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FOURIER-FEYNMAN TRANSFORMS FOR FUNCTIONALS IN A GENERALIZED FRESNEL CLASS

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ABSTRACT. Huffman, Park and Skoug introduced various results for the L_p analytic Fourier-Feynman transform and the convolution for functionals on classical Wiener space which belong to some Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick. Also Chang, Kim and Yoo extended the above results to an abstract Wiener space for functionals in the Fresnel class $\mathcal{F}(B)$ which corresponds to $\mathcal{S}$. Recently Kim, Song and Yoo investigated more generalized relationships between the Fourier-Feynman transform and the convolution product for functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2'}$ containing $\mathcal{F}(B)$.

In this paper, we establish various interesting relationships and expressions involving the first variation and one or two of the concepts of the Fourier-Feynman transform and the convolution product for functionals in $\mathcal{F}_{A_1,A_2}$.

1. Introduction

The concept of an L_1 analytic Fourier-Feynman transform for functionals on classical Wiener space $(C_0[0,T],m)$ was introduced by Brue in [3]. In [4], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [12], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ that extended the results in [3,4] and gave various relationships between the L_1 and L_2 theories. Also Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they obtained various results on the Fourier-Feynman transform and the convolution product [9, 10, 11]. In [17], Park, Skoug and Storvick investigated various relationships among the first variation, the convolution product and the Fourier-Feynman transform for functionals on classical Wiener space which belong to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [5]. For a detailed survey of previous work on Fourier-Feynman transform and related topics, see [18].

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The concept of abstract Wiener space (H, B, ν) was introduced by Gross in [8]. Ahn, Chang, Kim and Yoo [1, 7] obtained the relationships among the Fourier-Feynman transform, the convolution and the first variation for functionals in the Fresnel class $\mathcal{F}(B)$ which corresponds to the Banach algebra $\mathcal{S}$. Moreover they [6] introduced an L_p analytic Fourier-Feynman transform for functionals on a product abstract Wiener space and established the relationships between the Fourier-Feynman transform and the convolution for functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing $\mathcal{F}(B)$ introduced by Kallianpur and Bromley [13]. Recently Kim, Song and Yoo [15] investigate more generalized relationships between the Fourier-Feynman transform and the convolution product for functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$.

In this paper, we establish various interesting relationships and expressions involving the first variation and one or two of the concepts of the Fourier-Feynman transform and the convolution product for functionals in $\mathcal{F}_{A_1,A_2}$.

2. Definitions and preliminaries

Let (H, B, ν) be an abstract Wiener space [16] and let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*, the dual of B. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ as follows:

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. It is well known [13, 14] that for each $h(\neq 0)$ in H, $(h, \cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$, that is,

$$\int_B \exp\{i(h,x)^\sim\}\, d\nu(x) = \exp\Big\{-\frac{1}{2}|h|^2\Big\}. \tag{2.2}$$

A subset E of a product abstract Wiener space B^2 is said to be scale-invariant measurable provided $\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $(\nu \times \nu)(\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equals s-a.e., we write $F \approx G$.

Let $\mathbb{C}$ denote the complex numbers and let

$$\Omega = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \operatorname{Re}\lambda_k > 0 \text{ for } k = 1, 2\}$$

and

$$\tilde{\Omega} = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_k \neq 0, \operatorname{Re}\lambda_k \geq 0 \text{ for } k = 1, 2\}.$$

Let F be a complex-valued function on B^2 such that the integral

$$J_F(\lambda_1, \lambda_2) = \int_{B^2} F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)\, d(\nu \times \nu)(x_1, x_2)$$

exists as a finite number for all real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J_F^*(\lambda_1, \lambda_2)$ analytic on Ω such that $J_F^*(\lambda_1, \lambda_2) = J_F(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J_F^*(\lambda_1, \lambda_2)$ is defined to be the analytic Wiener integral of F over B^2 with parameter $\vec{\lambda} = (\lambda_1, \lambda_2)$, and for $\vec{\lambda} \in \Omega$ we write

$$\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2) = J_F^*(\lambda_1, \lambda_2).$$

Let q_1 and q_2 be nonzero real numbers and let F be a functional on B^2 such that $\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2)$ exists for all $\vec{\lambda} \in \Omega$. If the following limit exists, then we call it the analytic Feynman integral of F over B^2 with parameter $\vec{q} = (q_1, q_2)$ and we write

$$\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2)$$
$$= \lim_{\vec{\lambda} \to (-iq_1, -iq_2)} \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2),$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ approaches $(-iq_1, -iq_2)$ through Ω.

