Commun. Korean Math. Soc. **22** (2007), No. 1, pp. 137–143

# EXPONENTIAL PROBABILITY INEQUALITY FOR LINEARLY NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, a Berstein-Hoeffding type inequality is established for linearly negative quadrant dependent random variables. A condition is given for almost sure convergence and the associated rate of convergence is specified.

## 1. Introduction

Lehmann [6] introduced a simple and natural definition of negative dependence: A sequence $\{X_i,\ 1 \le i \le n\}$ of random variables is said to be pairwise negative quadrant dependent (pairwise NQD) if for any real $x_i,\ x_j$ and $i \neq j$, $P(X_i > x_i,\ X_j > x_j) \le P(X_i > x_i)P(X_j > x_j)$. Much stronger concept than NQD was considered by Joag-Dev and Proschan [4]: A sequence $\{X_i,\ 1 \le i \le n\}$ is said to be negatively associated(NA) if for any disjoint subsets, $A,\ B \subset \{1,\ 2, \ldots, n\}$ and any real coordinatewise increasing functions $f$ on $\mathbb{R}^A$ and $g$ on $\mathbb{R}^B$, $\mathrm{Cov}(f(X_i,\ i \in A),\ g(X_i,\ i \in B)) \le 0$.

Instead of negative association, Newman [8] noticed that his method of proof yielding the central limit theorem for negatively associated sequence requires only that positive linear combinations of the random variables are NQD, i.e., the random variables are linearly negative quadrant dependent (LNQD). This notion of negative dependence was formulated by Newman [8] as follows: $\{X_n,\ n \in \mathbb{N}\}$ is a sequence of LNQD random variables if for any disjoint subsets $A,\ B$ of $\mathbb{N}$ and positive $r_i$, the random vector $(\sum_{i\in A} r_i X_i,\ \sum_{i\in B} r_i X_i)$ is NQD.

Negatively associated sequence are LNQD and LNQD sequences are not necessarily NA, as it can be seen from examples in Newman [8] or Joag-Dev [3].

Received September 22, 2006.

2000 *Mathematics Subject Classification.* 60F15.

*Key words and phrases.* exponential inequality, negatively associated, linearly negative quadrant dependent, almost sure convergence.

[1]This work was supported by the SRC/ERC program of MOST/KOSEF(R11-2000-073-00000).

[2]Supported by KOSEF-R01-2005-000-10696-0.

We note also that negative association and its weaker concepts are of considerable use in probability and statistics (cf. Joag-Dev and Proschan [4]; Newman [8] and the references there in).

Newmann [8] was first to establish a central limit theorem for LNQD random variables, Zhang (2000) proved a functional central limit theorem for LNQD random fields and Kim, Ko and Ryu (2004) derived a general central limit theorem for weighted sum of LNQD random variables.

Let $X_1, X_2, \ldots$ be random variables defined on the underlying probability space($\Omega$, $A$, $P$) and set $S_n$ for sum of the first $n$ random variables, $\sum_{i=1}^{n} X_i$, and $\overline{S}_n$ for $S_n/n$. The problem of providing exponential bounds for the probabilities $P(|S_n| \geq \epsilon)(\epsilon > 0)$ is of paramount importance, both in probability and statistics. From a statistical viewpoint, such inequalities can be used, among other things, for the purpose of providing rates of convergence (both in the probability sense and almost surely) for estimates of various quantities. Especially so in a nonparametric setting, where the advantages of structure are not available to the investigator. Exponential inequalities for various kinds of random variables were studied extensively. Some of these exponential inequalities were studied by Devroye [1], Roussas [9] and Shao [10].

In this paper, a Bernstein-Hoeffding-type inequality is established for linearly negative quadrant dependent random variables. A condition is also given for almost sure convergence, and the associated rate of convergence is considered.

