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SLANT SUBMANIFOLDS OF QUATERNION KAEHLER MANIFOLDS

Bayram Sahin

Abstract. This paper has two objectives. The first objective is to study slant submanifolds of quaternion Kaehler manifolds. We give characterization theorems and examples of slant submanifolds. For the second objective, we introduce the notion of semi-slant submanifolds which are different from the definition of N. Papaghiuc [15]. We obtain characterization theorems, examples of semi-slant submanifolds and investigate the geometry of leaves of distributions which are involved in the definition of semi-slant submanifolds.

1. Introduction

The geometry of a submanifold M of a quaternion Kaehler manifold $\bar{M}$ is mainly based on the action of the local basis $\{J_1, J_2, J_3\}$ on each tangent space to M. More precisely, a submanifold M of a quaternion Kaehler manifold $\bar{M}$ is called a quaternion submanifold(resp. totally real submanifold) if $J_a(TM) \subset TM$ (resp. $J_a(TM) \subset TM^{\perp}$), $a = 1, 2, 3$, where TM and $TM^{\perp}$ denote the tangent and normal bundle of M, respectively. A submanifold of $\bar{M}$ is called quaternion CR-submanifold (see, [1]) if there exist two orthogonal complementary distributions D and $D^{\perp}$ on M such that D is invariant under J_a, i.e., $J_a D \subseteq D$, $a = 1, 2, 3$ and $D^{\perp}$ is totally real, i.e. $J_a D^{\perp} \subseteq TM^{\perp}$, $a = 1, 2, 3$. It is clear that quaternion CR-submanifolds contain quaternion and totally real submanifold as subcases. On the other hand, we can easily observe that quaternion CR-submanifolds do not include real hypersurfaces of quaternion Kaehler manifolds. A submanifold M of a quaternion Kaehler manifold is called QR-submanifold ([4]) if there exists a vector subbundle of the normal bundle $TM^{\perp}$ of M such that we have $J_a(\upsilon_p) = \upsilon_p$ and $J_a(\upsilon_p^{\perp}) \subset T_pM$ for each $p \in M$ and $a = 1, 2, 3$, where $\upsilon^{\perp}$ is the complementary orthogonal subbundle to υ in $TM^{\perp}$. It is known that every real hypersurface of a quaternion Kaehler manifold is a QR-submanifold [4].

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Slant submanifolds of Kaehler manifolds were defined by B. Y. Chen [10] and then these submanifolds were studied by many authors in Kaehler Manifolds (See: [11] and its references). Moreover, slant submanifolds of contact manifolds were studied in [6], [7], [13] and [14]. Recently, slant submanifolds of $S-$ manifold were also studied in [8] and [9]. It is known that slant submanifolds are the generalization of invariant and totally real (anti-invariant) submanifolds in complex geometry as well as in contact geometry.

On the other hand, in [15], N. Papaghiuc has introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifold, such that these submanifolds include proper CR-submanifolds and proper slant submanifolds as particular classes. Thus, since every real hypersurface of a Kaehler manifold is a CR-submanifold, it follows that a real hypersurface is a semi-slant submanifold with the slant angle $\theta = \frac{\pi}{2}$. Semi-slant submanifolds have been also studied in Sasakian geometry in [5].

However, the concept of slant submanifolds of quaternion Kaehler manifolds has not been studied as yet, as far as we know. Therefore, in the present paper, we study slant submanifolds of quaternion Kaehler manifolds. We show that quaternion and totally real submanifolds are slant submanifolds with slant angle 0 and $\frac{\pi}{2}$. This result agrees with the theory of slant submanifolds of Kaehler manifolds. Then, we introduce semi-slant submanifolds of quaternion Kaehler manifolds. We note that if one consider the semi-slant submanifolds for quaternion Kaehler manifolds in sense of N. Papaghiuc, then it will be easy to see that definition does not contain real hypersurfaces. Therefore, we give a new definition for semi-slant submanifolds which contain real hypersurfaces (more generally, QR-submanifold) of a quaternion Kaehler manifold. Thus our present definition fulfills the main purpose for which semi-slant submanifolds of Kaehler manifolds were designed.

