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HYBRID MEAN VALUE OF GENERALIZED BERNOULLI NUMBERS, GENERAL KLOOSTERMAN SUMS AND GAUSS SUMS

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ABSTRACT. The main purpose of this paper is to use the properties of primitive characters, Gauss sums and Ramanujan's sum to study the hybrid mean value of generalized Bernoulli numbers, general Kloosterman sums and Gauss sums, and give two asymptotic formulae.

1. Introduction

Let χ be a non-principal Dirichlet character modulo q. The generalized Bernoulli numbers $B_{k,\chi}$ are defined by the following:

$$\sum_{a=1}^{q}\chi(a)\frac{te^{at}}{e^{qt}-1}=\sum_{k=0}^{+\infty}\frac{B_{k,\chi}}{k!}t^k.$$

This sequence of numbers has considerable fascination and importance. The definition and basic properties of generalized Bernoulli numbers can be found in [4].

It is surprising that generalized Bernoulli numbers enjoy good value distribution properties in some problems of weighted mean value. The authors [5] used the properties of primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of Gauss sums and generalized Bernoulli numbers, and give a sharper asymptotic formula.

It might be interesting to study the hybrid mean value of $B_{k,\chi}$ and other arithmetical functions. For any integers m and n, the general Kloosterman sums $K(m,n,\chi;q)$ are defined by:

$$K(m,n,\chi;q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\chi(a)e\left(\frac{ma+n\bar{a}}{q}\right),$$

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where χ denotes a Dirichlet character modulo q, $a\bar{a} \equiv 1(\bmod q)$ and $\mathrm{e}(y) = e^{2\pi iy}$. This summation is a generalization of the classical Kloosterman sums

$$K(m,n;q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \mathrm{e}\left(\frac{ma+n\bar{a}}{q}\right).$$

For $q = p$ be a prime, S. Chowla [2] and A. V. Malyshev [6] obtained a sharper upper bound estimation for $K(m,n,\chi;p)$. That is

$$|K(m,n,\chi;p)| \ll (m,n,p)^{\frac{1}{2}} p^{\frac{1}{2}+\epsilon},$$

where (m,n,p) denotes the greatest common divisor of m, n and p, and ϵ is any fixed positive number. But for arbitrary composite number q, one does not know how large $|K(m,n,\chi;q)|$ is.

The general Kloosterman sums also enjoy good distribution properties. In [11] and [12], the second author studied the fourth power mean of $K(m,n,\chi;q)$. Moreover, he researched the mean value of $K(m,n,\chi;q)$ with the weight of Dirichlet L- functions, and gave a few interesting formulae (see [14] and [9]).

The first purpose of this paper is to use the properties of Gauss sums and Ramanujan's sum to study the mean square value of general Kloosterman sums and generalized Bernoulli numbers, and give a sharper asymptotic formula. That is the following:

Theorem 1.1. *Let $q \geq 3$ be an integer. Then for any given integers m, n and k with $(mn,q) = 1$ and $k > 1$, we have*

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} |K(m,n,\chi;q)|^2 \, |B_{k,\chi}|^2 = \frac{2(k!)^2\zeta(2k)}{(2\pi)^{2k}} q^{2(k-1)}\phi^3(q) + O\left(q^{2k+\frac{1}{2}+\epsilon}\right),$$

where $\sum_{\chi\neq\chi_0 \chi \bmod q}$ denotes the summation over all non-principal characters modulo q, $\zeta(s)$ is the Riemann zeta function, $\phi(q)$ is the Euler function, and ϵ is any fixed positive number.

Let $q \geq 3$ be an integer, and let χ denote a Dirichlet character modulo q. For any integer n, the Gauss sum $G(n,\chi)$ is defined as the following:

$$G(n,\chi) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \chi(a)\mathrm{e}\left(\frac{an}{q}\right).$$

When $\chi = \chi_0$ is the principal character, $G(n,\chi_0) = C_q(n)$ is the Ramanujan's sum. Especially for $n = 1$, we write $\tau(\chi) = \sum_{a=1 (a,q)=1}^{q} \chi(a)\mathrm{e}\left(\frac{a}{q}\right)$. The various properties and applications of $\tau(\chi)$ appear in many analytic number theory books (see [1]).

