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MONOTONE ITERATION SCHEME FOR A FORCED DUFFING EQUATION WITH NONLOCAL THREE-POINT CONDITIONS

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ABSTRACT. In this paper, we apply the generalized quasilinearization technique to a forced Duffing equation with three-point mixed nonlinear nonlocal boundary conditions and obtain sequences of upper and lower solutions converging monotonically and quadratically to the unique solution of the problem.

1. Introduction

Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of Duffing equation is in the field of the prediction of diseases. A careful measurement and analysis of a strongly chaotic voice has the potential to serve as an early warning system for more serious chaos and possible onset of disease. This chaos is stimulated with the help of Duffing equation. In fact, the success at analyzing and predicting the onset of chaos in speech and its simulation by equations such as the Duffing equation has enhanced the hope that we might be able to predict the onset of arrhythmia and heart attacks someday. Such predictions are based on the numerical solutions of the Duffing equation. However, there do exist a number of powerful procedures for obtaining approximate solutions of nonlinear problems such as Newton-Raphson method, Galerkins method, expansion methods, iterative techniques, method of upper and lower solutions to name a few. The monotone iterative method and Newton's method are known to be two efficient techniques for finding roots of nonlinear equations. The first one applies to equations involving monotone operators and produces a sequence converging monotonically to a solution. The Newton method has the advantage over the monotone iterative method that it provides quadratically convergent sequences. Applied to nonlinear differential equations, Newton's

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method is known as the quasilinearization method. The origin of the quasilinearization lies in the theory of dynamic programming [6-7, 20]. This method applies to semilinear equations with convex or concave nonlinearities and provides an explicit analytic representation of approximate solution of the given problem. However, the concavity/convexity assumption proved to be a stumbling block for further development of the theory. The nineties brought new dimensions to this technique. The most interesting new idea was introduced by Lakshmikantham [17-18] who generalized the method of quasilinearization by relaxing the convexity assumption. This extension, now known as generalized quasilinearization, consists of the method of lower and upper solutions and monotone iterative technique together with differential inequalities and comparison results. This development was so significant that it attracted the attention of many researchers and the method was extensively developed and applied to a wide range of initial and boundary value problems [1, 3-4, 8-9, 19] and references therein. Some real-world applications of the quasilinearization technique can be found in [23-26].

Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [10], have been addressed by many authors, for example, Kiguradze and Lomtatidze [16], Gupta [13], Gupta and Trofimchuck [14], Ma [20-21], Bai and Fang [5], and Eloe and Ahmad [11]. Eloe and Gao [12] discussed the quasilinearization method for a three-point boundary value problem. Ahmad [2] developed the generalized quasilinearization method for a general three-point nonlinear boundary value problem.

In this paper, we consider a forced Duffing equation with nonlinear nonlocal three-point mixed boundary conditions and develop a monotone iteration scheme by relaxing the convexity assumption on the function involved in the differential equation and the concavity assumption on nonlinearities in the boundary conditions. In fact, we obtain monotone sequences of iterates (approximate solutions) converging quadratically to the unique solution of the three-point boundary value problem.

2. Preliminaries

We consider a three-point boundary value problem for the forced Duffing equation with nonlocal conditions

$$x'' + kx' + f(t,x) = 0, \tag{2.1}$$

$$px(0) - qx'(0) = g_1(x(\sigma)), \quad px(1) + qx'(1) = g_2(x(\sigma)), \ 0 < \sigma < 1, \tag{2.2}$$

where f is continuous with $f_x < 0$ on $[0,1] \times R$, $k \in R$ such that $k \neq 0, p, q > 0$ with $p > 1$ and $g_i : R \longrightarrow R$, $i = 1, 2$ are continuous.