The organization of paper as follows: In section 2, we briefly present the basic information needed for this paper. In section 3, we give two characterization theorems and examples for slant submanifolds and find a necessary and sufficient condition for proper slant submanifolds to be quaternion slant. In section 4, we define semi-slant submanifolds and show that this class contains QR-submanifolds and quaternion submanifolds as subcases. We give characterizations and examples of semi-slant submanifolds and study the geometry of leaves of distributions.

2. Preliminaries

Let $\bar{M}$ be a $4m-$dimensional Riemannian manifold and g be Riemann metric on $\bar{M}$, $m \geq 1$. Then $\bar{M}$ is called quaternion Kaehler manifold [12] if there exist a 3-dimensional vector bundle V of tensors of type (1,1) with local basis Hermitian structures J_1,J_2 and J_3 (that is, $g(J_aX, J_aY) = g(X,Y)$, $a = 1,2,3$ and $X, Y \in \Gamma(T\bar{M})$) satisfying

$$J_1 o J_2 = -J_2 o J_1 = J_3 \tag{2.1}$$

and

$$\bar{\nabla}_X J_a = \sum_{b=1}^{3} Q_{ab}(X) J_b, a = 1, 2, 3 \tag{2.2}$$

for all vector fields X tangent to $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection and Q_{ab} are certain 1−forms locally defined on $\bar{M}$ such that $Q_{ab} + Q_{ba} = 0$. Let U and U' be arbitrary intersecting coordinate neighborhoods in $\bar{M}$ and let $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ be canonical local bases of E over U and U', respectively. Then J'_1, J'_2 and J'_3 are linear combinations of J_1, J_2 and J_3 in $U \cap U'$, that is,

$$J'_1 = S_{11}J_1 + S_{12}J_2 + S_{13}J_3$$
$$J'_2 = S_{21}J_1 + S_{22}J_2 + S_{23}J_3 \quad , \quad J'_3 = S_{31}J_1 + S_{32}J_2 + S_{33}J_3$$

with functions $S_{\beta\gamma}$, $(\beta, \gamma = 1, 2, 3)$ in $U \cap U'$. The coefficients $S_{\beta\gamma}$ appearing in the above equations form an element $S_{UU'} = (S_{\beta\gamma})$ of the proper orthogonal group $SO(3)$ of dimension 3 because both $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'\}$ satisfy (2.1).

Let $\bar{M}$ be a quaternion Kaehler manifold and M be any submanifold of $\bar{M}$. The formulae of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.4}$$

for vector fields X, Y tangent to M and any vector field V normal to M, where ∇ is the induced Riemann connection on M, h is the second fundamental form, $A_V$is the fundamental tensor field of Weingarten with respect to the normal section V and $\nabla^{\perp}$ is the normal connection. Moreover, we have the relation

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.5}$$

3. Slant submanifolds of quaternion Kaehler manifolds

In this section, we introduce the notion of slant submanifolds in quaternion Kaehler manifolds and give characterization theorems. We also give some examples of slant submanifolds.

Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then for any $X \in \Gamma(TM)$, we write

$$J_a X = T_a X + F_a X \tag{3.1}$$

where $T_a X$ is the tangential components of $J_a X$ and $F_a X$ is the normal component of $J_a X$. Similarly, for any $V \in \Gamma(TM^{\perp})$, we have

$$J_a V = B_a V + C_a V \tag{3.2}$$

where $B_a V$ denotes the tangential component and $C_a V$ denotes the normal component of $J_a V$.

Definition 3.1. Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we say that M is a slant submanifold if for each non-zero vector X tangent to M at x, the angle $\theta(X)$ between J_aX and T_xM, $(a = 1, 2, 3)$ is constant, i.e., it does not depend on choice of $x \in M$ and $X \in T_xM$.

First of all we have the following.

Proposition 3.1. *Quaternion submanifolds and totally real submanifolds are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.*

Proof. If M is an invariant submanifold of $\bar{M}$. Then $J_a(TM) = TM$, $a = 1, 2, 3$. This implies that $\theta = 0$. The other assertion can be verified in a similar way.

We say that a slant submanifold of a quaternion Kaehler manifold is proper if it is neither quaternion submanifold nor totally real.

The proposition 3.1 implies that there are several examples of slant submanifolds of quaternion Kaehler manifolds. Next we are going to give two examples of proper slant submanifolds. Before this, we recall the canonical quaternion Hermitian structure for $\mathbb{R}^{4m}$, $m \geq 1$.