Maybe the most important property of $\tau(\chi)$ is that if χ is a primitive character modulo q, then

$$|\tau(\chi)| = \sqrt{q}.$$

Even if χ is a non-primitive character modulo q, $\tau(\chi)$ also has many good value distribution properties in some problems of weighted mean value. For example, Y. Yi [8] studied the $2k$-th power mean of inversion of L-functions with the weight of Gauss sums, and gave some interesting formulae.

The second purpose of this paper is to use the properties of primitive characters and the estimate for classical Kloosterman sums to study the hybrid mean value of general Kloosterman sums, Gauss sums and generalized Bernoulli numbers, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem 1.2. *Let $q \geq 3$ be an integer. Then for any fixed integers m, n, k and h with $(mn, q) = 1$, $k > 1$ and $h > 0$, we have*

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} |K(m,n,\chi;q)|^2 \tau^h(\overline{\chi}) B_{k,\chi}^h$$
$$= \frac{2^{h-1}(k!)^h q^{kh-1}\phi^3(q)}{(-1)^{(k-1)h}(2\pi i)^{kh}} \prod_{p \| q} \left(1 - \frac{p^{h-1}-1}{p^{h-1}(p-1)^2}\right) + O\left(q^{kh+\frac{3}{2}+\epsilon}\right),$$

where $\prod_{p\|q}$ denotes the product over all prime divisors p of q with $p \mid q$ and $p^2 \nmid q$.

2. Some lemmas

To complete the proof of the theorems, we need the following lemmas.

Lemma 2.1. *Let $q \geq 3$ be an integer. Then for any given integer $k > 1$, we have*

$$\sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} G(r_1,\chi)\overline{G(r_2,\chi)} = 2\zeta(2k)\phi^2(q) + O\left(q^{1+\epsilon}\right).$$

Proof. From the orthogonality relations for character sums we have

$$\sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} G(r_1,\chi)\overline{G(r_2,\chi)}$$
$$= \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\chi \bmod q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(a) e\left(\frac{ar_1}{q}\right)$$

$$\times \sum_{\substack{b=1\\(b,q)=1}}^{q} \overline{\chi}(b)\mathrm{e}\left(-\frac{br_2}{q}\right) - \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2$$

$$= \phi(q) \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{a=1\\(a,q)=1}}^{q} \mathrm{e}\left(\frac{a(r_1-r_2)}{q}\right) - \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2$$

$$= \phi(q) \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{C_q(r_1-r_2)}{r_1^k r_2^k} - \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2 .$$

Note that

$$C_q(r) = \sum_{d|(q,r)} d\mu\left(\frac{q}{d}\right), \tag{2.1}$$

where $\mu(n)$ is the Möbius function. Then we have

$$\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k} = \sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{1}{r^k} \sum_{d|(q,r)} d\mu\left(\frac{q}{d}\right) = \sum_{d|q} \sum_{\substack{l=-\infty\\l\neq 0}}^{+\infty} \frac{d\mu\left(\frac{q}{d}\right)}{(ld)^k} \ll q^{\epsilon} \tag{2.2}$$

and

$$\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{C_q(r_1-r_2)}{r_1^k r_2^k} = \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{d|(q,r_1-r_2)} d\mu\left(\frac{q}{d}\right)$$

$$= \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0\\r_1\equiv r_2 \bmod d}}^{+\infty} \frac{1}{r_1^k r_2^k}$$

$$= \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{1}{r^{2k}} + \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \sum_{\substack{l=-\infty\\ld+r\neq 0\\l\neq 0}}^{+\infty} \frac{1}{r^k(ld+r)^k}$$

$$= 2\zeta(2k)\phi(q) + O\left(q^{\epsilon}\right).$$

So we have

$$\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}} G(r_1,\chi)\overline{G(r_2,\chi)} = 2\zeta(2k)\phi^2(q) + O\left(q^{1+\epsilon}\right).$$

□

Lemma 2.2. [3] *Let m, n and q be integers with $q \geq 3$. Then we have the estimate*

$$K(m, n; q) \ll (m, n, q)^{\frac{1}{2}} q^{\frac{1}{2}} d(q),$$

where $d(q)$ is the divisor function.