Notation 2.1. (i) For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$ and $(y_1, y_2) \in B^2$, let

$$T_{\vec{\lambda}}(F)(y_1, y_2) = \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1 + y_1, x_2 + y_2)\, d(\nu \times \nu)(x_1, x_2). \tag{2.3}$$

(ii) Let $1 < p < \infty$ and let $\{G_n\}$ and G be scale-invariant measurable functionals such that, for each $\alpha > 0$ and $\beta > 0$,

$$\lim_{n \to \infty} \int_{B^2} |G_n(\alpha x_1, \beta x_2) - G(\alpha x_1, \beta x_2)|^{p'}\, d(\nu \times \nu)(x_1, x_2) = 0, \tag{2.4}$$

where p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$. Then we write

$$\underset{n \to \infty}{\mathrm{l.\,i.\,m.}} (w_s^{p'})(G_n) \approx G \tag{2.5}$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter $\vec{\lambda}$.

Definition 2.2. Let q_1 and q_2 be nonzero real numbers. For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$ of F on B^2 by the formula ($\vec{\lambda} \in \Omega$)

$$T_{\vec{q}}^{(p)}(F)(y_1, y_2) = \underset{\vec{\lambda} \to (-iq_1, -iq_2)}{\mathrm{l.\,i.\,m.}} (w_s^{p'})(T_{\vec{\lambda}}(F))(y_1, y_2), \tag{2.6}$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_{\vec{q}}^{(1)}(F)$ of F by ($\vec{\lambda} \in \Omega$)

$$T_{\vec{q}}^{(1)}(F)(y_1, y_2) = \lim_{\vec{\lambda} \to (-iq_1, -iq_2)} T_{\vec{\lambda}}(F)(y_1, y_2), \tag{2.7}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Definition 2.3. Let F and G be functionals on B^2. For $\vec{q} = (q_1, q_2)$ with nonzero real numbers q_1 and q_2, we define their convolution product (if it exists) by

$$(2.8)\quad \begin{aligned}&(F * G)_{\vec{q}}(y_1, y_2)\\ &= \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F\Big(\frac{y_1 + x_1}{\sqrt{2}}, \frac{y_2 + x_2}{\sqrt{2}}\Big)\, G\Big(\frac{y_1 - x_1}{\sqrt{2}}, \frac{y_2 - x_2}{\sqrt{2}}\Big) d(\nu \times \nu)(x_1, x_2).\end{aligned}$$

Definition 2.4. Let F be a functional on B^2 and let $w_1, w_2 \in B$. Then

$$(2.9)\quad \delta F(x_1, x_2 | w_1, w_2) = \frac{\partial}{\partial t} F(x_1 + t w_1, x_2)\big|_{t=0} + \frac{\partial}{\partial t} F(x_1, x_2 + t w_2)\big|_{t=0}$$

(if it exists) is called the first variation of $F(x_1, x_2)$ in the direction (w_1, w_2).

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [2].

Now we state the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ introduced by Kallianpur and Bromley [13]. Let A_1 and A_2 be bounded, non-negative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}$ be the space of all s-equivalence classes of functionals F on B^2 which have the form

$$(2.10)\quad F(x_1, x_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2} h, x_j)^{\sim}\Big\}\, d\sigma(h)$$

for some complex-valued countably additive Borel measure σ on H.

As is customary, we will identify a functional with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on B^2 rather than as a collection of equivalence classes. Moreover the map $\sigma \mapsto [F]$ defined by (2.10) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$ [13].

Let $\mathcal{F}(B)$ denote the Fresnel class of functionals F on B of the form

$$(2.11)\quad F(x) = \int_H \exp\{i(h, x)^{\sim}\}\, d\sigma(h)$$

for some $\sigma \in M(H)$. If A_1 is the identity operator on H and $A_2 = 0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

Let $1 \le p < \infty$, $\vec{q} = (q_1, q_2)$ and $\vec{q_j} = (q_{j1}, q_{j2})$ where q_j, q_{j1} and q_{j2} are nonzero extended real numbers for $j = 1, 2$ throughout this paper. We adopt the convention $\frac{1}{\pm\infty} = 0$. Thus if $q_j = \pm\infty$ for $j = 1, 2$, then

$$(2.12)\quad T_{\vec{q}}^{(p)}(F)(y_1, y_2) = F(y_1, y_2)$$

and

$$(2.13)\quad (F * G)_{\vec{q}}(y_1, y_2) = F\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big) G\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}\Big).$$

We finish this section by introducing three results on the existence of the Fourier-Feynman transform and convolution product in [15] which play an important role in this paper.

Theorem 2.5. *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by (2.10). Then the analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}(F)$ exists, belongs to $\mathcal{F}_{A_1,A_2}$ and is given by the formula*

$$T_{\vec{q}}^{(p)}(F)(y_1,y_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^{\sim}\Big\}\, d\sigma_t(h) \tag{2.14}$$

for s-a.e. $(y_1,y_2) \in B^2$, where σ_t is the measure defined by

$$\sigma_t(E) = \int_E \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h) \tag{2.15}$$

for $E \in \mathcal{B}(H)$.