Let $\mathbb{R}^{4m}$, $(m > 1)$ be a Euclidean space. Then, the canonical complex structures J_1, J_2, J_3 of $\mathbb{R}^{4m}$ and the Hermitian metric g are given by

$$J_1(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m) = (-y_1, x_1, -w_1, z_1, \ldots, -y_m, x_m, -w_m, z_m)$$
$$J_2(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m) = (-z_1, w_1, x_1, -y_1, \ldots, -z_m, w_m, x_m, -y_m)$$
$$J_3(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m) = (-w_1, -z_1, y_1, x_1, \ldots, -w_m, -z_m, y_m, x_m)$$

and

$$g((x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m), (u_1, v_1, t_1, s_1, \ldots, u_m, v_m, t_m, s_m))$$
$$= x_1u_1 + y_1v_1 + z_1t_1 + w_1s_1 + \cdots + x_mu_m + y_mv_m + z_mt_m + w_ms_m.$$

Example 3.1. For any $\theta \in (0, \frac{\pi}{2})$, consider a surface given by

$$\chi(u, v) = (u \cos \theta, v, v, v, u \sin \theta, 0, 0, 0).$$

Then TM is spanned by

$$Z_1 = \cos \theta \, \partial x_1 + \sin \theta \, \partial x_5, \; Z_2 = \partial x_2 + \partial x_3 + \partial x_4.$$

Then it is easy to see that M is a slant surface of $\mathbb{R}^8$ with respect to the canonical complex structures J_1, J_2 and J_3 with slant angle α such that $\cos \alpha = \frac{1}{\sqrt{3}} \cos \theta$.

Example 3.2. For any $k > 0$, consider in $\mathbb{R}^{12}$ the submanifold given by

$$\chi(t, s) = (t + 2s, t + s, t + s, t + s, k \cos t, k \sin t, 0, 0, k \cos s, k \sin s, 0, 0).$$

Then TM is spanned by

$$Z_1 = \partial x_1 + \partial x_2 + \partial x_3 + \partial x_4 - k \sin t \, \partial x_5 + k \cos t \, \partial x_6$$
$$Z_2 = 2 \partial x_1 + \partial x_2 + \partial x_3 + \partial x_4 - k \sin s \, \partial x_9 + k \cos s \, \partial x_{10}$$

Hence, it is easy to see that M is a slant submanifold of $\mathbb{R}^{12}$ with respect to J_1, J_2 and J_3 with slant angle $\theta = \cos^{-1} \frac{1}{\sqrt{1+k^2}\sqrt{7+k^2}}$.

□

Theorem 3.1. *Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$T_b T_a X = \lambda X, \quad a, b = 1, 2, 3. \tag{3.3}$$

Furthermore, in such case, if θ is the slant angle of M, it satisfies $\lambda = -\cos^2 \theta$.

Proof. For any $X \in \Gamma(TM)$, from (2.1) and (3.1) we have

$$\cos \theta(X) = \frac{g(J_b X, T_a X)}{|X||T_a X|} = -\frac{g(X, J_b T_a X)}{|X||T_a X|} = -\frac{g(X, T_b T_a X)}{|X||T_a X|}. \tag{3.4}$$

On the other hand, we have $\cos \theta(X) = \frac{|T_a X|}{|J_a X|}$, thus using (3.4) we obtain

$$\cos^2 \theta(X) = -\frac{g(X, T_b T_a X)}{|X|^2}. \tag{3.5}$$

Thus, if (3.3) is satisfied then we have $\cos^2 \theta(X) = -\lambda$. Conversely, if M is a slant submanifold, then $\theta(X)$ is constant. Then right side of (3.5) must be a constant. This implies that (3.3) is satisfied. □

Corollary 3.1. *Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then, for $X, Y \in \Gamma(TM)$, we have*

$$g(T_b X, T_a Y) = \cos^2 \theta g(X, Y) \tag{3.6}$$

$$g(F_b X, F_a Y) = g(X, T_c Y) - \cos^2 \theta g(X, Y) \tag{3.7}$$

and, in particular

$$g(F_a X, F_a Y) = \sin^2 \theta g(X, Y) \tag{3.8}$$

Proof. From (2.1) and (3.3), we obtain (3.6). With regards to equation (3.7), using (3.1) we have $g(F_b X, F_a Y) = g(J_b X, F_a Y)$. Hence $g(F_b X, F_a Y) = g(J_b X, J_a Y) - g(J_b X, T_a Y)$. Thus using (2.1) we obtain $g(F_b X, F_a Y) = g(X, J_c Y) + g(X, T_b T_a Y)$. Then from (3.4) we get (3.7): (3.8) is clear. □

Now, using Theorem 3.1 we obtain an another characterization for slant submanifolds of quaternion Kaehler manifolds.

