Commun. Korean Math. Soc. **22** (2007), No. 1, pp. 15–18

THE STRONG PERRON INTEGRAL IN $\mathbb{R}^n$ REVISITED

VALENTIN A. SKVORTSOV AND PIOTR SWOROWSKI

ABSTRACT. It is shown that α-regular McShane and strong Perron integrals considered in [1] are equivalent to the usual McShane integral.

This note is related to the recent paper [1], where several Perron and McShane-type integrals in $\mathbb{R}^n$ are introduced and compared. We show here that all those integrals are in fact equivalent to the usual McShane integral.

We borrow all the notation and terminology from [1]. The n-dimensional interval $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is said to be α-regular, $\alpha \in (0,1)$, if

$$r(I) = \frac{\min_i (b_i - a_i)}{\max_i (b_i - a_i)} > \alpha.$$

Let I_0 be a fixed interval in $\mathbb{R}^n$ and $\mathcal{I}$ the family of all subintervals of I_0. With $\mathcal{F}$ we denote the free full interval basis $\mathcal{F} = \{(I,x) : I \in \mathcal{I}, x \in I_0\}$. For a given function $\delta : I_0 \to (0, \infty)$, called a *gauge*, and a given $\alpha \in (0,1)$ we define

$$\mathcal{F}_\delta = \{(I,x) \in \mathcal{F} : I \subset U(x, \delta(x))\},$$
$$\mathcal{F}_\delta^\alpha = \{(I,x) \in \mathcal{F} : r(I) > \alpha, I \subset U(x, \delta(x))\}.$$

A finite subset $\mathcal{P}$ of $\mathcal{F}_\delta$ (of $\mathcal{F}_\delta^\alpha$) is called an $\mathcal{F}_\delta$*-division* (an $\mathcal{F}_\delta^\alpha$*-division* respectively) if for distinct pairs (I_1, x_1) and (I_2, x_2) in $\mathcal{P}$, the intervals I_1 and I_2 are nonoverlapping. If, moreover, $\bigcup_{(I,x)\in\mathcal{P}} I = I_0$, then $\mathcal{P}$ is called respectively an $\mathcal{F}_\delta$*-partition* and an $\mathcal{F}_\delta^\alpha$*-partition of* I_0.

We recall the definition of the McShane integral.

Definition 1. A point function f on I_0 is *McShane integrable* (*M-integrable*, in brief), with the integral A, if for each $\epsilon > 0$ there exists a gauge δ such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \epsilon$$

for every $\mathcal{F}_\delta$-partition π of I_0.

Received December 19, 2005.
2000 *Mathematics Subject Classification.* 26A39.
Key words and phrases. McShane integral, Perron integral.
Supported by RFFI-05-01-00206.

Substituting in this definition $\mathcal{F}_\delta$-partition by $\mathcal{F}_\delta^\alpha$-partition, with $\alpha \in (0,1)$, we obtain M_α-integral as it is introduced in [1, Definition 2.5]. A Perron-type version of M_α-integral, called SP_α-integral, is also given in [1, Definition 2.3]. To show that all these integrals, for any α, are equivalent to the McShane integral, the following (rather folklore) construction is needed.

Lemma 2. *Each n-dimensional interval $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ can be split into finite number of intervals all of any given regularity $\alpha \in (0,1)$.*

Proof. For each $i = 2, \ldots, n$ pick a point $\hat{b}_i$ so that $\frac{\hat{b}_i - a_i}{b_1 - a_1} \in \mathbb{Q}$ and

$$1 \le \frac{\hat{b}_i - a_i}{b_i - a_i} < \frac{1}{\alpha}. \tag{1}$$

Let L be a coordinate-wise linear mapping of I onto $\hat{I} = [a_1, b_1] \times [a_2, \hat{b}_2] \times \cdots \times [a_n, \hat{b}_n]$. Since the intervals $[a_1, b_1], [a_2, \hat{b}_2], \ldots, [a_n, \hat{b}_n]$ are equi-rational, the interval $\hat{I}$ can be split into finite number of cubes. Let

$$\hat{J} = [\hat{c}_1, \hat{d}_1] \times \cdots \times [\hat{c}_n, \hat{d}_n]$$

be one of them. Consider the interval

$$L^{-1}(\hat{J}) = [c_1, d_1] \times \cdots \times [c_n, d_n].$$

Note that $d_1 - c_1 = \hat{d}_i - \hat{c}_i$ for each $i = 1, \ldots, n$. Then, (1) implies

$$1 \ge \frac{d_i - c_i}{\hat{d}_i - \hat{c}_i} > \alpha$$

and so $1 \ge \frac{d_i - c_i}{d_1 - c_1} > \alpha$. As $[c_1, d_1]$ is the longest edge of $L^{-1}(\hat{J})$, $r(L^{-1}(\hat{J})) > \alpha$. In this way, the interval $I = L^{-1}(\hat{I})$ is split into finite number of α-regular intervals of the kind $L^{-1}(\hat{J})$. □