Lemma 2.3. *Let $q \geq 3$ be an integer. Then for any integers m and n with $(mn, q) = 1$, we have*

$$\sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{q}\right) \ll q^{\frac{1}{2}+\epsilon}(a-1, q)^{\frac{1}{2}},$$

$$\sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1, q)^{\frac{1}{2}} \ll q^{1+\epsilon},$$

and

$$\sum_{\substack{a=2 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{q}\right) \ll q^{\frac{3}{2}+\epsilon}.$$

Proof. Note that $(m, q) = (n, q) = (a, q) = 1$, and

$$(a-1, q) = (a\bar{a} - a, q) = (a(\bar{a}-1), q) = (\bar{a}-1, q),$$

we have

$$(m(a-1), n(\bar{a}-1), q) = (a-1, q).$$

Then from Lemma 2.2 we get

$$\sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{q}\right) \ll (m(a-1), n(\bar{a}-1), q)^{\frac{1}{2}} q^{\frac{1}{2}} d(q)$$

$$\ll q^{\frac{1}{2}+\epsilon}(a-1, q)^{\frac{1}{2}}.$$

On the other hand,

$$\sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1, q)^{\frac{1}{2}} \ll \sum_{d|q} \sum_{\frac{1}{d} \leq l \leq \frac{q}{d}} d^{\frac{1}{2}} \ll q^{1+\epsilon}.$$

Then we have

$$\sum_{\substack{a=2 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1) + n\bar{b}(\bar{a}-1)}{q}\right) \ll q^{\frac{1}{2}+\epsilon} \sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1, q)^{\frac{1}{2}}$$

$$\ll q^{\frac{3}{2}+\epsilon}.$$

□

Lemma 2.4. *Let $q \geq 3$ be an integer. Then for any given integers m, n and k with $(mn, q) = 1$ and $k > 1$, we have*

$$\begin{aligned}\Psi &:= \sum_{\substack{a=2\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)\\ &\quad\times \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \chi(a) G(r_1,\chi)\overline{G(r_2,\chi)}\\ &\ll q^{\frac{5}{2}+\epsilon}.\end{aligned}$$

Proof. From the properties of character sums we have

$$\begin{aligned}\Psi &= \sum_{\substack{a=2\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k}\\ &\quad\times \sum_{\chi \bmod q} \chi(a) \sum_{\substack{s=1\\(s,q)=1}}^{q} \chi(s) e\left(\frac{sr_1}{q}\right) \sum_{\substack{t=1\\(t,q)=1}}^{q} \overline{\chi}(t) e\left(-\frac{tr_2}{q}\right)\\ &\quad- \sum_{\substack{a=2\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2\\ &= \phi(q) \sum_{\substack{a=2\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)\\ &\quad\times \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty} \frac{C_q(r_1-ar_2)}{r_1^k r_2^k}\\ &\quad- \sum_{\substack{a=2\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2.\end{aligned}$$

Then from (2.2) and Lemma 2.3 we easily get

$$\sum_{a=2}^{q}{}' \sum_{b=1}^{q}{}' e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \left(\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty} \frac{C_q(r)}{r^k}\right)^2 \ll q^{\frac{3}{2}+\epsilon}.$$

On the other hand, by Lemma 2.3 and formula (2.1) we also have

$$\sum_{\substack{a=2\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{C_q(r_1-ar_2)}{r_1^k r_2^k}$$

$$= \sum_{\substack{a=2\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)$$

$$\times \sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{1}{r_1^k r_2^k}\sum_{d|(q,r_1-ar_2)} d\mu\left(\frac{q}{d}\right)$$

$$\ll q^{\frac{1}{2}+\epsilon}\sum_{d|q} d \sum_{\substack{a=2\\(a,q)=1}}^{q}(a-1,q)^{\frac{1}{2}}\sum_{\substack{r_1=-\infty\\r_1\neq 0\\r_1\equiv ar_2 \bmod d}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{1}{r_1^k r_2^k}$$