Theorem 3.2. *Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is slant if and only if there exists a constant $\mu \in [0, 1]$ such that*

$$B_b F_a X = -T_c X + \mu X, \quad a, b, c = 1, 2, 3 \tag{3.9}$$

for $X \in \Gamma(TM)$.

Proof. Applying J_b to (3.1), using (3.1) and (3.2) we have

$$-J_cX = T_bT_aX + F_bT_aX + B_bF_aX + C_bF_aX.$$

Taking the tangential parts we get

$$-T_cX = T_bT_aX + B_bF_aX \tag{3.10}$$

for any X tangent to M. Now, if M is slant submanifold, from (3.3) we get $B_bF_aX = -T_cX + \cos^2\theta\, X$. Thus (3.9) is satisfied. Conversely, if (3.9) holds, then from (3.10) we obtain

$$-T_cX = T_bT_aX - T_cX + \mu\, X.$$

Hence $T_bT_aX = -\mu\, X$, $\mu \in [0,1]$. Put $-\mu = \lambda$ so that $\lambda \in [-1,0]$ Then proof follows from Theorem 3.1. □

From Theorem 3.1 and Theorem 3.2 we have the following results:

Corollary 3.2. *Let M be a slant submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we have*

$$T_a^2X = -\cos^2\theta\, X \tag{3.11}$$

$$B_a\,F_aX = -\sin^2\theta\, X \tag{3.12}$$

for any $X \in \Gamma(TM)$.

In the rest of this section, we will pay attention to proper slant submanifolds satisfying

$$(\nabla_X T_a)Y = Q_{ab}(X)T_bY + Q_{ac}(X)T_cY \tag{3.13}$$

for any $X, Y \in \Gamma(TM)$. In fact, one can thought that this kind submanifolds are the quaternion version of Kaehlerian slant submanifolds [11]. Therefore we say that a proper slant submanifold M is a quaternion slant submanifold if it satisfies the condition (3.13). Now, we will find a necessary and sufficient condition for this kind submanifolds.

Theorem 3.3. *Let M be a proper slant submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is a quaternion slant submanifold if and only if*

$$A_{F_aY}Z = A_{F_aZ}Y$$

for $Y, Z \in \Gamma(TM)$ and $a = 1, 2, 3$.

Proof. Using (2.2), (3.1), (3.2), (2.3), (2.4) and taking the tangential parts we obtain

$$(\nabla_X T_a)Y = A_{F_aY}X + B_ah(X,Y) + Q_{ab}T_bY + Q_{ac}(X)T_cY$$

for any $X, Y \in \Gamma(TM)$. Thus M is a quaternion slant if and only if $A_{F_aY}X + B_ah(X,Y) = 0$. On the other hand, for $X, Y, Z \in \Gamma(TM)$, we get

$$g(A_{F_aY}X + B_ah(X,Y), Z) = g(A_{F_aY}X, Z) + g(J_ah(X,Y), Z).$$

Thus using quaternion structures J_a and (3.2) we have

$$g(A_{F_aY}X + B_ah(X,Y), Z) = g(A_{F_aY}X, Z) - g(h(X,Y), F_aZ).$$

Then from (2.5) we obtain

$$g(A_{F_aY}X + B_ah(X,Y), Z) = g(A_{F_aY}X, Z) - g(A_{F_aZ}Y, X).$$

Since A_{F_aZ} is self-adjoint we get

$$g(A_{F_aY}X + B_ah(X,Y), Z) = g(A_{F_aY}Z - A_{F_aZ}Y, X).$$

Thus $A_{F_aY}X + B_ah(X,Y) = 0$ if and only if $A_{F_aY}Z = A_{F_aZ}Y$. This proves theorem. □

4. Semi-slant submanifolds of quaternion Kaehler manifolds

In this section we introduce semi-slant submanifolds of quaternion Kaehler manifolds and give two characterizations. We also investigate the geometry of leaves of distributions which are involved in the definition of semi-slant submanifolds.

