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SUBTRACTION ALGEBRAS WITH ADDITIONAL CONDITIONS

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ABSTRACT. Subtraction algebras with additional conditions, so called complicated subtraction algebras, are introduced, and several properties are investigated. In a complicated subtraction algebra, characterizations of ideals are provided, and showed that the set of all ideals in a complicated subtraction algebra is a complete lattice.

1. Introduction

B. M. Schein [7] considered systems of the form $(\Phi; \cdot, \setminus)$, where Φ is a set of functions closed under the composition "$\cdot$" of functions (and hence $(\Phi; \cdot)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [8] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [6] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In this paper, we discuss subtraction algebras with additional conditions, so called complicated subtraction algebra (c-subtraction algebra, for short), and investigate several properties. We provide characterizations of ideals in c-subtraction algebras, and showed that the set of all ideals in a complicated subtraction algebra is a complete lattice.

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2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X;-)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x-(y-x)=x$;
(S2) $x-(x-y)=y-(y-x)$;
(S3) $(x-y)-z=(x-z)-y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a-(a-b)$; the complement of an element $b \in [0,a]$ is $a-b$; and if $b, c \in [0,a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a-((a-b) \wedge (a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [5, 6]):

(a1) $(x-y)-y=x-y$.
(a2) $x-0=x$ and $0-x=0$.
(a3) $(x-y)-x=0$.
(a4) $x-(x-y) \leq y$.
(a5) $(x-y)-(y-x)=x-y$.
(a6) $x-(x-(x-y))=x-y$.
(a7) $(x-y)-(z-y) \leq x-z$.
(a8) $x \leq y$ if and only if $x=y-w$ for some $w \in X$.
(a9) $x \leq y$ implies $x-z \leq y-z$ and $z-y \leq z-x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x-y=x \wedge (z-y)$.
(a11) $(x \wedge y)-(x \wedge z) \leq x \wedge (y-z)$.
(a12) $(x-y)-z=(x-z)-(y-z)$.

Definition 2.1. [5] A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(1) $0 \in A$
(2) $(\forall x \in X)(\forall y \in A)(x-y \in A \Rightarrow x \in A)$.

Lemma 2.2. [6] *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Complicated subtraction algebras

Let X be a subtraction algebra. For any $a, b \in X$, we define

$$\mathscr{G}(a,b)=\{x \in X \mid x-a \leq b\}.$$

Note that $\mathscr{G}(a,b)$ is nonempty since $0, a, b \in \mathscr{G}(a,b)$.

Example 3.1. (1) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows.

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $\mathscr{G}(a,b) = \{0,a,b\}$, $\mathscr{G}(a,c) = \{0,a,c\}$, $\mathscr{G}(a,d) = \{0,a,d\}$, $\mathscr{G}(a,a) = \{0,a\}$, $\mathscr{G}(b,b) = \{0,b\}$, etc.

(2) Let $X = \{0, a, b, c\}$ be a subtraction algebra with the following Cayley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $\mathscr{G}(0,0) = \{0\}$, $\mathscr{G}(0,a) = \{0,a\} = \mathscr{G}(a,a)$, $\mathscr{G}(0,b) = \{0,b\} = \mathscr{G}(b,b)$, $\mathscr{G}(0,c) = X = \mathscr{G}(a,b) = \mathscr{G}(a,c) = \mathscr{G}(b,c) = \mathscr{G}(c,c)$.

Definition 3.2. A subtraction algebra X is said to be *complicated* if for any $a, b \in X$ the set $\mathscr{G}(a,b)$ has the greatest element.

The greatest element of $\mathscr{G}(a,b)$ is denoted by $a + b$.

Example 3.3. The subtraction algebra X in Example 3.1(2) is a complicated subtraction algebra. In fact, $0+0 = 0$, $0+a = a$, $0+b = b$, $0+c = c$, $a+a = a$, $a+b = c$, $a+c = c$, $b+b = b$, $b+c = c$, and $c+c = c$. But the subtraction algebra X in Example 3.1(1) is not a complicated subtraction algebra since $\mathscr{G}(a,b)$ has no the greatest element.

Proposition 3.4. *If X is a complicated subtraction algebra (c-subtraction algebra, for short), then*

(i) $(\forall a, b \in X)\ (a \leq a+b,\ b \leq a+b)$,
(ii) $(\forall a \in X)\ (a+0 = a = 0+a)$,
(iii) $(\forall a, b \in X)\ (a+b = b+a)$.

Proof. (i) and (ii) are straightforward.

