^d, t \in [0, T],$$

we can determine a positive function $\phi(t)$ so that

$$(2.1) \quad v(x, t) \leq \hat{v}(x, t).$$

Indeed, for all $x \in \{x \in \mathbb{R}^d : |x|^2 \leq \phi(t)\}$, we have

$$\begin{aligned} L[\hat{v}] &= \phi'(t) - \sum_{j=1}^d F'_j(u) x_j - 4\varepsilon |x|^2 + 2\varepsilon(m-1)\phi(t) \left(1 - \frac{|x|^2}{\phi(t)}\right)^+ \\ &\geq \phi'(t) - 2K\sqrt{\phi(t)} - 4\varepsilon\phi(t), \end{aligned}$$

where $K = K(T) > 0$ is determined in the following way:

It is well-known that

$$|u(x, t)| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \text{ a.e. in } \mathbb{R}^d \times [0, \infty).$$

Hence, it is natural to assume

$$K = \max_{j=1, \dots, d} \sup\{|F'_j(u)| : |u| \leq \|u_0\|_{L^\infty}\}$$

and denote by

$$\phi_0 = \sup\{|x| : x \in \text{supp}(u_0)\}.$$

Then if we choose $\phi(t)$ as the solution of the following initial value problem

$$\begin{cases} \phi'(t) - 2K\sqrt{\phi(t)} - 4\varepsilon\phi(t) = 0 \\ \phi(0) = \phi_0, \end{cases}$$

we obtain $L[\hat{v}] \geq 0$. By the maximum principle, this implies that (2.1) holds. \square

Now we obtain some estimates on approximate solutions u^ε of (1.3).

Lemma 2.2. *Suppose that $F_j \in C^1(\mathbb{R})$, $j = 1, \dots, d$ and assume that the initial data $u_0^\varepsilon(x)$ is continuous and has a compact support, then we have*

$$(2.2) \quad \sup_{x, t} |u^\varepsilon(x, t)| \leq \sup_x |u_0^\varepsilon(x)|$$

If in addition $u_0 \in L^2(\mathbb{R}^d)$, then

$$(2.3) \quad \sup_t \|u^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \|u_0^\varepsilon(x)\|_{L^2(\mathbb{R}^d)}$$

and for all $T > 0$,

$$(2.4) \quad \varepsilon m \sum_{j=1}^d \iint_{\mathbb{R}^d \times (0, T)} |u^\varepsilon(x, t)|^{m-1} (\partial_{x_j} u^\varepsilon(x, t))^2 dx dt \leq \|u_0^\varepsilon(x)\|_{L^2(\mathbb{R}^d)}^2.$$

Proof. Let $u(x, t) = u^\varepsilon(x, t)$. Let $k > 0$ be given to be determined later. First, we multiply the equation (1.3) by the test function $(|u(x, t)| - k)^+$. In the interior of $\{(x, t) : u(x, t) \geq k\}$, we have

$$(2.5) \quad \begin{aligned} & (u - k) \partial_t u + (u - k) \sum_{j=1}^d \partial_{x_j} F_j(u) \\ &= \frac{1}{2} \partial_t (u - k)^2 + \sum_{j=1}^d \partial_{x_j} Q_j(u) \\ &= \varepsilon (u - k) \sum_{j=1}^d (u^m)_{x_j x_j} \\ &= \varepsilon \sum_{j=1}^d \partial_{x_j} \left((u - k) (u^m)_{x_j} \right) - \varepsilon \sum_{j=1}^d u_{x_j} (u^m)_{x_j} \\ &= \varepsilon m \sum_{j=1}^d \partial_{x_j} \left((u - k) u^{m-1} u_{x_j} \right) - \varepsilon m \sum_{j=1}^d u^{m-1} (u_{x_j})^2, \end{aligned}$$

where $Q'_j = (u - k) F'_j(u)$.