Definition 4.1. Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we say that M is a semi-slant submanifold if there exist two orthogonal vector subbundles υ and $\upsilon^\perp$ of the normal bundle $TM^\perp$ such that

(1) $TM^\perp = \upsilon \oplus \upsilon^\perp$
(2) The vector subbundle $\upsilon^\perp$ is an anti-invariant with respect to J_a, $a = 1, 2, 3$.
(3) The vector subbundle υ is slant with respect to J_1, J_2, J_3 with slant angle $\theta \neq \frac{\pi}{2}$, i.e., for any non-zero vector $V \in \upsilon_x$, $x \in M$, the angle between J_aV, $a = 1, 2, 3$ and vector space υ_x is constant, that is , it is independent of the choice of $x \in M$ and $V \in \upsilon_x$.

In this case, we call the angle θ the slant angle of submanifold M. Now, we recall that a submanifold M in a quaternion Kaehler manifold is said to be an anti-quaternion submanifold ([2]) if

$$J_a(TM^\perp) \subset TM, \ a = 1, 2, 3.$$

It is clear that an anti-quaternion submanifold of a quaternion Kaehler manifold is a particular case of a QR-submanifold with $\upsilon = \{0\}$.

Proposition 4.1. *Let M be a QR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is a semi-slant submanifold of $\bar{M}$ with slant angle $\theta = 0$. Moreover, an anti-quaternion submanifold of $\bar{M}$ is a semi-slant submanifold with $\upsilon = \{0\}$. Furthermore, if $\dim(\upsilon^\perp) = 0$ and $\theta = 0$ then a semi-slant submanifold is a quaternion submanifold.*

Proof. Let M be a QR-submanifold of $\bar{M}$. Then we know that there exists a vector subbundle $\upsilon^\perp$ of the normal bundle $TM^\perp$ such that $J_a(\upsilon_x^\perp) \subset T_xM$ and $J_a(\upsilon_x) = \upsilon_x$, where υ is the orthogonal complementary to $\upsilon^\perp$ in $TM^\perp$.

Thus since υ is invariant with respect to J_1, J_2 and J_3, $\theta = 0$. Hence the conditions of Definition 4.1 are satisfied. So a QR-submanifold is a semi-slant submanifold with $\theta = 0$. Since an anti-quaternion submanifold M of $\bar{M}$ is a QR-submanifold, it follows that M is a semi-slant submanifold with $\upsilon = \{0\}$. Finally, if $\dim(\upsilon^{\perp}) = 0$, then $TM^{\perp} = \upsilon$. Moreover $\theta = 0$ implies that υ is invariant with respect to J_a, $a = 1, 2, 3$. Hence $TM^{\perp}$ is invariant. Thus TM is invariant and M is a quaternion submanifold. □

Remark 1. We note that our definition for semi-slant submanifolds is different from Papaghiuc's definition which was given for Kaehler manifolds [15]. If we considered Papaghiuc's definition for quaternion Kaehler manifolds, one can see that it does not include QR-submanifolds as a subcase.

We also note that real hypersurfaces of a quaternion Kaehler manifold $\bar{M}$ are QR-submanifolds [3]. Thus a real hypersurface, anti-quaternion submanifolds and QR-submanifolds are all examples of semi-slant submanifolds.

We say that M is a proper semi-slant submanifold if $\theta \neq 0, \frac{\pi}{2}$ and $n_1 n_2 \neq 0$, where $\dim(\upsilon) = n_1$ and $\dim(\upsilon^{\perp}) = n_2$.

Next we give an example of a proper semi-slant submanifold of a quaternion Kaehler manifold.