## 2. Main results

For the formulation of the result to be made in this paper, the introduction of some notation is required. This notation is closely related to the way the proofs are carried out. Namely, for positive integers $1 \leq p = p(n) < n$ and $p \to \infty$, divide the set $\{1, 2, \ldots, n\}$ into successive groups each containing $p$ elements. Let $r = r(n)$ be the largest integer with:

$$0 < r < n, \quad r \to \infty, \quad \text{and} \quad 2pr \leq n, \tag{2.1}$$

which implies that $n/2pr \to 1$. Thus, the set $\{1, 2, \ldots, n\}$ is split into $2r$ groups, each consisting of $p$ elements; the remaining $n - 2pr < p$ elements constitute a set which may be empty.

The basic assumption under which the main result in this paper is obtained is that the random variables $X_i{}'s$ are mean zero LNQD and bounded, i.e., $|X_i| \leq M,\ i \geq 1$.

Define $\overline{S}_n$ and $\epsilon_n$ by

$$\overline{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \epsilon_n = \left(\frac{\alpha M^2}{2}\right)^{\frac{1}{2}} \left(\frac{\log n}{r}\right)^{\frac{1}{2}}, \tag{2.2}$$

where $M$ is as in assumption and $r$ is as in (2.1), and $\alpha$ is an arbitrary constant $> 1$ (see discussion just prior to relation (3.12)). Then the exponential inequality to be established is the following:

**Theorem 2.1.** *Let $\{X_i,\ i \geq 1\}$ be a sequence of LNQD random variables which are bounded, i.e., $|X_i| \leq M$ and $EX_i = 0$, and let $\overline{S}_n$ and $\epsilon_n$ be defined by (2.2). Then,*

$$P(|\overline{S}_n| \geq \epsilon_n) \leq 4\exp(-cr{\epsilon_n}^2), \quad c = \frac{2}{9M^2} \tag{2.3}$$

*for all sufficiently large $n$, $n \geq n_0$. Furthermore, $\overline{S}_n \to 0$ a.s. at the rate $1/\epsilon_n$.*

Finally we introduce an extension of Theorem 2 in Hoeffding [2] to the LNQD case.

**Theorem 2.2.** *Let $\{X_i,\ i \geq 1\}$ be a sequence of LNQD random variables such that $a_i \leq X \leq b_i$ and $EX_i = 0$ for all $i \geq 1$. Then, for every $t > 0$,*

$$P(|S_n| \geq t) \leq 2\exp[-2n^2t^2/(b_i - a_i)^2]. \tag{2.4}$$

## 3. Proof of the main results

**Lemma 3.1.** *Let $\{X_i,\ 1 \leq i \leq n\}$ be a sequence of LNQD random variables. Then*

$$E(\Pi_{i=1}^n \exp(X_i)) \leq \Pi_{i=1}^n E(\exp X_i). \tag{3.1}$$

*Proof.* Since $\{X_i,\ 1 \leq i \leq n\}$ is LNQD, $X_i$ and $X_{i+1} + \cdots + X_n$, $i = 1, 2, \ldots, n-1$, are NQD. Note that $\exp(X_i)$ and $\exp(X_{i+1} + \cdots + X_n)$ are also NQD by Lemma 2 of Matula [7]. Hence,

$$\begin{aligned} E(\Pi_{i=1}^n \exp(X_i)) &= E[\exp(X_1)\exp(X_2 + \cdots + X_n)] \\ &\leq E(\exp X_1)E(\Pi_{i=2}^n \exp(X_i)) \\ &\leq \Pi_{i=1}^n (E\exp(X_i)) \text{ by induction.} \end{aligned}$$

□

Set $S_n = \sum_{i=1}^n X_i$ and $\overline{S}_n = S_n/n$. With $p$ and $r$ as in the previous section, define the random variables $U_i$, $V_i$, $i = 1, \ldots, r$ and $W_n$ by

$$U_i = X_{2(i-1)p+1} + \cdots + X_{(2i-1)p}, \tag{3.2}$$

$$V_i = X_{(2i-1)p+1} + \cdots + X_{2ip}, \tag{3.3}$$

$$W_n = X_{2pr+1} + \cdots + X_n, \tag{3.4}$$

and

$$\overline{U}_n = \frac{1}{n}\sum_{i=1}^r U_i,\ \overline{V}_n = \frac{1}{n}\sum_{i=1}^r V_i,\ \overline{W}_n = \frac{W_n}{n}, \tag{3.5}$$

so that

$$\overline{S}_n = \overline{U}_n + \overline{V}_n + \overline{W_n}. \tag{3.6}$$

We may now formulate the next lemma.