By Green's function method, the solution, $x(t)$ of (2.1)-(2.2) can be written as

$$x(t) = g_1(x(\sigma))[\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}]$$
$$+ g_2(x(\sigma))[\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}] + \int_0^1 G_k(t,s)f(s,x(s))ds,$$

where

$$G_k(t,s)$$
$$= \frac{pe^{ks}}{k[(p-kq) - (p+kq)e^k)]} \begin{cases} [\frac{p-kq}{p} - e^{k(1-s)}][\frac{p+kq}{p} - e^{-kt}], & 0 \le t \le s, \\ [\frac{p-kq}{p} - e^{k(1-t)}][\frac{p+kq}{p} - e^{-ks}], & s \le t \le 1. \end{cases}$$

We say that $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem (2.1)-(2.2) if

$$\alpha''(t) + k\alpha'(t) + f(t,\alpha(t)) \ge 0, \ t \in [0,1],$$
$$p\alpha(0) - q\alpha'(0) \le g_1(\alpha(\sigma)), \ p\alpha(1) + q\alpha'(1) \le g_2(\alpha(\sigma)),$$

and $\beta \in C^2[0,1]$ is an upper solution of (2.1)-(2.2) if

$$\beta''(t) + k\beta'(t) + f(t,\beta(t)) \le 0, \ t \in [0,1]$$
$$p\beta(0) - q\beta'(0) \ge g_1(\beta(\sigma)), \ p\beta(1) + q\beta'(1) \ge g_2(\beta(\sigma)).$$

Now, we present comparison and existence results related to (2.1)-(2.2) which play a pivotal role in proving the main result.

Theorem 2.1. *Assume that f is continuous with $f_x < 0$ on $[0,1] \times R$ and g is continuous on R satisfying a one-sided Lipschitz condition:*

$$g_i(x) - g_i(y) \le L_i(x-y), \ 0 \le L_i < 1, \ i = 1,2.$$

Let β and α be the upper and lower solutions of (2.1)-(2.2), *respectively. Then* $\alpha(t) \le \beta(t), \ t \in [0,1]$.

Proof. Define $h(t) = \alpha(t) - \beta(t)$. For the sake of contradiction, we suppose that $h(t) > 0$ for some $t \in [0,1]$. First we take $t_0 \in (0,1)$. Then by the definition of lower and upper solutions and the assumption $f_x < 0$, we obtain

$$h''(t_0) + kh'(t_0) = \alpha''(t_0) + k\alpha'(t_0) - \beta''(t_0) - k\beta'(t_0)$$
$$\ge -f(t_0,\alpha(t_0)) + f(t_0,\beta(t_0)) > 0.$$

Now, employing a standard procedure [15] in the applications of upper and lower solutions, let $h(t)$ have a local positive maximum at $t_0 \in (0,1)$, then $h'(t_0) = 0$ and $h''(t_0) \le 0$, which contradicts the above inequality. Thus, for $t_0 \in (0,1)$, we have $\alpha(t) \le \beta(t)$. Now, suppose that $h(t)$ has a local positive maximum at $t_0 = 1$, then $h'(1) = 0$ and $h''(1) < 0$. On the other hand, using the definition of lower and upper solutions together with the fact that g_2 satisfies a one sided Lipschitz condition, we find that

$$\begin{aligned} ph(1)+qh'(1) &= p\alpha(1)+q\alpha'(1)-(p\beta(1)+q\beta'(1)) \le g_2(\alpha(\sigma))-g_2(\beta(\sigma)) \\ &< \alpha(\sigma)-\beta(\sigma)=h(\sigma), \end{aligned}$$

Thus $ph(1) < h(\sigma)$ or $h(1) < h(\sigma)$ for $p > 1$, which is a contradiction. Similarly, we get a contradiction for $t_0 = 0$. Hence we conclude that $\alpha(t) \le \beta(t)$ on $[0, 1]$. □

Theorem 2.2. *Assume that f is continuous on $[0,1] \times R$ with $f_x < 0$ and g_i are continuous on R satisfying one-sided Lipschitz condition:*

$$g_i(x) - g_i(y) \le L_i(x-y), \ 0 \le L_i < 1, \ i = 1, 2.$$

Further, we assume that there exist an upper solution β and a lower solution α of (2.1)-(2.2) such that $\alpha(t) \le \beta(t), \ t \in [0,1]$. Then there exists a solution $x(t)$ of (2.1)-(2.2) satisfying $\alpha(t) \le x(t) \le \beta(t), \ t \in [0,1]$.

Proof. Let us define F and G by

$$F(t,x) = \begin{cases} f(t,\beta) - \frac{x-\beta}{1+x-\beta}, & \text{if } x(t) > \beta(t), \\ f(t,x), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ f(t,\alpha) - \frac{x-\alpha}{1+|x-\alpha|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$\hat{g}_i(x) = \begin{cases} g_i(\beta(\sigma)), & \text{if } x > \beta(\sigma), \\ g_i(x), & \text{if } \alpha(\sigma) \le x \le \beta(\sigma), \\ g_i(\alpha(\sigma)), & \text{if } x < \alpha(\sigma), \end{cases}$$

for $i = 1, 2$.