Example 4.1. Let M be a submanifold of $\mathbb{R}^{12}$ given by equations

$$x_1 = -u_1 - u_2, \qquad x_2 = u_1 - u_3$$
$$x_3 = u_2 + u_3, \qquad x_4 = u_1 - u_2 + u_3$$
$$x_5 = u_4 \cos\theta + u_8 \cos\theta + u_9 \sin\theta, \qquad x_6 = -u_4 \sin\theta + u_8 \cot\theta \cos\theta - u_9 \tan\theta \sin\theta$$
$$x_7 = u_6 \sin\theta + u_9 \sin\theta, \qquad x_8 = u_4 \operatorname{cosec}\theta - u_6 \cos\theta + u_9 \tan\theta \sin\theta$$
$$x_9 = u_5 \cos\theta + u_8 \cos\theta, \qquad x_{10} = u_5 \sin\theta - u_8 \cot\theta \cos\theta$$
$$x_{11} = u_7 \cos\theta, \qquad x_{12} = -u_7 \sin\theta$$

for $\theta \in (0, \frac{\pi}{2})$. Then the tangent bundle TM is spanned by

$$\begin{aligned}
Z_1 &= -\partial x_1 + \partial x_2 + \partial x_4 \\
Z_2 &= -\partial x_1 + \partial x_3 - \partial x_4 \\
Z_3 &= -\partial x_2 + \partial x_3 + \partial x_4 \\
Z_4 &= \cos\theta\, \partial x_5 - \sin\theta\, \partial x_6 + \operatorname{cosec}\theta\, \partial x_8 \\
Z_5 &= \cos\theta\, \partial x_9 + \sin\theta\, \partial x_{10} \\
Z_6 &= \sin\theta\, \partial x_7 - \cos\theta\, \partial x_8 \\
Z_7 &= \cos\theta\, \partial x_{11} - \sin\theta\, \partial x_{12} \\
Z_8 &= \cos\theta\, \partial x_5 + \cot\theta \cos\theta\, \partial x_6 + \cos\theta\, \partial x_9 - \cot\theta \cos\theta\, \partial x_{10} \\
Z_9 &= \sin\theta\, \partial x_5 - \tan\theta \sin\theta\, \partial x_6 + \sin\theta\, \partial x_7 + \tan\theta \sin\theta\, \partial x_8.
\end{aligned}$$

The normal bundle $TM^{\perp}$ is spanned by

$$\begin{aligned} V_1 &= \partial x_1 + \partial x_2 + \partial x_3 \\ V_2 &= \sin\theta\,\partial x_5 + \cos\theta\,\partial x_6 - \sin\theta\,\partial x_9 + \cos\theta\,\partial x_{10} \\ V_3 &= \cos\theta\,\partial x_5 - \sin\theta\,\partial x_6 - \cos\theta\,\partial x_7 - \sin\theta\,\partial x_8 \\ &\quad + \sin\theta\,\partial x_{11} + \cos\theta\,\partial x_{12}. \end{aligned}$$

Then it is easy to see that $J_1(V_1) = -Z_1$ $J_2(V_1) = -Z_2$ and $J_3(V_1) = -Z_3$, thus $v^{\perp} = \text{span}\{V_1\}$ is an anti-invariant distribution. On the other hand, we can easily obtain that $v = \text{span}\{V_2, V_3\}$ is a slant vector subspace with slant angle $\theta = \cos^{-1}(\frac{1}{\sqrt{6}})$. Thus M is a semi-slant submanifold of $\mathbb{R}^{12}$.

Now, we give the semi-slant version of Theorem 3.1.

Theorem 4.1. *Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is a semi-slant submanifold of $\bar{M}$ if and only if there exists a constant $\kappa \in [-1, 0]$ such that*

(a) *$v = \{V \in \Gamma(v) \mid C_b C_a = \kappa V\}$ is a vector subbundle of $TM^{\perp}$*

(b) *For any $W \in \Gamma(TM^{\perp})$, orthogonal to v, $C_a W = 0$,*

$a, b \in \{1, 2, 3\}$.

Proof. From (3.2) we have

$$\cos\theta(V) = \frac{g(J_b V, C_a V)}{|V||C_a V|} = -\frac{g(V, J_b C_a V)}{|V||C_a V|} = -\frac{g(V, C_b C_a V)}{|V||C_a V|}.$$

Then following the proof of Theorem 3.1 we get

$$\cos^2\theta(V) = -\frac{g(V, C_b C_a V)}{|V|^2}. \tag{4.1}$$

Thus, if M is a semi-slant submanifold, it is enough to put $\kappa = -\cos^2\theta$. This proves (a). (b) is clear. Conversely, we can consider the direct decomposition $TM^{\perp} = v \oplus v^{\perp}$. Since $C_a(v) \subseteq v$, from (b), it is clear that $J_a(v^{\perp}) \subset TM$. On the other hand, (4.1) and (a) imply that v is a slant vector subbundle with slant angle θ. □

From Theorem 4.1 we have the following.

