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ON 3-ADDITIVE MAPPINGS AND COMMUTATIVITY IN CERTAIN RINGS

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ABSTRACT. Let R be a ring with left identity e and suitably-restricted additive torsion, and $Z(R)$ its center. Let $H : R \times R \times R \to R$ be a symmetric 3-additive mapping, and let h be the trace of H. In this paper we show that (i) if for each $x \in R$,

$$\langle h(x), x\rangle_n = \langle\langle \cdots \langle h(x), x\rangle, x\rangle, \ldots, x\rangle \in Z(R)$$

with $n \geq 1$ fixed, then h is commuting on R. Moreover, h is of the form

$$h(x) = \lambda_0 x^3 + \lambda_1(x)x^2 + \lambda_2(x)x + \lambda_3(x) \quad \text{for all } x \in R,$$

where $\lambda_0 \in Z(R)$, $\lambda_1 : R \to R$ is an additive commuting mapping, $\lambda_2 : R \to R$ is the commuting trace of a bi-additive mapping and the mapping $\lambda_3 : R \to Z(R)$ is the trace of a symmetric 3-additive mapping; (ii) for each $x \in R$, either $\langle h(x), x\rangle_n = 0$ or $\langle\langle h(x), x\rangle_n, x^m\rangle = 0$ with $n \geq 0$, $m \geq 1$ fixed, then $h = 0$ on R, where $\langle y, x\rangle$ denotes the product $yx+xy$ and $Z(R)$ is the center of R. We also present the conditions which implies commutativity in rings with identity as motivated by the above result.

1. Introduction

Throughout, R will represent an associative ring, and $Z(R)$ will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We define the $(n+1)$-tuple $\langle y, x_1, \ldots, x_n\rangle$ as follows: $\langle y, x_1\rangle := yx_1 + x_1y$ and $\langle y, x_1, \ldots, x_{n-1}, x_n\rangle := \langle\langle y, x_1, \ldots, x_{n-1}\rangle, x_n\rangle$. In particular, in the case $x_1 = x_2 = \cdots = x_n = x$, $\langle y, x\rangle_n$ will stand for the $(n+1)$-tuple $\langle y, x, \ldots, x\rangle$ and let $\langle y, x\rangle_0 = y$. We will also make extensive use of the following basic properties: for any $x, y, z \in R$, $[xy, z] = x[y, z] + [x, z]y$, $[\langle y, x\rangle, x] = \langle [y, x], x\rangle$.

A mapping $f : R \to R$ is said to be commuting on R if $[f(x), x] = 0$ for all $x \in R$. Similarly f is called skew-commuting (resp. skew-centralizing) on R if $\langle f(x), x\rangle = 0$ (resp. $\langle f(x), x\rangle \in Z(R)$) for all $x \in R$. By analogy with the definition of n-commutativity introduced in [3], for $n \geq 2$ we define a mapping $f : R \to R$ to be n-skew-commuting (resp. n-skew-centralizing) on R if

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$\langle f(x), x^n\rangle = 0$ (resp. $\langle f(x), x^n\rangle \in Z(R)$) for all $x \in R$. An 1-skew-commuting mapping (resp. 1-skew-centralizing) is called simply a skew-commuting mapping (resp. skew-centralizing).

A map $H : R\times R\times R \to R$ is said to be symmetric if $H(x_1, x_2, x_3)=H(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ for all $x_1, x_2, x_3 \in R$ for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $h : R \to R$ defined by $h(x) = H(x, x, x)$ for all $x \in R$, where $H : R\times R\times R \to R$ is a symmetric map, is called the trace of H. It is obvious that, in case when $H : R \times R \times R \to R$ is a symmetric map which is also 3-additive (i.e., additive in each argument), the trace h of H satisfies the relation

$$h(x + y) = h(x) + h(y) + 3H(x, x, y) + 3H(x, y, y) \quad \text{for all} \quad x,\ y \in R.$$

Bell and Lucier [1] obtained some results for skew-commuting and skew-centralizing additive maps in rings with left identity and recently, in [4] we obtained the similar results for bi-additive maps in rings with left identity.