Theorem 2.6. *Let $F, G \in \mathcal{F}_{A_1,A_2}$ be given by (2.10) with corresponding finite Borel measures σ and ρ in $M(H)$. Then the convolution product $(F*G)_{\vec{q}}$ exists, belongs to $\mathcal{F}_{A_1,A_2}$ and is given by the formula*

$$(F*G)_{\vec{q}}(y_1,y_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^{\sim}\Big\}\, d\mu_c(h) \tag{2.16}$$

for s-a.e. $(y_1,y_2) \in B^2$, where $\mu_c = \mu \circ \phi^{-1}$ with $\phi : H^2 \to H$ is a function defined by $\phi(A_j^{1/2}h, A_j^{1/2}k) = \frac{1}{\sqrt{2}}A_j^{1/2}(h+k)$ and μ is the measure defined by

$$\mu(E) = \int_E \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k) \tag{2.17}$$

for $E \in \mathcal{B}(H^2)$.

Since $T_{\vec{q}_1}^{(p)}(F)$ and $T_{\vec{q}_2}^{(p)}(G)$ belong to $\mathcal{F}_{A_1,A_2}$ by Theorem 2.5, we apply Theorem 2.6 to obtain the following corollary.

Corollary 2.7. *Let F and G be given as in Theorem 2.6. Then*

$$(T_{\vec{q}_1}^{(p)}(F) * T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}(y_1,y_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^{\sim}\Big\}\, d\mu_{tc}(h) \tag{2.18}$$

for s-a.e. $(y_1,y_2) \in B^2$, where $\mu_{tc} = \mu_t \circ \phi^{-1}$ and ϕ is given as in Theorem 2.6 and μ_t is the measure defined by

$$\begin{aligned}\mu_t(E) = \int_E \exp\Big\{\sum_{j=1}^{2}\Big[&-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2 \\ &- \frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k)\end{aligned} \tag{2.19}$$

for $E \in \mathcal{B}(H^2)$.

3. The first variation of functionals in a generalized Fresnel class

In this section we establish various interesting relationships and expressions involving the first variation and one or two of the concepts of the Fourier-Feynman transform and the convolution product for functionals in $\mathcal{F}_{A_1,A_2}$.

We begin this section by showing the existence of the first variation for functionals in $\mathcal{F}_{A_1,A_2}$.

Theorem 3.1. *Let* $F \in \mathcal{F}_{A_1,A_2}$ *be given by* (2.10), *where* σ *satisfies the condition*

$$\int_H |A_j^{1/2}h|\, d|\sigma|(h) < \infty \tag{3.1}$$

for $j = 1, 2$. *Then, for s-a.e.* $(w_1, w_2) \in B^2$, *the first variation* δF *exists, belongs to* $\mathcal{F}_{A_1,A_2}$ *as a function of* (y_1, y_2) *and is given by the formula*

$$\delta F(y_1, y_2|w_1, w_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim\Big\}\, d\sigma_v(h) \tag{3.2}$$

for s-a.e. $(y_1, y_2) \in B^2$, *where* σ_v *is the measure defined by*

$$\sigma_v(E) = \int_E \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^\sim\Big]\, d\sigma(h) \tag{3.3}$$

for $E \in \mathcal{B}(H)$.

Proof. For s-a.e. $(w_1, w_2) \in B^2$, we have

$$\begin{aligned}
&\delta F(y_1, y_2|w_1, w_2)\\
&= \frac{\partial}{\partial t}\Big(\int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim + it(A_1^{1/2}h, w_1)^\sim\Big\}\, d\sigma(h)\Big)\Big|_{t=0}\\
&\quad + \frac{\partial}{\partial t}\Big(\int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim + it(A_2^{1/2}h, w_2)^\sim\Big\}\, d\sigma(h)\Big)\Big|_{t=0}
\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$. Then we obtain

$$\delta F(y_1, y_2|w_1, w_2) = \int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim\Big\}\, d\sigma(h)$$

provided we can interchange the differentiations and the integrations above. But this is justified because

$$\int_H |(A_j^{1/2}h, w_j)^\sim|\, d|\sigma|(h) < \infty$$

for s-a.e. $(w_1, w_2) \in B^2$ and for $j = 1, 2$, which we know easily from the fact that

$$\int_B \int_H |(A_j^{1/2} h, w_j)^\sim|\, d|\sigma|(h)\, d\nu(w_j) = \Big(\frac{2}{\pi}\Big)^{1/2} \int_H |A_j^{1/2} h|\, d|\sigma|(h)$$

which is finite by (3.1). Now it is easy to see that $\delta F(y_1, y_2 | w_1, w_2)$ can be rephrased as (3.2) where σ_v is the measure defined by (3.3). □

In view of Theorems 2.5, 2.6, 3.1 and Corollary 2.7, many of the functionals that occur in this paper are elements of $\mathcal{F}_{A_1,A_2}$ as a function of $(y_1, y_2) \in B^2$. For example, let F and G be any functionals in $\mathcal{F}_{A_1,A_2}$. Then,