Repeating the same calculus on the interior of $\{(x, t) : u(x, t) \leq -k\}$, we obtain

$$(2.6) \quad \begin{aligned} & \partial_t \eta_k(u) + \sum_{j=1}^d \partial_{x_j} Q_{j,k}(u) \\ &= \varepsilon m \sum_{j=1}^d \partial_{x_j} \left((|u| - k)^+ |u|^{m-1} u_{x_j} \right) - \varepsilon m \sum_{j=1}^d |u|^{m-1} (u_{x_j})^2 \end{aligned}$$

where

$$\begin{aligned} \eta_k(u) &= \frac{1}{2} \left[(|u| - k)^+ \right], \\ Q'_{j,k}(u) &= (|u| - k)^+ F'_j(u). \end{aligned}$$

Integrating on $\{(x, t) : |u(x, t)| \geq k\}$ and using Proposition 2.1, we get

$$(2.7) \quad \frac{d}{dt} \int_{\mathbb{R}^d} \eta_k(u(x, t)) dx = -\varepsilon m \sum_{j=1}^d \int_{\{|u| > k\}} |u|^{m-1} (u_{x_j})^2 dx.$$

An integration on t gives

$$(2.8) \quad \begin{aligned} & \int_{\mathbb{R}^d} \eta_k(u(x, t)) dx \\ &= \int_{\mathbb{R}^d} \eta_k(u_0(x)) dx - \varepsilon m \sum_{j=1}^d \int_0^t \int_{\{|u|>k\}} |u|^{m-1} (u_{x_j})^2 dx dt. \end{aligned}$$

Now, to show (2.2), we choose

$$k = \sup_x |u_0(x)|,$$

then we have

$$\eta_k(u(x, t)) \leq \eta_k(u_0(x)) \equiv 0, \text{ a.e. in } \mathbb{R}^d \times (0, T),$$

that is (2.2).

Finally, if we choose $k = 0$, we have $\eta_k(u) = u^2$ which shows (2.3) and (2.4). \square

3. Convergence

In preparation, recall that η - q with $q = (q_j(u))_{j=1, \dots, d}$ is an entropy-entropy flux pair if $q'_j = a_j \eta'$. Such pairs describe the nonlinear structure of (1.1) and are represented in terms of the kernel $\mathbb{1}(u, \xi)$ by the formulas

$$(3.1) \quad \begin{aligned} \eta(u) - \eta(0) &= \int_{\xi} \mathbb{1}(u, \xi) \eta'(\xi) d\xi, \\ q_j(u) - q_j(0) &= \int_{\xi} \mathbb{1}(u, \xi) a_j(\xi) \eta'(\xi) d\xi, \end{aligned}$$

where

$$(3.2) \quad \mathbb{1}_u(\xi) = \mathbb{1}(u, \xi) = \begin{cases} \mathbb{1}_{0 < \xi < u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u < \xi < 0} & \text{if } u < 0. \end{cases}$$

Remark 3.1. Let $\mathbb{1}(u, \xi)$ be the entropy kernel. Since $u^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ we have for K compact subset of $\mathbb{R}^d \times \mathbb{R}^+$

$$\int_K \left(\int_{\xi} |\mathbb{1}(u^\varepsilon, \xi)| d\xi \right)^2 dx dt = \int_K |u^\varepsilon|^2 dx dt \leq C$$

and thus $\mathbb{1}(u^\varepsilon, \xi) \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^+; L^1(\mathbb{R}))$.

Also, we use the limiting case of the averaging lemma proved in Perthame-Souganidis [18], see also [17] :

Theorem 3.2. *Let $\{f_n\}, \{g_{i,n}\}$ be two sequences of solutions to the transport equation*

$$(3.3) \quad \partial_t f_n + a(\xi) \cdot \nabla_x f_n = \partial_t \partial_\xi^k g_{0,n} + \sum_{i=1}^d \partial_{x_i} \partial_\xi^k g_{i,n}$$

where $k \in N$. Assume that $a(\xi) \in C^\infty(\mathbb{R})$ satisfies the non-degeneracy condition: for $R > 0$

$$(3.4) \quad \omega(\beta) = \sup_{\alpha \in \mathbb{R}, \omega \in S^{d-1}} \int_{\{|\xi| \leq R\}} \left(\left| \alpha + \frac{a(\xi) \cdot \omega}{\beta} \right|^2 + 1 \right)^{-1} d\xi \rightarrow 0, \quad \text{as } \beta \rightarrow 0.$$

If $\{f_n\}$ is bounded in $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for some $1 < q < \infty$, and $\{g_{i,n}\}$ is precompact in $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, then the average

$$\int_{\mathbb{R}} \psi(\xi) f_n(t, x, \xi) d\xi \quad \text{is precompact in } L^q(\mathbb{R}^d \times \mathbb{R}^+),$$

for any $\psi \in C_c^\infty(\mathbb{R})$.