Since $F(t,x)$ and $\hat{g}_i(x)$ are continuous and bounded, a standard application of Schauder's fixed point theorem ensures the existence of a solution, x of the problem

$$x''(t) + kx'(t) + F(t, x(t)) = 0, \ t \in [0,1],$$

$$px(0) - qx'(0) = \hat{g}_1(x(\sigma)), \ px(1) + qx'(1) = \hat{g}_2(x(\sigma)).$$

In order to complete the proof, we need to show that $\alpha(t) \le x(t) \le \beta(t)$ on $[0,1]$. For that, we set $h(t) = \alpha(t) - x(t)$. For the sake of the contradiction, let $h(t) > 0$ for some $t \in [0,1]$. We define

$$t_0 = \inf\{\tau \in [0,1] : h(\tau) \ge h(t), 0 \le t \le 1\},$$

and note that $0 < t_0$ by continuity. As $\hat{g}_2$ satisfies a one-sided Lipschitz condition on $[\alpha(\frac{1}{2}), \beta(\frac{1}{2})]$, it follows that

$$\begin{aligned} ph(1)+qh'(1) &= p\alpha(1)+q\alpha'(1)-(px(1)+qx'(1)) \le \hat{g}_2(\alpha(\sigma))-\hat{g}_2(x(\sigma)) \\ &< (\alpha(\sigma)-x(\sigma)) = h(\sigma). \end{aligned}$$

As in the proof of Theorem 2.1, let $h(t)$ have a local maximum at $t_0 \in (0,1)$ implying that $h'(t_0) = 0$ and $h''(t_0) \le 0$. On the other hand, by the definition of upper and lower solutions together with the assumption $F_x < 0$, we have

$$\begin{aligned} h''(t_0) + kh'(t_0) &= \alpha''(t_0) + k\alpha'(t_0) - (x''(t_0) + kx'(t_0)) \\ &\ge -F(t_0, \alpha(t_0)) + F(t_0, x(t_0)) > 0. \end{aligned}$$

This contradicts our supposition. Hence $\alpha(t) - x(t) \le 0$. Similarly, it can be shown that $x(t) \le \beta(t)$. Thus, it follows that $\alpha(t) \le x(t) \le \beta(t),\ t \in [0,1]$. □

3. Main result

Theorem 3.1. *Assume that*

($\mathbf{A_1}$) α_0, β_0 *are lower and upper solutions of* (2.1)-(2.2), *respectively.*

($\mathbf{A_2}$) $f(t,x) \in C([0,1]\times R)$ *be such that* $f_x < 0$ *and* $(f_{xx}(t,x) + \phi_{xx}(t,x)) \ge 0$, *where* $\phi_{xx}(t,x) \ge 0$ *for some continuous function* $\phi(t,x)$ *on* $[0,1] \times R$.

($\mathbf{A_3}$) *For* $i = 1,2$, $g_i(x), g_i'(x), g_i''(x)$ *are continuous on* R *with* $0 \le g_i' \le 1$ *and* $g_i''(x) + \psi_i''(x) \le 0$ *with* $\psi_i'' \le 0$ *on* R *for some continuous functions* $\psi_i(x)$.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *that converge quadratically in the space of continuous functions on* $[0,1]$ *to the unique solution* x *of* (2.1)-(2.2).

Proof. Define $F : [0,1] \times R \to R$ by

$$F(t,x) = f(t,x) + \phi(t,x),$$

and $G_i : R \to R$ by

$$G_i(x) = g_i(x) + \psi_i(x),\ i = 1,2.$$

Using the generalized mean value theorem together with (A_2) and (A_3), we obtain

$$f(t,x) \ge f(t,y) + F_x(t,y)(x-y) + \phi(t,y) - \phi(t,x), \tag{3.1}$$

$$g_i(x) \le g_i(y) + G_i'(y)(x-y) + \psi_i(y) - \psi_i(x),\ i = 1,2. \tag{3.2}$$