(1) by Theorem 2.5, the functionals $T_{\vec{q}_1}^{(p)}(F)$ and $T_{\vec{q}_2}^{(p)}(F)$ belong to $\mathcal{F}_{A_1,A_2}$,
(2) by Theorem 2.6, the functional $(T_{\vec{q}_1}^{(p)}(F) * T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}$ belongs to $\mathcal{F}_{A_1,A_2}$,
(3) finally, by Theorem 3.1, the functional $\delta(T_{\vec{q}_1}^{(p)}(F) * T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}(y_1, y_2 | w_1, w_2)$ belongs to $\mathcal{F}_{A_1,A_2}$ as a function of (y_1, y_2).

By similar arguments, all of the functionals that arise in (3.4), (3.6), (3.7), (3.9), (3.10), (3.12) and (3.14) below belong to $\mathcal{F}_{A_1,A_2}$ as a function of (y_1, y_2).

Applying Theorem 3.1 to the functional in (2.14) we have the following result.

Corollary 3.2. *Let F be given as in Theorem 3.1. Then, for s-a.e. $(w_1, w_2) \in B^2$, the first variation $\delta T_{\vec{q}}^{(p)}(F)$ exists and is given by the formula*

$$\delta T_{\vec{q}}^{(p)}(F)(y_1, y_2 | w_1, w_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2} h, y_j)^\sim\Big\}\, d\sigma_{tv}(h) \tag{3.4}$$

for s-a.e. $(y_1, y_2) \in B^2$, where σ_{tv} is the measure defined by

$$\sigma_{tv}(E) = \int_E \Big[\sum_{j=1}^{2} i(A_j^{1/2} h, w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i}{2q_j}|A_j^{1/2} h|^2\Big]\Big\}\, d\sigma(h) \tag{3.5}$$

for $E \in \mathcal{B}(H)$.

Applying Corollary 3.2 repeatedly, we obtain the following result.

Corollary 3.3. *Let n be a natural number and let $F \in \mathcal{F}_{A_1,A_2}$ with*

$$\int_H |A_1^{1/2} h|^{n_1} |A_2^{1/2} h|^{n_2}\, d|\sigma|(h) < \infty,$$

for all nonnegative integers n_1 *and* n_2 *with* $n_1 + n_2 = 1, 2, \ldots, n$. *Then, for s-a.e.* $(w_{l1}, w_{l2}) \in B^2$, $l = 1, \ldots, n$, *we have*

$$\begin{aligned}
&\delta^n T_{\vec{q}}^{(p)}(F)(\cdot,\cdot|w_{11},w_{12})\cdots(\cdot,\cdot|w_{(n-1)1},w_{(n-1)2})(y_1,y_2|w_{n1},w_{n2}) \\
&= \int_H \prod_{l=1}^{n} \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_{lj})^{\sim}\Big] \\
&\quad \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}h, y_j)^{\sim} - \frac{i}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h)
\end{aligned} \tag{3.6}$$

for s-a.e. $(y_1, y_2) \in B^2$.

From now on, we show various interesting relationships and expressions involving the concepts of the Fourier-Feynman transform and the first variation.

In our next two theorems, we show that the Fourier-Feynman transforms of $\delta T_{\vec{q}_2}^{(p)}(F)$ with respect to the first and second argument of variation equal to the variation of the Fourier-Feynman transforms.

Theorem 3.4. *Let* F *be given as in Theorem 3.1. Then, for s-a.e.* $(w_1, w_2) \in B^2$

$$T_{\vec{q}_1}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(F)(\cdot,\cdot|w_1,w_2)](y_1,y_2) = \delta T_{\vec{q}}^{(p)}(F)(y_1,y_2|w_1,w_2) \tag{3.7}$$

for $(y_1, y_2) \in B^2$, *where* $\vec{q} = (q_1, q_2)$ *with* $1/q_j = 1/q_{j1} + 1/q_{j2}$ *for* $j = 1, 2$.

Proof. Since $\delta T_{\vec{q}_2}^{(p)}(F)(y_1,y_2|w_1,w_2)$ belongs to $\mathcal{F}_{A_1,A_2}$ as a function of (y_1, y_2), we apply Theorem 2.5 to the expression (3.4) to obtain

$$\begin{aligned}
&T_{\vec{q}_1}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(F)(\cdot,\cdot|w_1,w_2)](y_1,y_2) \\
&= \int_H \exp\Big\{\sum_{j=1}^{n}\Big[i(A_j^{1/2}h, y_j)^{\sim} - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma_{tv}(h).
\end{aligned}$$

Using the definition of the measure σ_{tv} in (3.5) and the relationship between q_{j1}, q_{j2} and q_j, we see that the last expression is equal to

$$\int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^{\sim}\Big] \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}h, y_j)^{\sim} - \frac{i}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h)$$

which is equal to the right hand side of (3.7), by Corollary 3.2, and this completes the proof. □

Theorem 3.5. *Let* F *be given as in Theorem 3.1. Then, for s-a.e.* $(w_1, w_2) \in B^2$

$$T_{\vec{q}_1}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(F)(y_1,y_2|\cdot,\cdot)](w_1,w_2) = \delta T_{\vec{q}_2}^{(p)}(F)(y_1,y_2|w_1,w_2) \tag{3.8}$$

for $(y_1, y_2) \in B^2$.