Remark 3.3. 1. The non-degeneracy condition (3.4) is equivalent to for all $R > 0$

$$(3.5) \quad \text{meas}\{\xi \in B_R \mid \alpha + a(\xi) \cdot \omega = 0\} = 0, \quad \forall \alpha \in \mathbb{R}, \omega \in S^{d-1},$$

where $B_R = \{|\xi| \leq R\}$. The condition (3.5) can be interpreted geometrically, and means that the curve $\xi \mapsto a(\xi) \cdot \omega + \alpha$ is not locally contained in any hyperplane.

2. An assumption on the behavior of $a(\xi)$ is necessary; there would no improvement of regularity in the case $a(\xi) = \text{constant}$.

3. By using cut-off functions, it is easy to show a variant of theorem 3.2 stating that under the same hypotheses if $\{f_n\}$ is bounded in $L_{loc}^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and $\{g_{i,n}\}$ are precompact in $L_{loc}^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ then the averages are precompact in $L_{loc}^q(\mathbb{R}^d \times \mathbb{R}^+)$ for any $\psi \in C_c^\infty(\mathbb{R})$.

Now, we state the main theorem of this paper:

Theorem 3.4. *Suppose $u_0(x) \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and F_j are C^1 -functions that satisfy the nondegeneracy condition (3.5) (or (3.4)). Then solutions u^ε of (1.3) converge to a function u in $L_{loc}^p(\mathbb{R}^d \times \mathbb{R}^+)$, $1 < p < \infty$ and the limiting u is the unique Kruzhkov entropy solution of (1.1).*

Proof. We multiply (1.3) by $\eta'(u^\varepsilon)$ and obtain

$$(3.6) \quad \begin{aligned} & \partial_t \eta(u^\varepsilon) + \text{div} q(u^\varepsilon) \\ &= \varepsilon \sum_{j=1}^d \partial_{x_j} (\eta'(u^\varepsilon) m |u^\varepsilon|^{m-1} \partial_{x_j} u^\varepsilon) - \varepsilon \eta''(u^\varepsilon) \sum_{j=1}^d m |u^\varepsilon|^{m-1} (\partial_{x_j} u^\varepsilon)^2. \end{aligned}$$

Let $\varphi(x, t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and let $\eta \in C_c^\infty(\mathbb{R})$ be viewed as a test function. By introducing the indicator function $\mathbb{1}(u^\varepsilon, \xi)$, we have

$$\begin{aligned}
(3.7) \quad & - \int_{x,t,\xi} \left(\mathbb{1}(u^\varepsilon, \xi) \partial_t \varphi(x, t) + \sum_{j=1}^d F'_j(\xi) \mathbb{1}(u^\varepsilon, \xi) \partial_{x_j} \varphi(x, t) \right) \eta'(\xi) d\xi dx dt \\
& = - \int_{x,t} \sum_{j=1}^d \left(\varepsilon m |u^\varepsilon|^{m-1} \partial_{x_j} u^\varepsilon \right) \eta'(u^\varepsilon) \partial_{x_j} \varphi(x, t) dx dt \\
& \quad - \int_{x,t} \eta''(u^\varepsilon) \sum_{j=1}^d \left(\varepsilon m |u^\varepsilon|^{m-1} (\partial_{x_j} u^\varepsilon)^2 \right) \varphi(x, t) dx dt
\end{aligned}$$

which is viewed as describing the action on tensor products $\varphi \otimes \eta'$.

We proceed to interpret (3.7) as an equation in $\mathcal{D}'_{x,t,\xi}$. Let

$$\begin{aligned}
(3.8) \quad & \chi^\varepsilon(x, t, \xi) = \mathbb{1}(u^\varepsilon, \xi) \\
& H_j^\varepsilon(x, t) = \varepsilon m |u^\varepsilon|^{m-1} \partial_{x_j} u^\varepsilon \\
& G^\varepsilon(x, t) = \varepsilon m \sum_{j=1}^d |u^\varepsilon|^{m-1} (\partial_{x_j} u^\varepsilon)^2.
\end{aligned}$$

$H_j^\varepsilon, G^\varepsilon \in_b L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$ from lemma 2.2. We wish to define $\delta(u^\varepsilon - \xi)G^\varepsilon$ as a distribution in $\mathcal{D}'_{x,t,\xi}$ by its action on tensor products

$$(3.9) \quad \langle \delta(u^\varepsilon - \xi)G^\varepsilon, \varphi \otimes \eta' \rangle = \int_{x,t} G^\varepsilon(x, t) \varphi(x, t) \eta'(u^\varepsilon(x, t)) dx dt.$$