Corollary 4.1. *Let M be a semi-slant submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we have*

$$g(C_b V, C_a W) = \cos^2\theta g(V, W) \tag{4.2}$$

$$g(B_b V, B_a W) = g(C_c V, W) - \cos^2 g(V, W) \tag{4.3}$$

and, in particular

$$g(B_a V, B_a W) = \sin^2\theta g(V, W) \tag{4.4}$$

for $V, W \in \Gamma(v^{\perp})$.

Theorem 4.2. *Let M be a submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is a semi-slant if and only if there exists a constant $\rho \in [0,1]$ such that*

(i) $\upsilon = \{V \in \Gamma(\upsilon) \mid F_b B_a V = -C_c V + \rho V\}$ *is a vector subbundle of* $TM^{\perp}$

(ii) *For any* $W \in \Gamma(TM^{\perp})$, *orthogonal to* υ, $C_a W = 0$.

Proof. Applying J_b to (3.1) we have

$$-J_c V = T_b B_a V + F_b B_a V + C_b C_a V + B_b C_a V.$$

Taking the normal parts we get

$$-C_c V = F_b T_a V + C_b C_a V \tag{4.5}$$

for any $V \in \Gamma(TM^{\perp})$. Now, if M is semi-slant submanifold, from (4.1) we get $F_b B_a X = -C_c V + \cos^2\theta\, V$. Thus (i) is satisfied. Conversely, if (i) holds, then from (4.5) we obtain

$$-C_c V = C_b C_a V - C_c V + \rho V.$$

Hence $C_b C_a V = -\rho V$, $\rho \in [0,1]$. Then proof follows from Theorem 4.1. □

From Theorem 3.1 and Theorem 3.2 we have the following results:

Corollary 4.2. *Let M be a semi- slant submanifold of quaternion Kaehler manifold $\bar{M}$. Then we have*

$$C_a^2 V = -\cos^2\theta\, V \tag{4.6}$$

$$F_a B_a V = -\sin^2\theta\, V \tag{4.7}$$

for any $V \in \Gamma(\upsilon)$.

Now, let M be a semi-slant submanifold of a quaternion Kaehler manifold $\bar{M}$. We denote $Da\,x = J_a(\upsilon_x^{\perp})$, $x \in M$ and remark that D_{1x}, D_{2x}, D_{3x} are mutually orthogonal vector subspace of $T_x M$. We consider

$$D_x^{\perp} = D_{1x} \oplus D_{2x} \oplus D_{3x},$$

then we obtain a $3n_2$ dimensional distribution $D^{\perp} : x \longrightarrow D_x^{\perp}$ globally defined on M. We also have

$$J_a(D_{ax}) = \upsilon_x^{\perp}, \quad J_a(D_{bx}) = D_{cx}$$

for each $x \in M$, where (a,b,c) is a cyclic permutation of $(1,2,3)$. We denote the orthogonal complementary distribution to $D^{\perp}$ in TM by D. Thus we get

$$TM = D \oplus D^{\perp}.$$

Now, we choose a local field of orthonormal frames $\{V_1, \ldots, V_{n_2}\}$ of $\upsilon^{\perp}$. Then we have the field of orthonormal frames of $D^{\perp}$

$$\{E_{11}, \ldots, E_{1n_2}, E_{21}, \ldots, E_{2n_2}, E_{31}, \ldots, E_{3n_2}\}$$

where $E_{ai} = J_a V_i$, $a = 1,2,3$ and $i = 1,2,\ldots,n_2$.