The main purpose of this paper is to investigate 3-additive mappings in rings with left identity under some conditions and is to obtain the conditions which implies commutativity in rings with identity by using them.

2. Main results

Let R be a ring with left identity e and let n be any positive integer. The resulting tuple after the substitutions $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{j-1} = x_{j+1} = \cdots = x_{k-1} = x_{k+1} = \cdots = x_n = e$ and $x_i = x_j = x_k = x$ in the $(n+1)$-tuple $\langle y, x_1, \ldots, x_n\rangle$ will be denoted by $T_{i,j,k}(y, x, e)$ for all $x_i,\ x_j,\ x_k,\ y \in R$, where $i, j, k = 1, 2, \ldots, n$ with $i \neq j \neq k$.

Similarly, the one after the substitutions $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = e$ and $x_i = x_j = x$ in the $(n+1)$-tuple $\langle y, x_1, \ldots, x_n\rangle$ will be denoted by $T_{i,j}(y, x, e)$ for all $x_i, x_j, y \in R$, where $i, j = 1, 2, \ldots, n$ with $i \neq j$. If $i = j$, then $T_{i,j}(y, x, e)$ stands for the tuple $\langle y, x_1, \ldots, x_n\rangle$ such that $x_i = x$ and $x_l = e$ for all $l \neq i$ and all $x_i, y \in R$, where $i, l = 1, 2, \ldots, n$.

We begin with the following result which is motivated by [1] and [2].

Theorem 1. *Let $n \geq 1$. Let R be a $(n+2)!$-torsion-free ring with left identity e. Let $H : R \times R \times R \to R$ be a symmetric 3-additive mapping and let h be the trace of H. If $\langle h(x), x\rangle_n \in Z(R)$ for all $x \in R$, then h is commuting on R. Moreover, h is of the form*

$$h(x) = \lambda_0 x^3 + \lambda_1(x)x^2 + \lambda_2(x)x + \lambda_3(x) \quad \textit{for all} \quad x \in R,$$

where $\lambda_0 \in Z(R)$, $\lambda_1 : R \to R$ is an additive commuting mapping, $\lambda_2 : R \to R$ is the commuting trace of a bi-additive mapping and the mapping $\lambda_3 : R \to Z(R)$ is the trace of a 3-additive mapping.

Proof. We first remark that the relation $[x, e]y = 0$ holds for all $x,\ y \in R$ since e is a left identity. Let $n \geq 1$. Since our assumption is

(1) $$\langle h(x), x\rangle_n = \langle\langle h(x), x\rangle_{n-1}, x\rangle \in Z(R) \quad \text{for all} \quad x \in R,$$

we have

$$\langle h(e), e\rangle_n = \langle h(e), e\rangle_{n-1}e + \langle h(e), e\rangle_{n-1} \in Z(R). \tag{2}$$

Commuting with e gives $\langle h(e), e\rangle_{n-1} = \langle h(e), e\rangle_{n-1}e$. It follows from (1) that $2\langle h(e), e\rangle_{n-1} \in Z(R)$, hence $\langle h(e), e\rangle_{n-1} \in Z(R)$. Continuing in the same manner with this expression, we arrive at $\langle h(e), e\rangle = \langle h(e), e\rangle_1 \in Z(R)$, that is,

$$h(e)e + h(e) \in Z(R). \tag{3}$$

Commuting with e yields $h(e) = h(e)e$; and by (3), $2h(e) \in Z(R)$, and so $h(e) \in Z(R)$.

Let t be any positive integer. Replacing x by $x + te$ in (1) and noting that $h(x+te) = h(x) + t^3h(e) + 3tH(x,x,e) + 3t^2H(x,e,e)$ for all $x \in R$, we obtain

$$tP_1(x,e) + t^2P_2(x,e) + \cdots + t^{n+2}P_{n+2}(x,e) \in Z(R) \quad \text{for all} \quad x \in R,$$

where $P_k(x,e)$ is the sum of terms involving x and e such that $P_k(x,te) = t^kP_k(x,e)$, $k = 1,2,\ldots,n+2$.