Now, we set

$$\begin{aligned} F(t,x;\alpha_0) &= f(t,\alpha_0) + F_x(t,\alpha_0)(x-\alpha_0) + \phi(t,\alpha_0) - \phi(t,x), \\ \overline{F}(t,x;\alpha_0,\beta_0) &= f(t,\beta_0) + F_x(t,\alpha_0)(x-\beta_0) + \phi(t,\beta_0) - \phi(t,x), \end{aligned}$$

and

$$h_i(x(\sigma);\alpha_0,\beta_0) = g_i(\alpha_0(\sigma)) + G_i'(\beta_0(\sigma))(x(\sigma) - \alpha_0(\sigma)) + \psi_i(\alpha_0(\sigma)) - \psi_i(x(\sigma)),$$

$$\hat{h}_i(x(\sigma);\beta_0) = g_i(\beta_0(\sigma)) + G_i'(\beta_0(\sigma))(x(\sigma) - \beta_0(\sigma)) + \psi_i(\beta_0(\sigma)) - \psi_i(x(\sigma)),$$

for $i = 1,2$.

We now consider the BVPs

$$x''(t) + kx'(t) + F(t,x;\alpha_0) = 0,\ t \in [0,1], \tag{3.3}$$

$$px(0) - qx'(0) = h_1(x(\sigma);\alpha_0,\beta_0),\ px(1) + qx'(1) = h_2(x(\sigma);\alpha_0,\beta_0), \tag{3.4}$$

and

$$x''(t) + kx'(t) + \overline{F}(t, x; \alpha_0, \beta_0) = 0, \ t \in [0, 1], \tag{3.5}$$

$$px(0) - qx'(0) = \hat{h}_1(x(\sigma), \beta_0), \ px(1) + qx'(1) = \hat{h}_2(x(\sigma), \beta_0). \tag{3.6}$$

Let us show that α_0 and β_0 are respectively lower and upper solutions of (3.3)-(3.4). By definition of lower solution and the fact that $F(t, \alpha_0; \alpha_0) = f(t, \alpha_0)$, we get

$$\begin{aligned}
\alpha_0'' + k\alpha_0' + F(t, \alpha_0; \alpha_0) &= \alpha_0'' + k\alpha_0' + f(t, \alpha_0) \geq 0, \\
p\alpha_0(0) - q\alpha_0'(0) &\leq g_1(\alpha_0(\sigma)) = h_1(\alpha_0(\sigma); \alpha_0; \beta_0), \\
p\alpha_0(1) + q\alpha_0'(1) &\leq g_2(\alpha_0(\sigma)) = h_2(\alpha_0(\sigma); \alpha_0; \beta_0),
\end{aligned}$$

which implies that α_0 is a lower solution of (3.3)-(3.4). Using (3.1) and the definition of upper solution, we have

$$\begin{aligned}
& \beta_0'' + k\beta_0' + F(t, \beta_0; \alpha_0) \\
= \ & \beta_0'' + k\beta_0' + f(t, \alpha_0) + F_x(t, \alpha_0)(\beta_0 - \alpha_0) + \phi(t, \alpha_0) - \phi(t, \beta_0) \\
\leq \ & \beta_0'' + k\beta_0' + f(t, \beta_0) \leq 0.
\end{aligned}$$

Using mean value theorem and the nonincreasing property of G_1', we have

$$\begin{aligned}
& g_1(\beta_0(\sigma)) - h_1(\beta_0(\sigma); \alpha_0, \beta_0) \\
= \ & g_1(\beta_0(\sigma)) - g_1(\alpha_0(\sigma)) - G_1'(\beta_0(\sigma))(\beta_0(\sigma) - \alpha_0(\sigma)) \\
& -\psi_1(\alpha_0(\sigma)) + \psi_1(\beta_0(\sigma)) \\
= \ & G_1(\beta_0(\sigma)) - G_1(\alpha_0(\sigma)) - G_1'(\beta_0(\sigma))(\beta_0(\sigma) - \alpha_0(\sigma)) \\
= \ & [G_1'(c_0) - G_1'(\beta_0(\sigma))](\beta_0(\sigma) - \alpha_0(\sigma)) \geq 0,
\end{aligned}$$

where $\alpha_0(\sigma) \leq c_0 \leq \beta_0(\sigma)$. Consequently, we have

$$p\beta_0(0) - q\beta_0'(0) \geq h_1(\beta_0(\sigma); \alpha_0, \beta_0).$$

Similarly, it can be shown that

$$p\beta_0(1) + q\beta_0'(1) \geq h_2(\beta_0(\sigma); \alpha_0, \beta_0).$$

Thus, β_0 is an upper solution of (3.3)-(3.4). Hence, by Theorem 2.2, there is a solution α_1 of (3.3)-(3.4) satisfying