Let M be a semi-slant submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we denote by P, Q_1, Q_2 the projections on the distribution D, the vector subbundle v and the vector subbundle $v^{\perp}$, respectively. Thus, for any $V \in \Gamma(TM^{\perp})$ we can write

$$V = Q_1V + Q_2V. \tag{4.8}$$

Applying J_a to (4.8) we obtain

$$J_aV = J_aQ_1V + B_aQ_2V + C_aQ_2V. \tag{4.9}$$

Hence we have

$$J_aQ_1V \in \Gamma(D^{\perp}),\ B_aQ_2V \in \Gamma(D) \quad \text{and} \quad C_aQ_2V \in \Gamma(v). \tag{4.10}$$

Similarly, for $X \in \Gamma(TM)$, we write

$$X = PX + \sum_{i=1}^{n_2} \omega_{bi}(X)E_{bi}, b = 1,2,3, \tag{4.11}$$

where

$$\omega_{bi}(X) = g(X, E_{bi}). \tag{4.12}$$

Applying J_a to (4.11) and using (3.1) we obtain

$$J_aX = T_aPX + F_aPX + \sum_{i=1}^{n_2} \omega_{bi}(X)E_{ci} - \omega_{ci}(X)E_{bi} - \omega_{ai}(X)V_i. \tag{4.13}$$

In the rest of this section, we will investigate the conditions under which leaves of distributions on a semi-slant submanifold M in a quaternion Kaehler manifold $\bar{M}$ are totally geodesic immersed in M.

Theorem 4.3. *Let M be a semi-slant submanifold of a quaternion Kaehler manifold $\bar{M}$. The distribution D defines a totally geodesic foliation on M if and only if*

$$\nabla^{\perp}_X F_aY + h(X, T_aPY) \in \Gamma(v)$$

for $X, Y \in \Gamma(D)$.

Proof. Using (2.2), (4.13), (4.9), Gauss-Weingarten formulas and taking the normal parts we obtain

$$\begin{aligned} & h(X, T_aPY) + \nabla^{\perp}_X F_aPY + \omega_{bi}(Y)h(X, E_{ci}) - \omega_{ci}(Y)h(X, E_{bi}) \\ & -X(\omega_{ai}(Y))V_i - \omega_{ai}(Y)\nabla^{\perp}_X V_i - F_aP\nabla_XY + \omega_{ai}(\nabla_XY)V_i \\ & -C_aQ_1h(X,Y) \\ = \ & Q_{ab}(C)F_bPY - Q_{ab}(X)\omega_{bi}(Y)V_i + Q_{ac}(X)F_cPY - Q_{ac}(X)\omega_{ci}(Y)V_i \end{aligned}$$

for $X, Y \in \Gamma(TM)$. Thus for $X, Y \in \Gamma(D)$ we get

$$\begin{aligned} Q_{ab}(X)F_bPY + Q_{ac}(X)F_cPY \ = \ & h(X, TaPY) + \nabla^{\perp}_X F_aPY - FaP\nabla_XY \\ & +\omega_{ai}(\nabla_XY)V_i - C_aQ_2h(X,Y). \end{aligned}$$

Hence we obtain the assertion of theorem. □

For the distribution $D^{\perp}$ we have the following theorem.

Theorem 4.4. *Let M be a semi-slant submanifold of a quaternion Kaehler manifold $\bar{M}$. Then the distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if*

$$T_a P A_{V_i} X = B_a Q_2 \nabla^{\perp}_X V_i$$

for $X \in \Gamma(D^{\perp})$ and $V_i \in \Gamma(v^{\perp})$.

Proof. From Gauss formula we have

$$\nabla_X E_{ai} = \bar{\nabla}_X E_{ai} - h(X, E_{ai})$$

for $X, E_{ai} \in \Gamma(D^{\perp})$. Hence we get

$$\nabla_X E_{ai} = (\nabla_X J_a) V_i + J_a \bar{\nabla}_X V_i - h(X, E_{ai}).$$

Thus, using (2.3), (4.9), (4.13) and taking the tangential parts we obtain

$$\begin{aligned} \nabla_X E_{ai} &= Q_{ab}(X) E_{bi} + Q_{ac}(X) E_{ci} - T_a P A_{V_i} X \\ &\quad -\omega_{bi}(A_{V_i} X) E_{ci} + \omega_{ci}(A_{V_i} X) E_{bi} + J_a Q_1 \nabla^{\perp}_X V_i \\ &\quad +B_a Q_2 \nabla^{\perp}_X V_i. \end{aligned}$$

Hence $\nabla_X E_{ai} \in \Gamma(D^{\perp}) \Leftrightarrow T_a P A_{V_i} X = B_a Q_2 \nabla^{\perp}_X V_i$. □

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Department of Mathematics
Inonu University
Malatya 44280, Turkey
E-mail address: bsahin@inonu.edu.tr