Replacing t by $1,2,\ldots,n+2$ in turn, and expressing the resulting system of $n+1$ non-homogeneous equations with the variables $P_1(x,e)$, $P_2(x,e),\ldots,$ $P_{n+2}(x,e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ n+2 & (n+2)^2 & \ldots & (n+2)^{n+2} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to $n+1$, and since R is $(n+2)!$-torsion free, it follows immediately that for each $k = 1,2,\ldots,n+2$,

$$P_k(x,e) \in Z(R) \quad \text{for all} \quad x \in R.$$

In particular, we have for all $x \in R$,

$$Z(R) \ni P_{n+2}(x,e) = \sum_{i=1}^{n} T_{i,i}(h(e),x,e) + 3\langle H(x,e,e), \overbrace{e,\ldots,e}^{n \text{ times}}\rangle, \tag{4}$$

$$\begin{aligned} Z(R) \ni P_{n+1}(x,e) &= 3\langle H(x,x,e), \overbrace{e,\ldots,e}^{n \text{ times}}\rangle + \sum_{i=1}^{n-1} T_{1,(i+1)}(h(e),x,e) \\ &\quad + \sum_{i=2}^{n-1} T_{i,(i+1)}(h(e),x,e) + \sum_{i=2}^{n-2} T_{i,(i+2)}(h(e),x,e) \\ &\quad + \sum_{i=1}^{n} 3T_{i,i}(H(x,e,e),x,e). \end{aligned} \tag{5}$$

and

$$Z(R) \ni P_n(x,e) = \langle h(x), \overbrace{e,\ldots,e}^{n \text{ times}}\rangle + \sum_{k=3}^{n}\sum_{i=2}^{n-(k-2)} T_{1,i,[i+(k-2)]}(h(e),x,e)$$

$$+\sum_{i=2}^{n-2} T_{2,(i+1),(i+2)}(h(e),x,e)$$

$$+\sum_{i=2}^{n-3} T_{2,(i+1),(i+3)}(h(e),x,e)$$

$$(6) \qquad +\sum_{i=2}^{n-3} T_{3,(i+2),(i+3)}(h(e),x,e)$$

$$+\sum_{k=2}^{n}\sum_{i=1}^{n-(k-1)} 3T_{(k-1),[i+(k-1)]}(H(x,e,e),x,e)$$

$$+\sum_{i=1}^{n} 3T_{i,i}(H(x,x,e),x,e).$$

Since $h(e) \in Z(R)$, the first sum in (4) becomes $n2^n xh(e)$. A simple calculation shows that the second term in (4) makes $3\{(2^n-1)H(x,e,e)e + H(x,e,e)\}$.

Hence we conclude that for all $x \in R$,

$$(7) \qquad P_{n+2}(x,e) = n2^n xh(e) + 3\{(2^n-1)H(x,e,e)e + H(x,e,e)\} \in Z(R).$$

By using $[x,e]y = 0$ for all x, $y \in R$, commuting with e gives

$$(8) \qquad [H(x,e,e),e] = 0 \quad \text{for all} \quad x \in R;$$

that is, $H(x,e,e) = H(x,e,e)e$ for all $x \in R$.

Now it follows from (7) that

$$(9) \qquad n2^n xh(e) + 3\cdot 2^n H(x,e,e) \in Z(R) \quad \text{for all} \quad x \in R;$$

and commuting with x in (9) yields

$$(10) \qquad 3\cdot 2^n[H(x,e,e),x] = 0 = [H(x,e,e),x] \quad \text{for all} \quad x \in R.$$

On the other hand, it follows from an easy calculation that the first term in (5) becomes $3\cdot(2^n-1)H(x,x,e)e + H(x,x,e)$. Note that the total number of all the terms in $P_{n+1}(x,e)$ is $\frac{n^2+n+2}{2}$ if $n > 1$ and is 2 if $n = 1$. Since the number of terms of the second sum in (5) is $n-1$ if $n > 1$, and the total number of terms of the third sum and the fourth sum in (5) is $\frac{(n-2)(n-1)}{2}$, we see that the total sum of terms of the second sum, the third sum and the fourth sum in (5), by considering $h(e) \in Z(R)$, amounts to $\frac{(n-1)n}{2}\, 2^n\, x^2 h(e)$. The number of terms of the fifth sum in (5) is n, and hence it follows from (8) and (10) that the sum of the terms comes to $3n\, 2^n\, xH(x,e,e)$.