This follows from the Schwartz kernel theorem (e.g. [5, Sec 5.2]) as follows: Define the linear map

$$\mathcal{K} : C_c^\infty(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^+) \quad \text{by} \quad \mathcal{K}\psi = G^\varepsilon(x, t)\psi(u^\varepsilon(x, t)).$$

If $\psi_j \rightarrow 0$ in $C_c^\infty(\mathbb{R})$ then $\mathcal{K}\psi_j \rightarrow 0$ in $\mathcal{D}'_{x,t}$. The kernel theorem implies that $\delta(u^\varepsilon - \xi)G^\varepsilon$ is well defined as a distribution in $\mathcal{D}'_{x,t,\xi}$ and acts on tensor products via (3.9). Moreover,

$$(3.10) \quad \langle \partial_\xi \delta(u^\varepsilon - \xi)G^\varepsilon, \varphi \otimes \eta' \rangle = - \int_{x,t} G^\varepsilon(x, t) \varphi(x, t) \eta''(u^\varepsilon(x, t)) dx dt.$$

Thus (3.7) is written as

$$\begin{aligned}
& \langle \partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon, \eta'(\xi) \varphi(x, t) \rangle \\
& = \left\langle \sum_{j=1}^d \partial_{x_j} H_j^\varepsilon \delta(u^\varepsilon - \xi), \eta'(\xi) \varphi(x, t) \right\rangle + \langle \partial_\xi (\delta(u^\varepsilon - \xi)G^\varepsilon), \eta'(\xi) \varphi(x, t) \rangle.
\end{aligned}$$

Since the subspace generated by the direct sum test functions $\varphi \otimes \eta'$ is dense in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, the bracket (3) is extended to test functions $\theta(x, t, \xi) \in$

$C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. So, we have

$$\begin{aligned}
 \partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon &= \sum_{j=1}^d \partial_{x_j} \left(H_j^\varepsilon(x, t) \delta(u^\varepsilon - \xi) \right) + \partial_\xi \left(G^\varepsilon(x, t) \delta(u^\varepsilon - \xi) \right) \\
 (3.11) \qquad \qquad \qquad &=: \sum_{j=1}^d \partial_{x_j} \pi_j^\varepsilon + \partial_\xi k^\varepsilon \quad \text{in } D'_{x, t, \xi}.
 \end{aligned}$$

We estimate first the terms π_j^ε . Let $\theta(x, t, \xi) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. Using the estimates in lemma 2.2, we see that

$$\begin{aligned}
 &|\langle H_j^\varepsilon \delta(u^\varepsilon - \xi), \theta(x, t, \xi) \rangle| \\
 &= \left| \int_{x, t} (\varepsilon m |u^\varepsilon|^{m-1} u_{x_j}^\varepsilon) \theta(x, t, u^\varepsilon(x, t)) dx dt \right| \\
 &\leq \varepsilon^{1/2} \| |u^\varepsilon|^{\frac{m-1}{2}} \|_{L^\infty} \int_{x, t} \varepsilon^{1/2} m |u^\varepsilon|^{\frac{m-1}{2}} |\partial_{x_j} u^\varepsilon| |\theta(x, t, u^\varepsilon)| dx dt \\
 &\leq C \varepsilon^{1/2} \left(\|\varepsilon^{1/2} m |u^\varepsilon|^{\frac{m-1}{2}} \partial_{x_j} u^\varepsilon\|_{L^2_{x, t}} \right) \|\theta\|_{L^2_{x, t}(H_\xi^1)} \\
 &\leq C \varepsilon^{1/2} \|\theta\|_{L^2_{x, t}(H_\xi^1)}.
 \end{aligned}$$

Here we used the following:

$$\begin{aligned}
 \int_{x, t} \theta^2(x, t, u^\varepsilon) dx dt &= \int_{x, t} \int_{-\infty}^{u^\varepsilon(x, t)} 2\theta \theta_\xi d\xi dx dt \\
 (3.12) \qquad \qquad \qquad &\leq 2 \int_{x, t} \left(\int_{-\infty}^{u^\varepsilon} \theta^2 d\xi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{u^\varepsilon} (\partial_\xi \theta)^2 d\xi \right)^{\frac{1}{2}} dx dt \leq \|\theta\|_{L^2_{x, t}(H_\xi^1)}^2.
 \end{aligned}$$