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t), \ t \in [0, 1]. \tag{3.7}$$

Note that Theorem 2.2 applies since $h_i' = g_i'(\beta_0(\sigma)), \ i = 1, 2$. Similarly, β_0 is an upper solution of (3.5)-(3.6) as

$$\begin{aligned}
&\overline{F}(t, \beta_0; \alpha_0; \beta_0) = f(t, \beta_0), \\
&g_1(\beta_0(\sigma)) = \hat{h}_1(\beta_0(\sigma); \beta_0), \\
&g_2(\beta_0(\sigma)) = \hat{h}_2(\beta_0(\sigma); \beta_0).
\end{aligned}$$

As before, using (3.1), we obtain

$$\begin{aligned} &\alpha_0'' + k\alpha_0' + \overline{F}(t, \alpha_0; \alpha_0, \beta_0) \\ = \ &\alpha_0'' + k\alpha_0' + f(t, \beta_0) + F_x(t, \alpha_0)(\alpha_0 - \beta_0) + \phi(t, \beta_0) - \phi(t, \alpha_0) \\ \geq \ &\alpha_0'' + k\alpha_0' + f(t, \alpha_0) \geq 0. \end{aligned}$$

Now, we will show that $p\alpha_0(0) - q\alpha_0'(0) \leq \hat{h}_1(\alpha_0(\sigma); \beta_0)$. By mean value theorem, we find that

$$\begin{aligned} &\hat{h}_1(\alpha_0(\sigma); \beta_0) - g_1(\alpha_0(\sigma)) \\ = \ &g_1(\beta_0(\sigma)) + G_1'(\beta_0(\sigma))(\alpha_0(\sigma) - \beta_0(\sigma)) \\ &+\psi_1(\beta_0(\sigma)) - \psi_1(\alpha_0(\sigma)) - g_1(\alpha_0(\sigma)) \\ = \ &G_1(\beta_0(\sigma)) - G_1(\alpha_0(\sigma)) + G_1'(\beta_0(\sigma))(\alpha_0(\sigma) - \beta_0(\sigma)) \\ = \ &[G_1'(c_1) - G_1'(\beta_0(\sigma))](\beta_0(\sigma) - \alpha_0(\sigma)) \geq 0, \end{aligned}$$

where $\alpha_0(\sigma) \leq c_1 \leq \beta_0(\sigma)$. Thus

$$p\alpha_0(0) - q\alpha_0'(0) \leq g_1(\alpha_0(\sigma)) \leq \hat{h}_1(\alpha_0(\sigma); \beta_0).$$

Similarly, it can be shown that

$$p\alpha_0(1) + q\alpha_0'(1) \leq \hat{h}_2(\alpha_0(\sigma); \beta_0).$$

Thus, α_0 is a lower solution of (3.5)-(3.6). Again, by Theorem 2.2, there exists a solution β_1 of (3.5)-(3.6) such that

$$\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t), \ t \in [0, 1]. \tag{3.8}$$

Now, we show that $\alpha_1 \leq \beta_1$, To do this we prove that α_1, β_1 are lower and upper solutions of (2.1)-(2.2), respectively. Using the fact that α_1 is a solution of (3.3)-(3.4), we get

$$\begin{aligned} &\alpha_1''(t) + k\alpha_1'(t) + f(t, \alpha_1) \\ \geq \ &\alpha_1''(t) + k\alpha_1'(t) + f(t, \alpha_0) + F_x(t, \alpha_0)(\alpha_1 - \alpha_0) + \phi(t, \alpha_0) - \phi(t, \alpha_1) \\ = \ &\alpha_1''(t) + k\alpha_1'(t) + F(t, \alpha_1; \alpha_0) = 0. \end{aligned}$$