Therefore we conclude that for all $x \in R$,

$$(11)\qquad \begin{aligned} P_{n+1}(x,e) &= 3\{(2^n-1)H(x,x,e)e + H(x,x,e)\} \\ &\quad + \frac{(n-1)n}{2}\, 2^n\, x^2 h(e) + n\, 2^n\, xH(x,e,e) \in Z(R). \end{aligned}$$

By again using $[x,\ e]y = 0$ for all $x, y \in R$, commuting with e gives

$$(12)\qquad [H(x,x,e),e] = 0 \text{ for all } x \in R,$$

that is, $H(x,x,e) = H(x,x,e)e$ for all $x \in R$.

Therefore we can rewrite (11) in the form

$$\begin{aligned} P_{n+1}(x,e) &= 3\cdot 2^n H(x,x,e) + \frac{(n-1)n}{2}\, 2^n\, x^2 h(e) \\ &\quad + n\, 2^n\, xH(x,e,e) \in Z(R) \end{aligned}$$

for all $x \in R$; thus commuting with x and using (10) give

$$(13)\qquad [H(x,x,e),x] = 0 \text{ for all } x \in R.$$

Finally, the first term in (6) is $(2^n-1)h(x)e + h(x)$. Using $h(e) \in Z(R)$, we see that the total sum of terms of the second sum, the third sum, the fourth sum and the fifth sum in (6) is $\alpha_n x^3 h(e)$, where $\alpha_n = \frac{n^2+3n-20}{2}$ if $n \geq 4$, $\alpha_n = 1$ if $n = 3$ and $\alpha_n = 0$ if $n = 1, 2$. From (8), (10), (12) and (13), it follows that the sixth sum in (6) is $3\beta_n x^2 H(x,e,e)$, where $\beta_n = \frac{(n-1)n}{2}$ and that the final sum in (6) is $n2^n xH(x,x,e)$.

Hence we have

$$(14)\qquad \begin{aligned} P_n(x,e) &= (2^n-1)h(x)e + h(x) + \alpha_n 2^n x^3 h(e) \\ &\quad + 3\beta_n 2^n x^2 H(x,e,e) + 3n2^n xH(x,x,e) \in Z(R). \end{aligned}$$

By using $[x,e]y = 0$ for all $x, y \in R$, commuting with e yields

$$[h(x),e] = 0 \text{ for all } x \in R,$$

that is, $h(x) = h(x)e$ for all $x \in R$.

We now can rewrite (14) in the form

$$(15)\qquad \begin{aligned} P_n(x,e) &= 2^n h(x) + \alpha_n 2^n x^3 h(e) + 3\beta_n 2^n x^2 H(x,e,e) \\ &\quad + 3n2^n xH(x,x,e) \in Z(R) \end{aligned}$$

for all $x \in R$. Thus commuting with x and using (10) and (13) give

$$[h(x),x] = 0 \text{ for all } x \in R.$$

Moreover, (15) implies that

$$(16)\qquad \begin{aligned} P_n(x,e) &= h(x) + \alpha_n h(e)x^3 + 3\beta_n H(x,e,e)x^2 \\ &\quad + 3nH(x,x,e)x \in Z(R). \end{aligned}$$

Let $\lambda_0 = -\alpha_n h(e)$. Then we have $\lambda_0 \in Z(R)$ since $h(e) \in Z(R)$.

The mapping $\lambda_1 : R \to R$ defined by $\lambda_1(x) = -3\beta_n H(x,e,e)$ for all $x \in R$ is additive and commuting by the additivity of $H(x,e,e)$ and $[H(x,e,e),x] = 0$ for all $x \in R$, respectively.