$$= q^{\frac{1}{2}+\epsilon}\sum_{d|q} d \sum_{\substack{a=2\\(a,q)=1}}^{q}(a-1,q)^{\frac{1}{2}}\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty}\frac{1}{a^k r^{2k}}$$

$$+q^{\frac{1}{2}+\epsilon}\sum_{d|q} d \sum_{\substack{a=2\\(a,q)=1}}^{q}(a-1,q)^{\frac{1}{2}}\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty}\sum_{\substack{l=-\infty\\ld+ar\neq 0\\l\neq 0}}^{+\infty}\frac{1}{r^k(ld+ar)^k}$$

$$\ll q^{\frac{3}{2}+\epsilon}\sum_{\substack{a=2\\(a,q)=1}}^{q}\frac{(a-1,q)^{\frac{1}{2}}}{a^k}+q^{\frac{1}{2}+\epsilon}\sum_{a=2}^{q}(a-1,q)^{\frac{1}{2}} \ll q^{\frac{3}{2}+\epsilon}.$$

So from the above we have

$$\Psi \ll q^{\frac{5}{2}+\epsilon}.$$

□

Lemma 2.5. [5] *Let $q \geq 3$ be an integer. Then for any positive integers k and h we have*

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \tau^h\left(\overline{\chi}\right) B_{k,\chi}^h$$

$$= \frac{2^{h-1}(k!)^h q^{kh-1}\phi^2(q)}{(-1)^{(k-1)h}\left(2\pi i\right)^{kh}}\prod_{p\|q}\left(1-\frac{p^{h-1}-1}{p^{h-1}\left(p-1\right)^2}\right)+O\left(q^{kh+\epsilon}\right).$$

It is well-known that (see [7]), for every character $\chi \bmod q$, there exists a unique positive integer q^*, and a unique primitive character $\chi^* \bmod q^*$ such that

$$\chi^*(n) = \chi(n) \quad \text{for all } n, (n,q) = 1.$$

We call q^* the conductor of χ, and χ^* the primitive character corresponding to χ. Conversely, suppose χ^* is a primitive character modulo q^*, q a positive integer, and $q^* \mid q$. Then there exists a unique character χ mod q such that

$$\chi(n) = \chi^*(n) \quad \text{for all } n,\ (n,q) = 1.$$

χ is said to be the character modulo q induced by χ^*. For convenience, the correlation between χ and χ^* is usually written in any of the notations:

$$\chi \mod q \longleftrightarrow \chi^* \bmod q^*, \qquad \chi_q \longleftrightarrow \chi^*_{q^*},$$

or, for short, $\chi \longleftrightarrow \chi^*$.

Lemma 2.6. [7] *For any integer $q \geq 3$, let χ be a non-primitive character modulo q, and let q^* denote the conductor of χ with $\chi \longleftrightarrow \chi^*$. If $(n,q) > 1$, we have*

$$G(n,\chi) = \begin{cases} \overline{\chi}^*\left(\frac{n}{(n,q)}\right)\chi^*\left(\frac{q}{q^*(n,q)}\right)\mu\left(\frac{q}{q^*(n,q)}\right)\phi(q)\phi^{-1}\left(\frac{q}{(n,q)}\right)\tau(\chi^*), & q^* = \frac{q_1}{(n,q_1)}; \\ 0, & q^* \neq \frac{q_1}{(n,q_1)}, \end{cases}$$

where q_1 is the largest divisor of q that has the same prime factors with q^*.

If $(n,q) = 1$, then we have

$$G(n,\chi) = \overline{\chi}^*(n)\chi^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\chi^*).$$

Lemma 2.7. *Let q and r be integers with $q \geq 3$ and $(r,q) = 1$, and let χ be a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d \mid (q, r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d \mid q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of primitive characters modulo q.