Now, in view of nonincreasing property of G_1', we obtain

$$\begin{aligned} &g_1(\alpha_1(\sigma)) - [p\alpha_1(0) - q\alpha_1'(0)] \\ = \ &g_1(\alpha_1(\sigma)) - g_1(\alpha_0(\sigma)) - G_1'(\beta_0(\sigma))(\alpha_1(\sigma) - \alpha_0(\sigma)) \\ &-\psi_1(\alpha_0(\sigma)) + \psi_1(\alpha_1(\sigma)) \\ = \ &[G_1'(c_2) - G_1'(\beta_0(\sigma))](\alpha_1(\sigma) - \alpha_0(\sigma)) \geq 0, \end{aligned}$$

where $c_2 \in (\alpha_0(\sigma), \alpha_1(\sigma))$, which in turn yields

$$p\alpha_1(0) - q\alpha_1'(0) \leq g_1(\alpha_1(\sigma)).$$

Similarly, it can be shown that $p\alpha_1(1) + q\alpha_1'(1) \leq g_2(\alpha_1(\sigma))$. This implies that α_1 is a lower solution of (2.1)-(2.2). Similarly, it can be shown that β_1 is an upper solution of (2.1)-(2.2). By Theorem 2.1, it follows that

$$\alpha_1(t) \leq \beta_1(t), \ t \in [0, 1]. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) yields

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \ t \in [0,1].$$

Continuing this process, by induction, one can prove that

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t), \ t \in [0,1], \ n = 0, 1, \ldots,$$

where α_{n+1} satisfies the problem

$$x''(t) + kx'(t) + F(t, x; \alpha_n) = 0, \ t \in [0,1].$$

$$px(0) - qx'(0) = h_1(x(\sigma); \alpha_n, \beta_n), \ px(1) + qx'(1) = h_2(x(\sigma); \alpha_n, \beta_n)$$

and β_{n+1} satisfies the BVP

$$x''(t) + kx'(t) + \overline{F}(t, x; \alpha_n, \beta_n) = 0, \ t \in [0,1],$$

$$px(0) - qx'(0) = \hat{h}_1(x(\sigma); \beta_n), \ px(1) + qx'(1) = \hat{h}_2(x(\sigma); \beta_n).$$

Since $[0,1]$ is compact and the convergence is monotone, it follows that the convergence of each sequence $\{\alpha_n\}$ and $\{\beta_n\}$ is uniform. Employing the standard arguments [15, 20], we conclude that x is the limit point of each of the two sequences and consequently, we get

$$\begin{aligned} x(t) = {} & g_1(x(\sigma))[\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}] \\ & + g_2(x(\sigma))[\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}] + \int_0^1 G_k(t,s) f(s, x(s)) ds. \end{aligned}$$

This proves that x is the unique solution of (2.1)-(2.2).

In order to prove that each of the sequences $\{\alpha_n\}, \{\beta_n\}$ converges quadratically, we set $q_n = \beta_n - x \ge 0$, $p_n = x - \alpha_n \ge 0$, where x denotes the unique solution of (2.1)-(2.2). We only show the quadratic convergence with p_n as the details for the quadratic convergence for q_n are similar. Applying the mean value theorem, there exist $\alpha_n \le c_3, c_4, c_5 \le x$ and $\alpha_n \le \zeta_1 \le \alpha_{n+1}$ such that

$$\begin{aligned} & p''_{n+1} + kp'_{n+1} \\ = & -f(t,x) + f(t,\alpha_n) + F_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) + \phi(t,\alpha_n) - \phi(t,\alpha_{n+1}) \\ = & -f_x(t,c_3)(x - \alpha_n) + F_x(t,\alpha_n)(\alpha_{n+1} - x + x - \alpha_n) - \phi_x(t,\zeta_1)(\alpha_{n+1} - \alpha_n) \\ = & [-F_x(t,c_3) + F_x(t,\alpha_n) + \phi_x(t,c_3) - \phi_x(t,\zeta_1)]p_n + [-F_x(t,\alpha_n) + \phi_x(t,\zeta_1)]p_{n+1} \\ \ge & [-F_x(t,x) + F_x(t,\alpha_n) + \phi_x(t,\alpha_n) - \phi_x(t,x)]p_n + [-F_x(t,\zeta_1) + \phi_x(t,\zeta_1)]p_{n+1} \\ = & -F_{xx}(t,c_4)p_n^2 - \phi_{xx}(t,c_5)p_n^2 - f_x(t,\zeta_1)p_{n+1} \\ \ge & -M\|p_n\|^2, \end{aligned}$$

where A is a bound on $\|F_{xx}\|$, B is a bound on $\|\phi_{xx}\|$ for $t \in [0,1]$ and $M = A+B$. Here $\|.\|$ denotes the supremum norm on $C[0,1]$. Also there exist $\alpha_n(\sigma) \le$