Setting $\lambda_2(x) = -3nH(x,x,e)$ for all $x \in R$, a mapping $\lambda_2 : R \to R$ is the commuting trace of a bi-additive mapping $G : R \times R \to R$ defined by $G(x,y) = -3nH(x,y,e)$ for all $x, y \in R$ since $[H(x,x,e),x] = 0$ for all $x \in R$.

Hence we can rewrite (16) in the desired structure

$$h(x) = \lambda_0 x^3 + \lambda_1(x)x^2 + \lambda_2(x)x + \lambda_3(x) \quad \text{for all} \quad x \in R,$$

where $\lambda_3(x) \in Z(R)$, that is, we define a mapping $\lambda_3 : R \to Z(R)$ which is the trace of a 3-additive mapping $C : R \times R \times R \to R$ defined by

$$C(x,y,z) = H(x,y,z) + \alpha_n h(e)xyz + 3\beta_n H(x,e,e)yz + 3nH(x,y,e)z$$

for all $x, y, z \in R$. The proof is complete. $\square$

Theorem 2. *Let $n \geq 1$. Let R be a $(n+2)!$-torsion-free ring with left identity e. Let $H : R \times R \times R \to R$ be a symmetric 3-additive mapping and let h be the trace of H. If $\langle h(x), x\rangle_n = 0$ for all $x \in R$, then we have $H = 0$.*

Proof. We follow the same argument as in the proof of Theorem 1. In the proof of Theorem 1, letting $Z(R) = \{0\}$ and then using (1), (2) and (3), we obtain $h(e) = 0$. Thus (7), in conjunction with (4), shows that

$$(17) \qquad 3\langle H(x,e,e), \overbrace{e,\dots,e}^{n \text{ times}}\rangle = 3\{(2^n - 1)H(x,e,e)e + H(x,e,e)\} = 0$$

for all $x \in R$. Right-multiply by e, obtaining

$$2^n H(x,e,e)e = 0 = H(x,e,e)e$$

for all $x \in R$, and so, by (17), $H(x,e,e) = 0$ for all $x \in R$. Consequently, (5) becomes

$$(18) \qquad 3\langle H(x,x,e), \overbrace{e,\dots,e}^{n \text{ times}}\rangle = 3\{(2^n - 1)H(x,x,e)e + H(x,x,e)\} = 0$$

for all $x \in R$. Right-multiplying by e gives

$$2^n H(x,x,e)e = 0 = H(x,x,e)e = 0$$

which means that, in view of (18), $H(x,x,e) = 0$ for all $x \in R$. Since we know that $h(e) = 0$, $H(x,e,e) = 0$ and $H(x,x,e) = 0$ for all $x \in R$, it follows that

$$(19) \qquad \langle h(x), \overbrace{e,\dots,e}^{n \text{ times}}\rangle = (2^n - 1)h(x)e + h(x) = 0 \quad \text{for all} \quad x \in R.$$

Again by right-multiplying by e, we obtain $2^n h(x)e = 0 = h(x)e$ for all $x \in R$, and so, by (19), $h(x) = 0$ for all $x \in R$ which implies that $H = 0$. $\square$

The following is a result concerning m-skew-commuting mappings.

Theorem 3. *Let $n \geq 0$ and $m \geq 1$. Let R be a $(n+m+2)!$-torsion-free ring with left identity e. Let $H : R \times R \times R \to R$ be a symmetric 3-additive mapping and let h be the trace of H. If the mapping $x \mapsto \langle h(x), x\rangle_n$ is m-skew-commuting on R, then we have $H = 0$.*

Proof. Suppose that

$$\langle\langle h(x), x\rangle_n, x^m\rangle = 0 \quad \text{for all} \quad x \in R. \tag{20}$$

Then we get

$$\langle\langle h(e), e\rangle_n, e^m\rangle = \langle h(e), e\rangle_n e + \langle h(e), e\rangle_n = 0; \tag{21}$$

and right-multiplying by e gives $2\langle h(e), e\rangle_n e = 0 = \langle h(e), e\rangle_n e$. Hence (21) yields $\langle h(e), e\rangle_n = 0$. Using similar approach as in the proof of Theorem 1, we obtain $h(e) = 0$.