Proof. This is Lemma 3 of [10]. Also, one can see Lemma 4 of [13]. □

Lemma 2.8. *Let $q = uv$, with $(u,v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then for any fixed integers m, n, k, h with $(mn,q) =$*

1, $k > 1$ and $h > 0$, we have

$$\begin{aligned}\Psi_1 \quad := \quad & \sum_{d|v}(ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\ & \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}} \chi(a) \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h \\ \ll \quad & q^{h+\frac{3}{2}+\epsilon}\end{aligned}$$

and

$$\begin{aligned}\Psi_2 \quad := \quad & \sum_{d|v}(ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\ & \times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \chi(a) \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h \\ \ll \quad & q^{h+\frac{3}{2}+\epsilon}.\end{aligned}$$

Proof. We only prove the second estimate, since similarly we can deduce the first one. Let $\tau_h(r)$ denote the h-th divisor function (i.e., the number of positive integer solutions of the equation $r = r_1 r_2 \cdots r_h$). Note that $J(u) = \phi^2(u)/u$, if u is a square-full number. Then using Lemma 2.7 and Lemma 2.3 we have

$$\begin{aligned}&\Psi_2 \\ &= \sum_{d|v}(ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\ &\times \sum_{t_1|\frac{v}{d}} \cdots \sum_{t_h|\frac{v}{d}} \sum_{r=1}^{+\infty} \frac{\mu(t_1)\cdots\mu(t_h)\phi(t_1)\cdots\phi(t_h)\tau_h(r)}{t_1^k\cdots t_h^k r^k} \\ &\times \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \chi(a)\overline{\chi}(t_1\cdots t_h r) \\ &= \frac{1}{2}\sum_{d|v}(ud)^h \sum_{s|ud} \mu\left(\frac{ud}{s}\right)\phi(s) \sum_{\substack{a=2 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)\end{aligned}$$

$$\times \sum_{t_1 | \frac{v}{d}} \cdots \sum_{t_h | \frac{v}{d}} \sum_{\substack{r=1 \\ (r,ud)=1 \\ t_1 \cdots t_h r \equiv a \bmod s}}^{+\infty} \frac{\mu(t_1)\cdots\mu(t_h)\phi(t_1)\cdots\phi(t_h)\tau_h(r)}{t_1^k \cdots t_h^k r^k}$$

$$-\frac{1}{2}\sum_{d|v}(ud)^h \sum_{s|ud} \mu\left(\frac{ud}{s}\right)\phi(s) \sum_{\substack{a=2 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right)$$

$$\times \sum_{t_1 | \frac{v}{d}} \cdots \sum_{t_h | \frac{v}{d}} \sum_{\substack{r=1 \\ (r,ud)=1 \\ t_1 \cdots t_h r \equiv -a \bmod s}}^{+\infty} \frac{\mu(t_1)\cdots\mu(t_h)\phi(t_1)\cdots\phi(t_h)\tau_h(r)}{t_1^k \cdots t_h^k r^k}$$

$$\ll q^{\frac{1}{2}+\epsilon}\sum_{d|v}(ud)^h \sum_{s|ud}\phi(s) \sum_{\substack{a=2 \\ (a,q)=1}}^{q} \frac{(a-1,q)^{\frac{1}{2}}}{a^{k-1+\epsilon}}$$

$$+ q^{\frac{1}{2}+\epsilon}\sum_{d|v}(ud)^h \sum_{s|ud}\phi(s) \sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1,q)^{\frac{1}{2}} \sum_{\substack{\frac{1-a}{s} \le l < +\infty \\ l \neq 0}} \frac{1}{(ls+a)^{k-1+\epsilon}}$$

$$+ q^{\frac{1}{2}+\epsilon}\sum_{d|v}(ud)^h \sum_{s|ud}\phi(s) \sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1,q)^{\frac{1}{2}} \sum_{\frac{1+a}{s} \le l < +\infty} \frac{1}{(ls-a)^{k-1+\epsilon}}$$

$$\ll q^{h+\frac{3}{2}+\epsilon} \sum_{\substack{a=2 \\ (a,q)=1}}^{q} \frac{(a-1,q)^{\frac{1}{2}}}{a^{k-1+\epsilon}} + q^{h+\frac{1}{2}+\epsilon} \sum_{\substack{a=2 \\ (a,q)=1}}^{q} (a-1,q)^{\frac{1}{2}}$$

$$\ll q^{h+\frac{3}{2}+\epsilon} \sum_{d|q} \sum_{\frac{1}{d} \le l \le \frac{q}{d}} \frac{d^{\frac{1}{2}}}{(ld+1)^{k-1+\epsilon}} + q^{h+\frac{3}{2}+\epsilon} \ll q^{h+\frac{3}{2}+\epsilon}.$$