Proof. Using (3.4) we have, for all $\vec{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$ and s-a.e. $(y_1, y_2) \in B^2$,

$$\begin{aligned}
&T_{\vec{\lambda}}[\delta T_{\vec{q}_2}^{(p)}(F)(y_1, y_2|\cdot,\cdot)](w_1, w_2) \\
&= \int_{B^2} \delta T_{\vec{q}_2}^{(p)}(F)(y_1, y_2|\lambda^{-1/2}x_1 + w_1, \lambda^{-1/2}x_2 + w_2)\, d(\nu \times \nu)(x_1, x_2) \\
&= \int_{B^2}\int_H \Big[\sum_{j=1}^2 i(A_j^{1/2}h, \lambda^{-1/2}x_j + w_j)^\sim\Big] \\
&\quad \exp\Big\{\sum_{j=1}^2 \Big[i(A_j^{1/2}h, y_j)^\sim - \frac{i}{2q_{2j}}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h)\, d(\nu\times\nu)(x_1, x_2).
\end{aligned}$$

Fubini theorem together with the fact that $\int_B (A_j^{1/2}h, x_j)^\sim\, d\nu(x_j) = 0$ for $j = 1, 2$ enable us to conclude that

$$\begin{aligned}
&T_{\vec{\lambda}}[\delta T_{\vec{q}_2}^{(p)}(F)(y_1, y_2|\cdot,\cdot)](w_1, w_2) \\
&= \int_H \Big[\sum_{j=1}^2 i(A_j^{1/2}h, w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^2 \Big[i(A_j^{1/2}h, y_j)^\sim - \frac{i}{2q_{2j}}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h).
\end{aligned}$$

But the right hand side of the last expression is independent of the vector $\vec{\lambda}$ and so $T_{\vec{q}_1}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(F)(y_1, y_2|\cdot,\cdot)](w_1, w_2)$ is given by the same expression. Finally by Corollary 3.2 we obtain (3.8). □

Using Corollary 2.7 and Theorem 3.1, we obtain an expression for the first variation of the convolution product as follows.

Theorem 3.6. *Let $F, G \in \mathcal{F}_{A_1,A_2}$ with corresponding finite Borel measures σ and ρ respectively, where σ and ρ satisfy the conditions*

$$\int_H |A_j^{1/2}h|\, d|\sigma|(h) < \infty, \quad \int_H |A_j^{1/2}k|\, d|\rho|(k) < \infty$$

for $j = 1, 2$. Then, for s-a.e. $(w_1, w_2) \in B^2$

$$\begin{aligned}
&\delta(T_{\vec{q}_1}^{(p)}(F) * T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2) \\
&= \int_{H^2} \Big[\sum_{j=1}^2 \frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^2 \Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^\sim \\
&\quad - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2 - \frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k)
\end{aligned} \tag{3.9}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Let μ_{tc} be the measure given as in Corollary 2.7. Then

$$\begin{aligned}\int_H |A_j^{1/2}h|\, d|\mu_{tc}|(h) &= \int_{H^2} \frac{1}{\sqrt{2}} |A_j^{1/2}(h+k)|\, d|\sigma|(h)\, d|\rho|(k) \\ &\leq \frac{1}{\sqrt{2}}\Big[\|\rho\| \int_H |A_j^{1/2}h|\, d|\sigma|(h) + \|\sigma\| \int_H |A_j^{1/2}k|\, d|\rho|(k)\Big]\end{aligned}$$

which is finite. Hence we apply Theorem 3.1 to the expression (2.18) to obtain

$$\begin{aligned}&\delta(T_{\vec{q}_1}^{(p)}(F) * T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2) \\ &= \int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^{\sim}\Big] \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^{\sim}\Big\}\, d\mu_{tc}(h).\end{aligned}$$

Finally by the definition of μ_{tc} in Corollary 2.7, we have the result. □

In our next two theorems, we obtain expressions for the convolution product with respect to the first and second argument of the first variation.