Let t be any positive integer. Replacing x by $x + te$ in (20) and considering $h(x+te) = h(x) + t^3 h(e) + 3tH(x,x,e) + 3t^2 H(x,e,e)$ for all $x \in R$, we obtain

$$tP_1(x,e) + t^2 P_2(x,e) + \cdots + t^{n+m+2} P_{n+m+2}(x,e) = 0 \quad \text{for all} \quad x \in H,$$

where $P_k(x,e)$ is the sum of terms involving x and e such that $P_k(x,te) = t^k P_k(x,e)$, $k = 1, 2, \ldots, n+m+2$. Replacing t by $1, 2, \ldots, n+m+2$ in turn, and expressing the resulting system of $n+m+2$ homogeneous equations with the variables $P_1(x,e), P_2(x,e), \ldots, P_{n+m+2}(x,e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n+m+2} \\ \vdots & \vdots & \vdots & \vdots \\ n+m+2 & (n+m+2)^2 & \cdots & (n+m+2)^{n+m+2} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to $n+m+2$, and since R is $(n+m+2)!$ -torsion free, it follows immediately that for each $k = 1, 2, \ldots, n+m+2$,

$$P_k(x,e) = 0 \quad \text{for all} \quad x \in R.$$

In particular, we have, by utilizing $h(e) = 0$ and $e^m = e$, for all $x \in R$,

$$0 = P_{n+m+2}(x,e) = 3\langle H(x,e,e), \overbrace{e, \ldots, e}^{n+1 \text{ times}} \rangle \quad \text{for all} \quad x \in R, \tag{22}$$

$$\begin{aligned} 0 &= P_{n+m+1}(x,e) \\ &= 3\langle H(x,x,e), \overbrace{e, \ldots, e}^{n+1 \text{ times}} \rangle + Q_{n+m+1} \quad \text{for all} \quad x \in R, \end{aligned} \tag{23}$$

where Q_{n+m+1} is the sum of all terms containing $H(x,e,e)$ in P_{n+m+1} (x,e) and

$$0 = P_{n+m}(x,e) = \langle h(x), \overbrace{e, \ldots, e}^{n+1 \text{ times}} \rangle + Q_{n+m} + R_{n+m}, \tag{24}$$

where Q_{n+m} and R_{n+m} are the sums of all terms containing $H(x,e,e)$ and $H(x,x,e)$, respectively, in $P_{n+m}(x,e)$.

We now obtain from (22) that

$$3\{(2^{n+1}-1)H(x,e,e)e+H(x,e,e)\}=0 \quad \text{for all} \quad x\in R; \tag{25}$$

and right-multiplying by e gives $3\cdot 2^{n+2}H(x,e,e)e=0=H(x,e,e)e$ for all $x\in R$, therefore, by (25), $H(x,e,e)=0$ for all $x\in R$. This forces (23) to

$$3\langle H(x,x,e), \overbrace{e,\ldots,e}^{n+1 \text{ times}} \rangle=0 \quad \text{for all} \quad x\in R. \tag{26}$$

By calculating (26), we get

$$3\{(2^{n+1}-1)H(x,x,e)e+H(x,x,e)\}=0 \quad \text{for all} \quad x\in R; \tag{27}$$

and the right-multiplication by e yields $3\cdot 2^{n+1}H(x,x,e)e=0=H(x,x,e)e$ for all $x\in R$, hence, by (27), $H(x,x,e)=0$ for all $x\in R$. Since $H(x,e,e)=0$ and $H(x,x,e)=0$ holds for all $x\in R$, it follows from (24) that

$$\langle h(x), \overbrace{e,\ldots,e}^{n+1 \text{ times}} \rangle=0$$

which implies that $(2^{n+1}-1)h(x)e+h(x)=0$ for all $x\in R$. As above, we obtain $h(x)=0$ for all $x\in R$. This completes the proof of the theorem. $\square$

Example 1. Let

$$R=\left\{\begin{pmatrix} w & 0 & 0\\ x & w & 0\\ y & z & w\end{pmatrix} : w,\, x,\, y,\, z\in\mathbb{C}\right\},$$

where $\mathbb{C}$ is the set of complex numbers. Then R is a noncommutative associative ring with left identity, i.e., the unit matrix under the usual matrix addition and multiplication. We define a mapping $f:R\to R$ by

$$f(X)=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ y & z & 0\end{pmatrix} \quad \text{for all} \quad X\in R.$$

It is obvious that f is additive.