This proves Lemma 2.8. □

3. Proof of the theorems

In this section, we complete the proof of the theorems. For any complex x the Bernoulli polynomials are defined by the equation

$$\frac{z\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \qquad \text{where} \quad |z| < 2\pi.$$

Let $q \geq 3$ be an integer, and χ be a Dirichlet character modulo q. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_{k,\chi} = q^{k-1} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(a) B_k\left(\frac{a}{q}\right).$$

From Theorem 12.19 of [1] we also have

$$B_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\mathrm{e}(rx)}{r^k}, \qquad \text{if } 0 < x \leq 1.$$

Therefore

$$\begin{aligned} B_{k,\chi} &= q^{k-1} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(a) \left[-\frac{k!}{(2\pi i)^k} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\mathrm{e}\left(\frac{ar}{q}\right)}{r^k} \right] \\ &= -\frac{k! q^{k-1}}{(2\pi i)^k} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{G(r,\chi)}{r^k}. \end{aligned} \tag{3.1}$$

On the other hand, from the properties of residue systems we have

$$\begin{aligned} |K(m,n,\chi;q)|^2 &= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(a)\mathrm{e}\left(\frac{ma+n\bar{a}}{q}\right) \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \overline{\chi}(b)\mathrm{e}\left(-\frac{mb+n\bar{b}}{q}\right) \\ &= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \chi(a)\mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\ &= \phi(q) + \sum_{\substack{a=2 \\ (a,q)=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \chi(a)\mathrm{e}\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right). \end{aligned} \tag{3.2}$$

Therefore

$$\begin{aligned} &\sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}} |K(m,n,\chi;q)|^2 |B_{k,\chi}|^2 \\ = \ &\frac{(k!)^2 q^{2(k-1)}\phi(q)}{(2\pi)^{2k}} \sum_{\substack{r_1=-\infty \\ r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2\neq 0}}^{+\infty} \frac{1}{r_1^k r_2^k} \sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}} G(r_1,\chi)\overline{G(r_2,\chi)} \end{aligned}$$

$$+\frac{(k!)^2q^{2(k-1)}}{(2\pi)^{2k}}\sum_{\substack{a=2\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\mathrm{e}\left(\frac{mb(a-1)+n\overline{b}(\overline{a}-1)}{q}\right)$$

$$\times\sum_{\substack{r_1=-\infty\\r_1\neq0}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq0}}^{+\infty}\frac{1}{r_1^kr_2^k}\sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}}\chi(a)G(r_1,\chi)\overline{G(r_2,\chi)}.$$

Then from Lemma 2.1 and Lemma 2.4 we have

$$\sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}}|K(m,n,\chi;q)|^2\left|B_{k,\chi}\right|^2=\frac{2(k!)^2\zeta(2k)}{(2\pi)^{2k}}q^{2(k-1)}\phi^3(q)+O\left(q^{2k+\frac{1}{2}+\epsilon}\right).$$

This proves Theorem 1.1.

Let $q=uv$, with $(u,v)=1$, u be a square-full number or $u=1$, v be a square-free number. Let q^* denote the conductor of χ with $\chi\longleftrightarrow\chi^*$. Then

$$\tau(\overline{\chi})=\overline{\chi}^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\overline{\chi}^*)\neq0$$

if and only if $q^*=ud$, where $d\mid v$. So from formulae (3.1), (3.2), Lemmas 2.5, 2.6 and the properties of primitive characters we can get

$$\sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}}|K(m,n,\chi;q)|^2\tau^h\left(\overline{\chi}\right)B_{k,\chi}^h$$