**Lemma 3.2.** *Let $\overline{U}_n$ be defined by (3.5), let $\epsilon_n > 0$, and suppose $X_i'$s are LNQD random variables with $EX_i = 0$ and $|X_i| \le M$ for all $i \ge 1$. Then*

$$P(\overline{U}_n \ge \epsilon_n) \;\le\; \exp\left(\frac{-2r{\epsilon_n}^2}{M^2}\right).$$

*Proof.* Note that the random variables $U_1, \ldots, U_r$ are LNQD and $|U_i| \le pM$ for all $i$. For some $\lambda > 0$ we have

$$Ee^{\lambda \overline{U}_n} \le \Pi_{i=1}^r (Ee^{(\lambda/n)U_i}) \tag{3.7}$$

by Lemma 3.1. At this point, we apply Lemma 1 in Devroye [1], which states that, if $EX = 0$ and $a \le x \le b$, then for every $\lambda > 0$, $E\exp(\lambda X) \le \exp[\lambda^2(b-a)^2/8]$. Take $X = U_i$, so that $|U_i| \le pM$ and $b - a = 2pM$. Then we obtain

$$E\exp((\lambda/n)U_i) \le \exp(\lambda^2 p^2 M^2/2n^2),$$

and hence

$$\Pi_{i=1}^r Ee^{(\lambda/n)U_i} \le e^{\lambda^2 M^2 p^2 r/2n^2} \le e^{\lambda^2 M^2/8r}, \tag{3.8}$$

because

$$\frac{\lambda^2 M^2 p^2 r}{2n^2} = \frac{\lambda^2 M^2}{8r}\left(\frac{2pr}{n}\right)^2 \le \frac{\lambda^2 M^2}{8r} \text{ by (2.1).}$$

From (3.7) and (3.8) it follows that

$$Ee^{\lambda \overline{U}_n} \le e^{\lambda^2 M^2/8r}. \tag{3.9}$$

Therefore, for $\epsilon_n > 0$

$$P(\overline{U}_n \ge \epsilon_n) \le e^{-\lambda\epsilon_n + (\lambda^2 M^2/8r)}. \tag{3.10}$$

Minimizing, with respect to $\lambda$, the right-hand side in (3.10), we obtain

$$P(\overline{U}_n \ge \epsilon_n) \le e^{-2r{\epsilon_n}^2/M^2} \text{ for } \lambda_0 = 4r\epsilon_n/M^2. \tag{3.11}$$

This completes the proof of the lemma. □

For the almost sure convergence purpose, we wish to have $2r{\epsilon_n}^2/M^2 = \log n^\alpha$ (for any arbitrary $\alpha > 1$), or equivalently

$$\epsilon_n = \left(\frac{\alpha M^2}{2}\right)^{\frac{1}{2}} \left(\frac{\log n}{r}\right)^{\frac{1}{2}}. \tag{3.12}$$

For this choice of $\epsilon_n$, $\lambda_0$ becomes

$$\lambda_0 = \left(\frac{8\alpha}{M^2}\right)^{\frac{1}{2}} (r \log n)^{\frac{1}{2}}. \tag{3.13}$$

Summarizing these observations, we have

**Lemma 3.3.** *Under assumptions of Theorem 2.1 and with* $\epsilon_n$ *specified by* (3.12), *it holds*

$$P(\overline{U}_n \geq \epsilon_n) \ \leq \ n^{-\alpha}.$$

*Remark* 3.1. It is obvious that $\overline{V}_n$, as defined in (3.5) satisfies the same inequalities as $\overline{U}_n$ in Lemmas 3.2 and 3.3.

The following observation is meant to explain that we may dispense with $\overline{W}_n$ as defined in (3.5).