Theorem 3.7. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1, w_2) \in B^2$*

$$\begin{aligned}&(\delta T_{\vec{q}_1}^{(p)}(F)(\cdot,\cdot|w_1, w_2) * \delta T_{\vec{q}_2}^{(p)}(G)(\cdot,\cdot|w_1, w_2))_{\vec{q}}(y_1, y_2) \\ &= \int_{H^2} \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^{\sim}\Big]\Big[\sum_{j=1}^{2} i(A_j^{1/2}k, w_j)^{\sim}\Big] \\ &\quad \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2 \\ &\quad - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2 - \frac{i}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k)\end{aligned} \tag{3.10}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Applying Theorem 2.6 to the expressions (3.4) for F and G, we see that the left hand side of (3.10) is given by

$$\int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^{\sim}\Big\}\, d\mu_c(h)$$

where $\mu_c = \mu \circ \phi^{-1}$ and μ is given by (2.17) with the measure σ and ρ are replaced by σ_{tv} and ρ_{tv} defined in (3.5). Hence we have the result. □

Theorem 3.8. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1, w_2) \in B^2$*

$$
\begin{aligned}
&(\delta T_{\vec{q}_1}^{(p)}(F)(y_1, y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1, y_2|\cdot,\cdot))_{\vec{q}}(w_1, w_2) \\
&= \int_{H^2} \Big\{ \Big[\sum_{j=1}^{2} \frac{i}{\sqrt{2}}(A_j^{1/2}h, w_j)^\sim\Big] \Big[\sum_{j=1}^{2} \frac{i}{\sqrt{2}}(A_j^{1/2}k, w_j)^\sim\Big] \\
&\quad + \sum_{j=1}^{2} \frac{i}{2q_j}\langle A_j^{1/2}h, A_j^{1/2}k\rangle \Big\} \exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}(h+k), y_j)^\sim \\
&\quad - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k)
\end{aligned} \tag{3.11}
$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. For all $\vec{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$ and s-a.e. (w_1, w_2) and (y_1, y_2), we have

$$
\begin{aligned}
&(\delta T_{\vec{q}_1}^{(p)}(F)(y_1, y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1, y_2|\cdot,\cdot))_{\lambda}(w_1, w_2) \\
&= \int_{B^2} \delta T_{\vec{q}_1}^{(p)}(F)\Big(y_1, y_2\Big|\frac{w_1 + \lambda_1^{-1/2}x_1}{\sqrt{2}}, \frac{w_2 + \lambda_2^{-1/2}x_2}{\sqrt{2}}\Big) \\
&\delta T_{\vec{q}_2}^{(p)}(G)\Big(y_1, y_2\Big|\frac{w_1 - \lambda_1^{-1/2}x_1}{\sqrt{2}}, \frac{w_2 - \lambda_2^{-1/2}x_2}{\sqrt{2}}\Big)\, d(\nu \times \nu)(x_1, x_2).
\end{aligned}
$$

But using the expressions (3.4) for F and G, we see that the last expression is equal to

$$
\begin{aligned}
&\int_{B^2}\int_{H^2} \Big[\sum_{j=1}^{2} i\Big(A_j^{1/2}h, \frac{w_j + \lambda_j^{-1/2}x_j}{\sqrt{2}}\Big)^\sim\Big] \Big[\sum_{j=1}^{2} i\Big(A_j^{1/2}k, \frac{w_j - \lambda_j^{-1/2}x_j}{\sqrt{2}}\Big)^\sim\Big] \\
&\exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}(h+k), y_j)^\sim - \frac{i}{2q_{1j}}|A_j^{1/2}h|^2 - \frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\} \\
&d\sigma(h)\, d\rho(k)\, d(\nu \times \nu)(x_1, x_2).
\end{aligned}
$$

Fubini theorem together with the facts that $\int_B (A_j^{1/2}h, x_j)^\sim\, d\nu(x_j) = 0$ and

$$
\int_B (A_j^{1/2}h, x_j)^\sim (A_j^{1/2}k, x_j)^\sim\, d\nu(x_j) = \langle A_j^{1/2}h, A_j^{1/2}k\rangle
$$

enable us to conclude that

$$\begin{aligned}&(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot)*\delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot))_\lambda(w_1,w_2)\\&=\int_{H^2}\Big\{\Big[\sum_{j=1}^2\frac{i}{\sqrt{2}}(A_j^{1/2}h,w_j)^\sim\Big]\Big[\sum_{j=1}^2\frac{i}{\sqrt{2}}(A_j^{1/2}k,w_j)^\sim\Big]\\&+\sum_{j=1}^2\frac{1}{2\lambda_j}\langle A_j^{1/2}h,A_j^{1/2}k\rangle\Big\}\exp\Big\{\sum_{j=1}^2\Big[i(A_j^{1/2}(h+k),y_j)^\sim\\&-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2-\frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k).\end{aligned}$$

The last expression is an analytic function of $\vec{\lambda}\in\Omega$. Hence letting $\vec{\lambda}\to(-iq_1,-iq_2)$ we have the result. □

In our next two theorems, we obtain relationships involving the Fourier-Feynman transform of $\delta(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}$ with respect to the first and second argument of the first variation.