(iii) Note from (S3) that

$$(\forall x, a, b \in X)\,(x - a \leq b \Leftrightarrow x - b \leq a).$$

Thus $\mathscr{G}(a,b) = \mathscr{G}(b,a)$, and so $a+b = b+a$. □

Proposition 3.5. *In a c-subtraction algebra X, the operator "+" is order preserving with respect to the ordering $\leq$, that is,*

$$(\forall x, y, z \in X)\,(x \leq y \Rightarrow x + z \leq y + z).$$

Proof. Let $x, y, z \in X$ satisfy $x \leq y$. Using (a9), we have

$$(x + z) - y \leq (x + z) - x \leq z,$$

and so $x + z \leq y + z$. □

Theorem 3.6. *If X is a c-subtraction algebra, then $(X, +, 0)$ is a commutative monoid.*

Proof. It is sufficient to show that X satisfies the associative law under the operation "+", that is,

$$(\forall a, b, c \in X)\,((a + b) + c = a + (b + c)).$$

Note that $((a + b) + c) - c \leq a + b$ for all $a, b, c \in X$. It follows from (S3) and (a9) that

$$(((a + b) + c) - b) - c = (((a + b) + c) - c) - b \leq (a + b) - b \leq a$$

so that $((a+b)+c)-b \leq c+a = a+c$. Hence $(a+b)+c \leq b+(a+c) = (a+c)+b$, and so $(a + b) + c = (a + c) + b$. Therefore

$$(a + b) + c = (b + a) + c = (b + c) + a = a + (b + c).$$

This completes the proof. □

Example 3.7. Consider the subtraction algebra X in Example 3.1(2). Then $(X, +, 0)$ is a commutative monoid, where + is given by the following Cayley table:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Remark 3.8. By means of Example 3.7, we know that a c-subtraction algebra has no group structure under the operation +.

Theorem 3.9. *Let X be a c-subtraction algebra and $a, b \in X$. Then the set*

$$\mathscr{H}(a, b) := \{x \in X \mid a \leq b + x\}$$

has the least element, and it is $a - b$.

Proof. The inequality $a - b \leq a - b$ implies that $a \leq b + (a - b)$, and so $a - b \in \mathscr{H}(a, b)$. Let $z \in \mathscr{H}(a, b)$. Then $a \leq b + z$, which implies from (a9) that $a - b \leq (b + z) - b \leq z$. Hence $a - b$ is the least element of $\mathscr{H}(a, b)$. □

Proposition 3.10. *Any c-subtraction algebra X satisfies the following axioms:*

(i) $(\forall x, y, z \in X)\ ((x - y) - z = x - (y + z))$,

(ii) $(\forall x, y, z \in X)$ $(x - y \leq (x - z) + (z - y))$,
(iii) $(\forall x, y, z \in X)$ $((x + z) - (y + z) \leq x - y \leq x + y)$,
(iv) $(\forall x, y \in X)$ $(x \leq y \Rightarrow x + y = y)$,
(v) $(\forall x, y, z \in X)$ $((x + y) - z = (x - z) + (y - z))$,
(vi) $(\forall x, y \in X)$ ($x + y$ is the least upper bound of x and y),
(vii) $(\forall x, y \in X)$ $(x + y = x + (y - x))$.

Proof. Let $x, y, z \in X$. Using (S3) and (a7), we have

$$(x - y) - (x - (y + z)) \leq (y + z) - y \leq z,$$

and so $(x - y) - z \leq x - (y + z)$. Now using (S3), (a4) and (a7), we have

$$(x - ((x - y) - z)) - z = (x - z) - ((x - y) - z) \leq x - (x - y) \leq y,$$

which implies that $x - ((x - y) - z) \leq z + y = y + z$. It follows from (S3) that

$$(x - (y + z)) - ((x - y) - z) = (x - ((x - y) - z)) - (y + z) = 0$$

so that $x - (y + z) \leq (x - y) - z$. Consequently, (i) is valid. Using (S3) and (a7), we get $(x - y) - (x - z) \leq z - y$, and so $x - y \leq (x - z) + (z - y)$. This proves (ii). Now, (a4) implies $x \leq (x - y) + y$. Proposition 3.5 and Theorem 3.6 induce

$$x + z \leq ((x - y) + y) + z = (x - y) + (y + z).$$

Hence

$$(x + z) - (y + z) \leq ((x - y) + (y + z)) - (y + z) \leq x - y \leq x = x + 0 \leq x + y.$$