On the other hand, putting

$$Z(R)=\left\{\begin{pmatrix} w & 0 & 0\\ 0 & w & 0\\ y & 0 & w\end{pmatrix} : w,\, y\in\mathbb{C}\right\},$$

it is immediate to see that $Z(R)$ is the center of R.

Now, defining a mapping $H:R\times R\times R\to R$ by

$$\begin{aligned} H(X,Y,Z) &= [f(X),Y]+[f(Y),X]+[f(Y),Z]\\ &\quad +[f(Z),Y]+[f(Z),X]+[f(X),Z]\end{aligned}$$

for all $X, Y, Z \in R$, we can easily check that H is 3-additive, and that the mapping h on R defined by $h(X) = H(X, X, X)$ for all $X \in R$ is the trace of H such that $\langle h(X), X\rangle_n \in Z(R)$ for all $X \in R$.

3. Some results concerning the commutativity of rings with identity

Let R be a ring. An additive mapping $d : R \to R$ is called a derivation if the Leibniz rule $d(xy) = d(x)y + xd(y)$ is valid for all $x, y \in R$.

In this section, we use the results in Section 2 to establish some results concerning the commutativity of rings with identity.

First, we need the following well-known lemma [5].

Lemma 4. *Let R be a prime ring. Let $d : R \to R$ be a nonzero derivation such that d is commuting on R. Then R is commutative.*

Theorem 5. *Let $n \geq 0$ and $m \geq 1$. Let R be a $(n+m+2)!$-torsion-free prime ring with identity. If there is a nonzero derivation $d : R \to R$ such that the mapping $x \mapsto \langle d(x), x\rangle_n$ is m-skew-centralizing on R, then R is commutative.*

Proof. We define a mapping $H : R \times R \times R \to R$ by

$$H(x, y, z) = [d(x), y] + [d(y), x] + [d(y), z] + [d(z), y] + [d(z), x] + [d(x), z]$$

for all $x, y, z \in R$. Then it is clear that H is symmetric and 3-additive, and that the mapping $h : R \to R$ defined by $h(x) = H(x, x, x) = 6[d(x), x]$ for all $x \in R$ is the trace of H.

Since it follows from the hypothesis that $\langle\langle d(x), x\rangle_n, x^m\rangle \in Z(R)$ for all $x \in R$, we have, by recalling $[\langle y, x\rangle, x] = \langle [y, x], x\rangle$,

$$[\langle d(x), x\rangle_n x^m + x^m\langle d(x), x\rangle_n, x] = 0 \quad \text{for all } x \in R,$$

which implies that $[\langle d(x), x\rangle_n, x]x^m + x^m[\langle d(x), x\rangle_n, x] = 0$ for all $x \in R$. This reduces to $\langle [d(x), x], x\rangle_n x^m + x^m\langle [d(x), x], x\rangle_n = 0$ for all $x \in R$, that is, $\langle h(x), x\rangle_n x^m + x^m\langle h(x), x\rangle_n = 0$ for all $x \in R$. Hence it follows from Theorem 3 that $h = 0$ on R and so d is commuting on R. In view of Lemma 4, this means that R is commutative. □

Theorem 6. *Let $n \geq 1$. Let R be a $(n+2)!$-torsion-free prime ring with identity. If there is a nonzero derivation $d : R \to R$ such that the mapping $x \mapsto \langle d(x), x\rangle_n$ is commuting on R, then R is commutative.*

Proof. Let us define the symmetric 3-additive mapping $H : R \times R \times R \to R$ and the trace $h : R \to R$ as in Theorem 5.