Theorem 3.9. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1,w_2)\in B^2$ and $(y_1,y_2)\in B^2$*

$$\begin{aligned}&T_{\vec{q}}^{(p)}[\delta(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}(\cdot,\cdot|w_1,w_2)](y_1,y_2)\\&=\delta T_{\vec{q}}^{(p)}[(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}](y_1,y_2|w_1,w_2)\\&=\delta T_{\vec{q}_1'}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big|\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big)T_{\vec{q}_2'}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\\&+T_{\vec{q}_1'}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\delta T_{\vec{q}_2'}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big|\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big)\end{aligned}\tag{3.12}$$

where $\vec{q_j'}=(q_{j1}',q_{j2}')$ with q_{jl}' is a nonzero extended real number such that $1/q_l+1/q_{jl}=1/q_{jl}'$ for $j,l=1,2$.

Proof. The first equality is obtained by Theorem 3.4. The second expression in (3.12) is equal to

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(T_{\vec{q}}^{(p)}[(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}](y_1+tw_1,y_2)\Big)\Big|_{t=0}\\&+\frac{\partial}{\partial t}\Big(T_{\vec{q}}^{(p)}[(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}](y_1,y_2+tw_2)\Big)\Big|_{t=0}.\end{aligned}$$

But by Theorem 3.13 of [15]

$$\begin{aligned}&T_{\vec{q}}^{(p)}[(T_{\vec{q}_1}^{(p)}(F)*T_{\vec{q}_2}^{(p)}(G))_{\vec{q}}](y_1+tw_1,y_2)\\&=T_{\vec{q}_1'}^{(p)}(F)\Big(\frac{y_1+tw_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)T_{\vec{q}_2'}^{(p)}(G)\Big(\frac{y_1+tw_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\end{aligned}$$

and

$$\begin{aligned}&T_{\vec{q}}^{(p)}[(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}](y_1, y_2 + tw_2)\\ =&T_{\vec{q_1}}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2 + tw_2}{\sqrt{2}}\Big)T_{\vec{q_2}}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}}, \frac{y_2 + tw_2}{\sqrt{2}}\Big).\end{aligned}$$

Hence we obtain the second equality in (3.12). □

Theorem 3.10. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1, w_2) \in B^2$ and $(y_1, y_2) \in B^2$*

$$\begin{aligned}&T_{\vec{q}}^{(p)}[\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|\cdot, \cdot)](w_1, w_2)\\ &= \delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2).\end{aligned} \tag{3.13}$$

Proof. Since $\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2)$ does not belong to $\mathcal{F}_{A_1,A_2}$ as a function of (w_1, w_2), we can not apply Theorem 3.5 in this case. But note that by Corollary 2.7 and Theorem 3.1

$$\begin{aligned}&\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2)\\ &= \int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim\Big\}\, d\mu_{tc}(h)\end{aligned}$$

where μ_{tc} is the measure given as in Corollary 2.7. Now for all $\vec{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$, we have

$$\begin{aligned}&T_\lambda[\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|\cdot, \cdot)](w_1, w_2)\\ &= \int_{B^2}\int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, \lambda^{-1/2}x_j + w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim\Big\}\\ &\quad d\mu_{tc}(h)\, d(\nu \times \nu)(x_1, x_2).\end{aligned}$$

Fubini theorem together with the facts that $\int_B (A_j^{1/2}h, x_j)^\sim\, d\nu(x_j) = 0$ for $j = 1, 2$ enable us to conclude that

$$\begin{aligned}&T_\lambda[\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|\cdot, \cdot)](w_1, w_2)\\ &= \int_H \Big[\sum_{j=1}^{2} i(A_j^{1/2}h, w_j)^\sim\Big] \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, y_j)^\sim\Big\}\, d\mu_{tc}(h)\end{aligned}$$

which is equal to the expression for $\delta(T_{\vec{q_1}}^{(p)}(F) * T_{\vec{q_2}}^{(p)}(G))_{\vec{q}}(y_1, y_2|w_1, w_2)$ given in the first part of this proof. Hence we obtain (3.13). □

In the following theorem, we obtain relationships involving the Fourier-Feynman transform of $(\delta T_{\vec{q_1}}^{(p)}(F)(\cdot, \cdot|w_1, w_2) * \delta T_{\vec{q_2}}^{(p)}(G)(\cdot, \cdot|w_1, w_2))_{\vec{q}}$ with respect to (y_1, y_2).