**Lemma 3.4.** *Under assumptions of Theorem 2.1 and with* $\epsilon_n$ *defined by* (3.12), $P(\overline{W}_n \geq \epsilon_n) = 0$ *for all sufficiently large* $n$.

*Proof.* $W_n$ consists of $n-2pr$ terms and $n-2pr < p$. Then $|\overline{W}_n| \leq pM/n$, so that $P(|\overline{W}_n| \geq \epsilon_n) \leq P(M > n\epsilon_n/p)$. The last expression, however, is 0, for all sufficiently large $n$, because

$$\frac{n\epsilon_n}{p} = \left(\frac{\alpha M^2}{2}\right)^{\frac{1}{2}} \left(\frac{n^2 \log n}{p^2 r}\right)^{\frac{1}{2}} = (2\alpha M^2)^{\frac{1}{2}} \left(\frac{n}{2pr}\right) (r \log n)^{\frac{1}{2}} \to \infty;$$

this is so because $n/2pr \to 1$. □

*Proof of Theorem 2.1.* It consists, essentially, in combining Lemma 3.2, Remark 3.1 and Lemma 3.4. The random variables $-X_i$, $i = 1, \ldots, n$, have the same properties as the random variables $X_i$, $i = 1, \ldots, n$. Thus, always

$$\begin{aligned} P(|\overline{U}_n| \geq \epsilon_n) &= P(\overline{U}_n \geq \epsilon_n) + P(-\overline{U}_n \geq \epsilon_n) \\ &\leq 2\exp(-2r{\epsilon_n}^2/M^2), \end{aligned}$$

and similarly for $P(|\overline{V}_n| \geq \epsilon_n)$. Therefore,

$$\begin{aligned} P(|\overline{S}_n| \geq 3\epsilon_n) &\leq P(|\overline{U}_n| \geq \epsilon_n) + P(|\overline{V}_n| \geq \epsilon_n) + P(|\overline{W}_n| \geq \epsilon_n) \\ &\leq P(|\overline{U}_n| \geq \epsilon_n|) + P(|\overline{V}_n| \geq \epsilon_n) \text{ (for every } n \geq n_0, \text{ say)} \\ &\leq 4\exp(-2r{\epsilon_n}^2/M^2). \end{aligned}$$

Replacing $\epsilon_n$ by $\frac{\epsilon_n}{3}$, we obtain, finally,

$$P(|\overline{S}_n \geq \epsilon_n|) \leq 4\exp(-cr{\epsilon_n}^2), \ c = \frac{2}{9M^2}, \ n \geq n_0.$$

□

*Remark* 3.2. The specification of $\epsilon_n$ by (3.12), leads to the convergence $\overline{S}_n \to 0$ a.s. at the rate of $1/\epsilon_n$.

Regarding the latter part of the theorem, proceed as follows. For the value of $\epsilon_n$ specified in (3.12), the rate of convergence is given by

$$\frac{1}{\epsilon_n} = \left(\frac{2}{\alpha M^2}\right)^{\frac{1}{2}} \left(\frac{r}{\log n}\right)^{\frac{1}{2}}. \tag{3.14}$$

*Proof of Theorem 2.2.* Note that

$$E\exp[\lambda X_i] \le \exp[\lambda^2(b_i - a_i)^2/8] \tag{3.15}$$

by Lemma 3.1 in Devroye [1] and

$$P(\overline{S_n} \ge t) \le \exp(-\lambda nt)E\prod_{i=1}^{n}\exp(\lambda X_i) \tag{3.16}$$

by Lemma 3.1. By (3.1), (3.15) and (3.16) we get

$$\begin{aligned} P(\overline{S_n} \ge t) \le\ & \exp(-\lambda nt)E\prod_{i=1}^{n}\exp(\lambda X_i) \\ \le\ & \exp(-\lambda nt)\prod_{i=1}^{n}E\exp(\lambda X_i) \\ \le\ & \exp\left[\frac{\lambda^2}{8}\sum_{i=1}^{n}(b_i - a_i)^2 - \lambda nt\right]. \end{aligned} \tag{3.17}$$

By minimizing (with respect to $\lambda$) the right hand side in (3.17), the desired result (2.4) follows. □

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