$$=\phi(q)\sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}}\tau^h\left(\overline{\chi}\right)B_{k,\chi}^h+\sum_{\substack{a=2\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\mathrm{e}\left(\frac{mb(a-1)+n\overline{b}(\overline{a}-1)}{q}\right)$$

$$\times\sum_{\substack{\chi\neq\chi_0\\\chi \bmod q}}\chi(a)\tau^h\left(\overline{\chi}\right)B_{k,\chi}^h$$

$$=\frac{2^{h-1}(k!)^hq^{kh-1}\phi^3(q)}{(-1)^{(k-1)h}\left(2\pi i\right)^{kh}}\prod_{p\|q}\left(1-\frac{p^{h-1}-1}{p^{h-1}\left(p-1\right)^2}\right)+O\left(q^{kh+1+\epsilon}\right)$$

$$+\sum_{d|v}\sideset{}{^*}\sum_{\chi \bmod ud}\sum_{\substack{a=2\\(a,\frac{v}{d})=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\mathrm{e}\left(\frac{mb(a-1)+n\overline{b}(\overline{a}-1)}{q}\right)\chi(a)\overline{\chi}^h\left(\frac{v}{d}\right)$$

$$\times\mu^h\left(\frac{v}{d}\right)\tau^h(\overline{\chi})\left[-\frac{k!q^{k-1}}{(2\pi i)^k}\sum_{t\mid\frac{v}{d}}\frac{\chi\left(\frac{v}{dt}\right)\mu\left(\frac{v}{dt}\right)\phi(q)\tau(\chi)}{t^k\phi\left(\frac{q}{t}\right)}\sum_{\substack{r=-\infty\\r\neq0}}^{+\infty}\frac{\overline{\chi}(r)}{r^k}\right]^h$$

$$:=\frac{2^{h-1}(k!)^hq^{kh-1}\phi^3(q)}{(-1)^{(k-1)h}\left(2\pi i\right)^{kh}}\prod_{p\|q}\left(1-\frac{p^{h-1}-1}{p^{h-1}\left(p-1\right)^2}\right)+O\left(q^{kh+1+\epsilon}\right)+\Omega.$$

Then from Lemma 2.8 we have

$$\begin{aligned}
&\Omega \\
&= \frac{(-1)^h (k!)^h q^{(k-1)h}}{(2\pi i)^{kh}} \sum_{d|v} \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\
&\times \sideset{}{^*}\sum_{\chi \bmod ud} (ud)^h \chi^h(-1)\chi(a) \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h \\
&= \frac{(-1)^h (k!)^h q^{(k-1)h}}{(2\pi i)^{kh}} \sum_{d|v} (ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) \\
&\times \sideset{}{^*}\sum_{\chi \bmod ud} \chi^h(-1)\chi(a)[1+\overline{\chi}(-1)(-1)^k]^h \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h \\
&= \begin{cases} \dfrac{(-1)^h 2^h (k!)^h q^{(k-1)h}}{(2\pi i)^{kh}} \displaystyle\sum_{d|v} (ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) & \\ \times \displaystyle\sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}} \chi(a) \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h, & \text{if } 2 \mid k; \\ \dfrac{2^h (k!)^h q^{(k-1)h}}{(2\pi i)^{kh}} \displaystyle\sum_{d|v} (ud)^h \sum_{\substack{a=2 \\ (a,\frac{v}{d})=1}}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{mb(a-1)+n\bar{b}(\bar{a}-1)}{q}\right) & \\ \times \displaystyle\sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \chi(a) \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^k} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^k}\right]^h, & \text{if } 2 \nmid k. \end{cases} \\
&\ll q^{kh+\frac{3}{2}+\epsilon}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} |K(m,n,\chi;q)|^2 \tau^h(\overline{\chi}) B_{k,\chi}^h \\
&= \frac{2^{h-1}(k!)^h q^{kh-1}\phi^3(q)}{(-1)^{(k-1)h}(2\pi i)^{kh}} \prod_{p\|q}\left(1-\frac{p^{h-1}-1}{p^{h-1}(p-1)^2}\right) + O\left(q^{kh+\frac{3}{2}+\epsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 1.2.

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