Theorem 3.11. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1, w_2) \in B^2$ and $(y_1, y_2) \in B^2$*

$$\begin{aligned}
&T_{\vec{q}}^{(p)}(\delta T_{\vec{q}_1}^{(p)}(F)(\cdot,\cdot|w_1,w_2) * \delta T_{\vec{q}_2}^{(p)}(G)(\cdot,\cdot|w_1,w_2))_{\vec{q}}(y_1,y_2)\\
&= T_{\vec{q}}^{(p)}[\delta T_{\vec{q}_1}^{(p)}(F)(\cdot,\cdot|w_1,w_2)]\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\\
&\quad T_{\vec{q}}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(G)(\cdot,\cdot|w_1,w_2)]\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\\
&= \delta T_{\vec{q_1'}}^{(p)}(F)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big|w_1,w_2\Big)\delta T_{\vec{q_2'}}^{(p)}(G)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big|w_1,w_2\Big)
\end{aligned} \tag{3.14}$$

where $\vec{q_j'} = (q'_{j1}, q'_{j2})$ with q'_{jl} is a nonzero extended real number such that $1/q_l + 1/q_{jl} = 1/q'_{jl}$ for $j, l = 1, 2$.

Proof. The first equality is obtained by Corollary 3.9 of [15] and the second equality is obtained by Theorem 3.4. □

In the following theorem, we obtain relationships involving the Fourier-Feynman transform of $(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot))_{\vec{q}}$ with respect to (w_1, w_2).

Theorem 3.12. *Let F and G be given as in Theorem 3.6. Then, for s-a.e. $(w_1, w_2) \in B^2$ and $(y_1, y_2) \in B^2$*

$$\begin{aligned}
&T_{\vec{q}}^{(p)}(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot))_{\vec{q}}(w_1,w_2)\\
&= \delta T_{\vec{q}_1}^{(p)}(F)\Big(y_1,y_2\Big|\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big)\delta T_{\vec{q}_2}^{(p)}(G)\Big(y_1,y_2\Big|\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big)\\
&= T_{\vec{q}}^{(p)}[\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot)]\Big(\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big)T_{\vec{q}}^{(p)}[\delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot)]\Big(\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}}\Big).
\end{aligned} \tag{3.15}$$

Proof. The second equality is obtained by Theorem 3.5. To prove the first equality let $\vec{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$. Then for s-a.e. (w_1, w_2) and (y_1, y_2), we have

$$\begin{aligned}
&T_{\lambda}(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot))_{\vec{q}}(w_1,w_2)\\
&= \int_{B^2}(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot) * \delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot)_{\vec{q}}\\
&\quad(\lambda_1^{-1/2}x_1 + w_1, \lambda_2^{-1/2}x_2 + w_2)\, d(\nu\times\nu)(x_1,x_2).
\end{aligned}$$

By Theorem 3.8 the last expression is equal to

$$\int_{B^2}\int_{H^2}\Big\{\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}h, \lambda_j^{-1/2}x_j + w_j)^{\sim}\Big]$$

$$\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}k,\lambda_j^{-1/2}x_j+w_j)^{\sim}\Big]+\sum_{j=1}^{2}\frac{i}{2q_j}\langle A_j^{1/2}h,A_j^{1/2}k\rangle\Big\}$$
$$\exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}(h+k),y_j)^{\sim}-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2-\frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}$$
$$d\sigma(h)\,d\rho(k)\,d(\nu\times\nu)(x_1,x_2).$$

By the same method as in the proof of Theorem 3.8, if we evaluate the Wiener integral in the last expression, we obtain

$$\int_{H^2}\Big\{\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}h,w_j)^{\sim}\Big]\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}k,w_j)^{\sim}\Big]$$
$$+\sum_{j=1}^{2}\Big(-\frac{1}{2\lambda_j}+\frac{i}{2q_j}\Big)\langle A_j^{1/2}h,A_j^{1/2}k\rangle\Big\}\exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}(h+k),y_j)^{\sim}$$
$$-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2-\frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k).$$

But the last expression is an analytic function of $\vec{\lambda}\in\Omega$. Hence letting $\vec{\lambda}\to(-iq_1,-iq_2)$ we have

$$T_{\vec{q}}^{(p)}(\delta T_{\vec{q}_1}^{(p)}(F)(y_1,y_2|\cdot,\cdot)*\delta T_{\vec{q}_2}^{(p)}(G)(y_1,y_2|\cdot,\cdot))_{\vec{q}}(w_1,w_2)$$
$$\int_{H^2}\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}h,w_j)^{\sim}\Big]\Big[\sum_{j=1}^{2}\frac{i}{\sqrt{2}}(A_j^{1/2}k,w_j)^{\sim}\Big]$$
$$\exp\Big\{\sum_{j=1}^{2}\Big[i(A_j^{1/2}(h+k),y_j)^{\sim}-\frac{i}{2q_{1j}}|A_j^{1/2}h|^2-\frac{i}{2q_{2j}}|A_j^{1/2}k|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k)$$

which is equal to the second expression of (3.15) by Corollary 3.2 and this completes the proof. □

Remark 3.13. As we commented in Section 2, if A_1 is the identity operator in H and $A_2=0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$. Hence we obtain all the results in Section 4 of [1] as corollaries of the results in this section.

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