Therefore (iii) is valid. Now let $x, y \in X$ satisfy $x \leq y$. Since $(x + y) - x \leq y$, it follows from (a2) and (a12) that

$$\begin{aligned} (x + y) - y &= ((x + y) - y) - 0 \\ &= ((x + y) - y) - (x - y) \\ &= ((x + y) - x) - y \\ &= 0 \end{aligned}$$

so that $x + y \leq y$. Since $y \leq x + y$ for all $x, y \in X$, we have $x + y = y$. Hence (iv) is valid. Form (iv), we get $x + x = x$ for all $x \in X$. Using (iii), we get

$$(x + y) - (y + z) = (x + y) - (z + y) \leq x - z,$$

and thus $(x + y) - (x - z) \leq y + z$. It follows from (S3), (a9) and (iii) that

$$\begin{aligned} ((x + y) - z) - (x - z) &= ((x + y) - (x - z)) - z \\ &\leq (y + z) - z \\ &= (y + z) - (z + z) \\ &\leq y - z \end{aligned}$$

so that $(x + y) - z \leq (x - z) + (y - z)$. Using Proposition 3.4(i) and (a9), we obtain $x - z \leq (x + y) - z$ and $y - z \leq (x + y) - z$ for all $x, y, z \in X$. Applying

Proposition 3.5, we get

$$\begin{aligned}(x-z)+(y-z) &\leq ((x+y)-z)+(y-z)\\ &\leq ((x+y)-z)+((x+y)-z)\\ &= (x+y)-z.\end{aligned}$$

Consequently, (v) is valid. By means of Proposition 3.4(i), $x+y$ is an upper bound of x and y. Let z be an upper bound of x and y. Then

$$(x+y)-z=(x-z)+(y-z)=0+0=0,$$

that is, $x+y \leq z$. Hence $x+y$ is the least upper bound of x and y. Theorem 3.9 implies that $y \leq x+(y-x)$ for all $x, y \in X$. Since $x+x=x$ for all $x \in X$, it follows from Proposition 3.5 that

$$x+y \leq x+(x+(y-x))=(x+x)+(y-x)=x+(y-x).$$

Combining (a3) and Proposition 3.5 induce $x+(y-x) \leq x+y$ for all $x, y \in X$. Hence $x+y=x+(y-x)$ for all $x, y \in X$, that is, (vii) is valid. This completes the proof. □

Remark 3.11. Note that a subtraction algebra X is a semilattice under the operation $\wedge$. By Proposition 3.10(vi), we know that a c-subtraction algebra X is a semilattice under the operation $+$. Hence any c-subtraction algebra X is a lattice with respect to operations $\wedge$ and $+$.

We provide characterizations of ideals in a c-subtraction algebra.

Theorem 3.12. *Let A be a nonempty subset of a c-subtraction algebra X. Then A is an ideal of X if and only if it satisfies the following conditions*:

(i) $(\forall x \in A)\ (\forall y \in X)\ (y \leq x \Rightarrow y \in A)$,
(ii) $(\forall x, y \in A)\ (\exists\, z \in A)\ (x \leq z,\ y \leq z)$.

Proof. Assume that A is an ideal of X. The condition (i) follows from Lemma 2.2. Let $x, y \in A$. Since $(x+y)-x \leq y$ and $y \in A$, it follows from (i) that $(x+y)-x \in A$ so that $x+y \in A$ because A is an ideal of X. If we take $z=x+y$, then $x \leq z$ and $y \leq z$ by Proposition 3.4(i) which proves (ii). Conversely, let A be a nonempty subset of X satisfying conditions (i) and (ii). Since A is nonempty, we have $0 \in A$ by (i) and (a2). Let $x, y \in X$ satisfy $y \in A$ and $x-y \in A$. Then, by (ii), there exists $z \in A$ such that $y \leq z$ and $x-y \leq z$. It follows from (a2) and (a12) that

$$x-z=(x-z)-0=(x-z)-(y-z)=(x-y)-z=0$$

so that $x \leq z$. Since $z \in A$, it follows from (i) that $x \in A$. Hence A is an ideal of X. □

Theorem 3.13. *Let A be a nonempty subset of a c-subtraction algebra X. Then A is an ideal of X if and only if it satisfies the following conditions*:

(i) $(\forall x \in A)\ (\forall y \in X)\ (y \leq x \Rightarrow y \in A)$,
(ii) $(\forall x, y \in X)\ (x, y \in A \Rightarrow x+y \in A)$.

Proof. The necessity is induced in the proof of Theorem 3.12. Let A be a nonempty subset of X satisfying conditions (i) and (ii). Obviously $0 \in A$ by (i) and (a2). Let $x, y \in X$ satisfy $y \in A$ and $x - y \in A$. Then $y + (x - y) \in A$ by (ii). Since $x \leq y + (x - y)$ by Theorem 3.9, it follows from (i) that $x \in A$ so that A is an ideal of X. □

Denote by $\mathscr{I}(X)$ the set of all ideals in a subtraction algebra X. Applying [3, Theorem 1], we have the following theorem.

Theorem 3.14. *If X is a c-subtraction algebra, then $(\mathscr{I}(X), \subseteq)$ is a complete lattice.*

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