This shows that $\pi_j^\varepsilon \rightarrow 0$ in $L^2_{x, t}(H_\xi^{-1})$ as $\varepsilon \rightarrow 0$, or in other words

$$\pi_j^\varepsilon = \bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon \quad \text{with } \bar{g}_j^\varepsilon, g_j^\varepsilon \rightarrow 0 \text{ in } L^2_{x, t, \xi}.$$

Next, consider the term $m^\varepsilon = G^\varepsilon \delta(u^\varepsilon - \xi)$. Observe that

$$\begin{aligned}
 |\langle m^\varepsilon, \theta \rangle| &= |\langle \delta(u^\varepsilon - \xi) G^\varepsilon, \theta \rangle| \\
 &\leq \sup_{x, t, \xi} |\theta(x, t, \xi)| \cdot \|\varepsilon m |u^\varepsilon|^{m-1} (\partial_{x_j} u^\varepsilon)^2\|_{L^1(\mathbb{R}^d \times \mathbb{R}^+)} \\
 &\leq C \|\theta\|_{C^0}
 \end{aligned}$$

and m^ε is positive bounded in measures $\mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. In summary, the function $\chi^\varepsilon = \mathbb{1}(u^\varepsilon, \xi)$ satisfies the (approximate) transport equation

$$(3.13) \quad \partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon = \sum_{j=1}^d \partial_{x_j} (\bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon) + \partial_\xi m^\varepsilon \quad \text{in } D'_{x, t, \xi},$$

where $\bar{g}_j^\varepsilon, g_j^\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and m^ε is positive bounded in measures $\mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and precompact in $W_{loc}^{-1, p}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \leq p < \frac{d+2}{d+1}$.

By the averaging lemma (Theorem 3.2),

$$\int_{\xi} \mathbb{1}(u^{\varepsilon}, \xi) \psi(\xi) d\xi \quad \text{is precompact in } L^p_{loc}, \quad 1 < p < \frac{d+2}{d+1}$$

for $\psi(\xi) \in C_c^\infty(\mathbb{R})$.

Let R be a large positive number and consider $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $(-R, R)$ and $0 \leq \psi \leq 1$. Then

$$\begin{aligned} \left| u^{\varepsilon} - \int_{\mathbb{R}} \mathbb{1}(u^{\varepsilon}, \xi) \psi(\xi) d\xi \right| &= \left| \int_{\mathbb{R}} \mathbb{1}(u^{\varepsilon}, \xi) (1 - \psi(\xi)) d\xi \right| \\ &\leq \int_R^{\infty} |\mathbb{1}(u^{\varepsilon}, \xi)| d\xi + \int_{-\infty}^{-R} |\mathbb{1}(u^{\varepsilon}, \xi)| d\xi \\ &= (u^{\varepsilon} - R)^+ + (u^{\varepsilon} + R)^-. \end{aligned}$$

Moreover,

$$\begin{aligned} \int (u^{\varepsilon} - R)^+ + (u^{\varepsilon} + R)^- dx dt &\leq \int_{|u^{\varepsilon}| > R} |u^{\varepsilon}| dx dt \\ &\leq \frac{1}{R} \int_0^T \int |u^{\varepsilon}|^2 dx dt \leq \frac{C}{R}. \end{aligned}$$

We conclude that $\{u^{\varepsilon}\}$ is Cauchy in $L^1_{loc, x, t}$.

Since $u^{\varepsilon} \in L^\infty_{x, t}$, it follows that (along subsequence) $u^{\varepsilon} \rightarrow u$ in L^p_{loc} , $p < \infty$, and almost everywhere and that $u \in L^\infty_{x, t}$. Next we pass to the limit $\varepsilon \rightarrow 0$ in (3.13). Along a further subsequence $m^{\varepsilon} \rightarrow m$ weak- \star in measure; it follows

$$(3.14) \quad \partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_{\xi} m \quad \text{in } D'_{x, t, \xi}.$$

So, the function $\chi = \mathbb{1}(u, \xi)$ satisfies the kinetic formulation of Lions-Perthame-Tadmor and thus u is the unique entropy solution of (1.1) (see [17]). \square

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