$c_6, r_1 \le c_7, r_2 \le x \le \beta_n$ and $\alpha_n \le \zeta_2, \zeta_3 \le \zeta_4, \zeta_5 \le \alpha_{n+1}$ such that

$$\begin{aligned}
p_{n+1}(t) &= [g_1(x(\sigma)) - h_1(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)](\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}) \\
&\quad +[g_2(x(\sigma)) - h_2(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)](\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}) \\
&\quad + \int_0^1 G_k(t,s)[f(s,x) - F(s, \alpha_{n+1}; \alpha_n)]ds \\
&= [g_1(x(\sigma)) - g_1(\alpha_n(\sigma) - G_1'(\beta_n(\sigma))(\alpha_{n+1}(\sigma) - \alpha_n(\sigma)) \\
&\quad -\psi_1(\alpha_n(\sigma)) + \psi_1(\alpha_{n+1}(\sigma))](\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}) \\
&\quad +[g_2(x(\sigma)) - g_2(\alpha_n(\sigma) - G_2'(\beta_n(\sigma))(\alpha_{n+1}(\sigma) - \alpha_n(\sigma)) \\
&\quad -\psi_2(\alpha_n(\sigma)) + \psi_2(\alpha_{n+1}(\sigma))](\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}) \\
&\quad - \int_0^1 G_k(t,s)[p_{n+1}'' + kp_{n+1}']ds \\
&= [g_1'(c_6)(x(\sigma) - \alpha_n(\sigma)) - G_1'(\beta_n(\sigma))(\alpha_{n+1} - \alpha_n(\sigma)) \\
&\quad +\psi_1'(\zeta_2)(\alpha_{n+1}(\sigma) - \alpha_n(\sigma))](\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}) \\
&\quad +[g_2'(r_1)(x(\sigma) - \alpha_n(\sigma)) - G_2'(\beta_n(\sigma))(\alpha_{n+1} - \alpha_n(\sigma)) \\
&\quad +\psi_2'(\zeta_3)(\alpha_{n+1}(\sigma) - \alpha_n(\sigma))](\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}) \\
&\quad - \int_0^1 G_k(t,s)[p_{n+1}'' + kp_{n+1}']ds \\
&\le [(G_1'(c_6) - G_1'(\beta_n(\sigma)) - (\psi_1'(c_6) - \psi_1'(\zeta_2)))p_n(\sigma) \\
&\quad +(G_1'(\beta_n(\sigma)) - \psi_1'(\zeta_2))p_{n+1}(\sigma)](\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}) \\
&\quad +[(G_2'(r_1) - G_2'(\beta_n(\sigma)) - (\psi_2'(r_1) - \psi_2'(\zeta_3)))p_n(\sigma) \\
&\quad +(G_2'(\beta_n(\sigma)) - \psi_2'(\zeta_3))p_{n+1}(\sigma)](\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}) \\
&\quad +M\|p_n\|^2 \int_0^1 |G_k(t,s)|ds \\
&\le [-G_1''(c_7)(\beta_n(\sigma) - c_6)p_n(\sigma) - \psi_1''(\zeta_4)p_n^2(\sigma) \\
&\quad +g_1'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p-kq)e^{-k} - pe^{-kt}}{p[(p-kq)e^{-k} - (p+kq)]}) \\
&\quad +[-G_2''(r_2)(\beta_n(\sigma) - r_1)p_n(\sigma) - \psi_2''(\zeta_5)p_n^2(\sigma) \\
&\quad +g_2'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p+kq) - pe^{-kt}}{p[(p+kq) - (p-kq)e^{-k}]}) + M_1\|p_n\|^2
\end{aligned}$$