As a corollary of the above lemma we get

Theorem 3. *For any $\alpha \in (0,1)$, a function f on an interval I_0 is M_α-integrable if and only if it is M-integrable. The values of integrals agree.*

Proof. It is clear that M-integrability implies M_α-integrability with the same integral. In the opposite direction, it is enough to notice that for any $(I, x) \in \mathcal{F}$,

$$f(x)|I| = f(x)|I_1| + \cdots + f(x)|I_N|,$$

where $I_1, \ldots, I_N$ form an α-regular partition of I (Lemma 2), and so if $(I, x) \in \mathcal{F}_\delta$ for some δ, then $(I_1, x), \ldots, (I_N, x) \in \mathcal{F}_\delta^\alpha$. □

Corollary 4. *All M_α-integrals, $\alpha \in (0,1)$, are equivalent.*

It is proved in [1] that M_α-integral is equivalent to SP_α-integral. Theorem 3 and the foregoing corollary imply that all SP_α-integrals coincide and are equivalent to the M-integral. Recalling a diagram obtained at the end of [1], we can get the following its simplification:

$$M = \bigcap_{\alpha\in(0,1)} M_\alpha = M_\alpha = SP_\alpha = \bigcap_{\alpha\in(0,1)} SP_\alpha.$$

Now, let us refer to some results from [3]. Consider an M-integrable function f on I_0. For $\epsilon > 0$ take a gauge δ suitable for ϵ according to Definition 1. For an interval $J \subset I_0$ we define

$$\Phi_\delta^f(J) = \sup_{\mathcal{P}} \sum_{(K,t)\in\mathcal{P}} f(t)|K|, \qquad \phi_\delta^f(J) = \inf_{\mathcal{P}} \sum_{(K,t)\in\mathcal{P}} f(t)|K|,$$

where sup and inf range over all $\mathcal{F}_\delta$-divisions $\mathcal{P}$ such that $\bigcup_{(K,t)\in\mathcal{P}} K = J$; note that $K \subset J$ but not necessarily $t \in J$. Due to a Saks-Henstock-type lemma we have $\Phi_\delta^f(I_0) - \phi_\delta^f(I_0) \leq 2\epsilon$. It follows from [3] that Φ_δ^f and ϕ_δ^f are respectively strong major and strong minor functions of f. So, f is strongly[1] Perron integrable (SP-integrable) [1, Definition 2.2]. In consequence, $M \subset SP$ and thus

$$M = SP.$$

That means, all the integrals considered in [1] coincide and are equivalent to the McShane integral.

We remark in the conclusion that in [3] it has been shown that Φ_δ^f and ϕ_δ^f are *additive* and *continuous* (in the standard sense). It implies that also in the multidimensional case, the Perron integral defined with strong major and strong minor functions which are not assumed to be either additive (only super-/sub-additive) or continuous, like in [1, Definition 2.2], is equivalent to the Perron integral defined with strong major and strong minor functions which are assumed to be both additive and continuous.

References

[1] J. M. Park, B. M. Kim, and D. H. Lee, *The strong Perron integral in the n-dimensional space* $\mathbb{R}^n$, Commun. Korean Math. Soc. **20** (2005), no. 2, 291–297.

[2] S. Saks, *Theory of the integral*, Dover, New York, 1964.

[3] V. A. Skvortsov and P. Sworowski, *On McShane-type integrals with respect to some derivation bases*, Mathematica Bohemica **131** (2006), no. 4, 365–378.

[1]Perhaps it would be better to keep the name *strong Perron* for the integral defined with *strong derivatives* in standard Saks' terminology [2]. In [3] we refer to an integral of this type as to *McShane-Perron* integral.

Valentin A. Skvortsov
Moscow State University
Department of Mathematics
11992 Moscow, Russia

Valentin A. Skvortsov and Piotr Sworowski
Casimirus the Great University
Department of Mathematics
Plac Weyssenhoffa 11
85–072 Bydgoszcz, Poland
E-mail address: vaskvor2000@yahoo.com
E-mail address: piotrus@ukw.edu.pl