By hypothesis, we have $[\langle d(x), x\rangle_n, x] = 0$ for all $x \in R$, and so we get $\langle h(x), x\rangle_n = \langle [d(x), x], x\rangle_n = [\langle d(x), x\rangle_n, x] = 0$ for all $x \in R$. Thus we obtain from Theorem 2 that $h = 0$ on R and so d is commuting on R. Lemma 4 yields that R is commutative. □

Theorem 7. *Let $n \geq 2$. Let R be a 2-torsion-free ring with identity such that*

$$\langle x_1x_2\cdots x_{n-1}x_n,\ x_nx_{n-1}\cdots x_2x_1\rangle \in Z(R)$$

holds for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$. Then R is commutative.

Proof. Replacing x_n by $x_n + 1$ in $\langle x_1x_2\cdots x_{n-1}x_n,\ x_nx_{n-1}\cdots x_2x_1\rangle \in Z(R)$, we get

$$\begin{aligned} Z(R) \ni\ & \langle x_1x_2\cdots x_{n-1}(x_n+1),\ (x_n+1)x_{n-1}\cdots x_2x_1\rangle \\ =\ & \langle x_1x_2\cdots x_{n-1}x_n + x_1x_2\cdots x_{n-1},\ x_nx_{n-1}\cdots x_2x_1 + x_{n-1}\cdots x_2x_1\rangle \\ =\ & \langle x_1x_2\cdots x_{n-1}x_n,\ x_nx_{n-1}\cdots x_2x_1\rangle \\ & +\langle x_1x_2\cdots x_{n-1}x_n, x_{n-1}\cdots x_2x_1\rangle + \langle x_1x_2\cdots x_{n-1},\ x_nx_{n-1}\cdots x_2x_1\rangle \\ & +\langle x_1x_2\cdots x_{n-1}, x_{n-1}\cdots x_2x_1\rangle \end{aligned}$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$. That is,

$$\begin{aligned} Z(R) \ni\ & \langle x_1x_2\cdots x_{n-1}x_n,\ x_nx_{n-1}\cdots x_2x_1\rangle + \langle x_1x_2\cdots x_{n-1}x_n, x_{n-1}\cdots x_2x_1\rangle \\ & +\langle x_1x_2\cdots x_{n-1},\ x_nx_{n-1}\cdots x_2x_1\rangle + \langle x_1x_2\cdots x_{n-1}, x_{n-1}\cdots x_2x_1\rangle \end{aligned}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in R$. Since we see that

$$\langle x_1x_2\cdots x_{n-1}x_n,\ x_nx_{n-1}\cdots x_2x_1\rangle \in Z(R),$$

we therefore have

$$\begin{aligned} Z(R) \ni\ & \langle x_1x_2\cdots x_{n-1}x_n,\ x_{n-1}\cdots x_2x_1\rangle + \langle x_1x_2\cdots x_{n-1}, x_nx_{n-1}\cdots x_2x_1\rangle \\ & +\langle x_1x_2\cdots x_{n-1},\ x_{n-1}\cdots x_2x_1\rangle \end{aligned}$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$.

Putting $-x_n$ instead of x_n now gives $\langle x_1x_2\cdots x_{n-1},\ x_{n-1}\cdots x_2x_1\rangle \in Z(R)$ for all $x_1, x_2, \ldots, x_{n-1} \in R$. Similarly, replacing x_{n-1} by $x_{n-1}+1$ in the just above relation and then letting $x_{n-1} := -x_{n-1}$ in the result, we obtain that

$$\langle x_1x_2\cdots x_{n-2},\ x_{n-2}\cdots x_2x_1\rangle \in Z(R)$$

for all $x_1, x_2, \ldots, x_{n-2} \in R$. Continuing in the similar processing with this relation, we finally arrive at $\langle x_1, x_1\rangle \in Z(R)$ for all $x_1 \in R$. This implies that $[y, x_1^2] = 0$ for all $x_1, y \in R$ and hence we see that $\langle [y, x_1], x_1\rangle = 0$ for all $x_1, y \in R$. Since the mapping $x_1 \mapsto [y, x_1]$ for any fixed $y \in R$ is additive on R, it follows from [1, Theorem 2] that $[y, x_1] = 0$ for all $x_1 \in R$. Since y is arbitrary, thus R is commutative. □

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