$$\begin{aligned}
\leq\ & [-G_1''(c_7)(\beta_n(\sigma)-\alpha_n(\sigma))p_n(\sigma)-\psi_1''(\zeta_4)p_n^2(\sigma) \\
& +g_1'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p-kq)e^{-k}-pe^{-kt}}{p[(p-kq)e^{-k}-(p+kq)]}) \\
& +[-G_2''(r_2)(\beta_n(\sigma)-\alpha_n(\sigma))p_n(\sigma)-\psi_2''(\zeta_5)p_n^2(\sigma) \\
& +g_2'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p+kq)-pe^{-kt}}{p[(p+kq)-(p-kq)e^{-k}]})+M_1\|p_n\|^2 \\
=\ & [-G_1''(c_7)(q_n(\sigma)+p_n(\sigma))p_n(\sigma)-\psi_1''(\zeta_4)p_n^2(\sigma) \\
& +g_1'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p-kq)e^{-k}-pe^{-kt}}{p[(p-kq)e^{-k}-(p+kq)]}) \\
& +[-G_2''(r_2)(q_n(\sigma)+p_n(\sigma))p_n(\sigma)-\psi_2''(\zeta_5)p_n^2(\sigma) \\
& +g_2'(\beta_n(\sigma))p_{n+1}(\sigma)](\frac{(p+kq)-pe^{-kt}}{p[(p+kq)-(p-kq)e^{-k}]})+M_1\|p_n\|^2 \\
\leq\ & [N_1(\frac{1}{2}q_n^2(\sigma)+\frac{3}{2}p_n^2(\sigma))+g_1'(\beta_n(\sigma))p_{n+1}(\sigma)+D_1p_n^2(\sigma)]N_2 \\
& +[M_2(\frac{1}{2}q_n^2(\sigma)+\frac{3}{2}p_n^2(\sigma))+g_2'(\beta_n(\sigma))p_{n+1}(\sigma)+D_2p_n^2(\sigma)]M_3+M_1\|p_n\|^2 \\
\leq\ & [(\frac{3}{2}N_1+D_1)p_n^2(\sigma)+\frac{N_1}{2}q_n^2(\sigma)+\lambda_1p_{n+1}(\sigma)]N_2 \\
& +[(\frac{3}{2}M_2+D_2)p_n^2(\sigma)+\frac{M_2}{2}q_n^2(\sigma)+\lambda_2p_{n+1}(\sigma)]M_3+M_1\|p_n\|^2 \\
\leq\ & (\frac{3}{2}N_1N_2+D_1N_2+\frac{3}{2}M_2M_3+D_2M_3+M_1)\|p_n\|^2 \\
& +(\frac{N_1}{2}N_2+\frac{M_2}{2}M_3)\|q_n\|^2+(\lambda_1+\lambda_2)\|p_{n+1}\|,
\end{aligned}$$

where $|g_1'|\leq\lambda_1<1,\ |g_2'|\leq\lambda_2<1,\ |G_1''|<N_1,\ |G_2''|<M_2,\ |\psi_1''|<D_1,\ |\psi_2''|<D_2,\ |\frac{(p-kq)e^{-k}-pe^{-kt}}{p[(p-kq)e^{-k}-(p+kq)]}|<N_1,\ |\frac{(p+kq)-pe^{-kt}}{p[(p+kq)-(p-kq)e^{-k}]}|<M_3$ and M_1 provides a bound on $M\int_0^1|G_k(t,s)|ds$. Letting $M_4=\frac{3}{2}N_1N_2+D_1N_2+\frac{3}{2}M_2M_3+D_2M_3+M_1,\ M_5=\frac{N_1}{2}N_2+\frac{M_2}{2}M_3,\ \lambda=\lambda_1+\lambda_2$ and solving algebraically for $\|p_{n+1}\|$, we obtain

$$\|p_{n+1}\|\leq\frac{1}{1-\lambda}[M_4\|p_n\|^2+M_5\|q_n\|^2].$$

□

4. Concluding remarks

We have developed the generalized quasilinearization method for a nonlocal three-point boundary value problem involving a forced Duffing equation. Several interesting results can be recorded as a special case of the work established in this paper, for example, if we take $g_1(x(\sigma))=a$ and $g_2(x(\sigma))=b$ (a and b are constants), the problem corresponds to a two-point problem involving a forced Duffing equation with separated boundary conditions. The classical method of quasilinearization for a forced Duffing equation with three-point nonlinear boundary conditions can be recorded by taking $\phi(t,x) \equiv 0$ and $\psi_i(x) \equiv 0$ in the assumptions (A_2) and (A_3) of Theorem 3.1.

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