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OPTIMUM PERFORMANCE AND DESIGN OF A RECTANGULAR FIN

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ABSTRACT–A rectangular fin with a fluid in the inside wall is analyzed and optimized using a two-dimensional analytical method. The influence of the fluid convection characteristic number in the inside wall and the fin base thickness on the fin base temperature is listed. For the fixed fin volumes, the maximum heat loss and the corresponding optimum fin effectiveness and dimensions as a function of the fin base thickness, convection characteristic numbers ratio, convection characteristic number over the fin, fluid convection characteristic number in the inside wall, and the fin volume are represented. One of the results shows that both the optimum heat loss and the corresponding fin effectiveness increase as the fin base thickness decreases.

KEY WORDS : Analytical method, Heat loss, Optimization, Convection characteristic number

NOMENCLATURE

Bi : Biot number, $(h\, l_h)/k$
h : heat transfer coefficient over the fin [$W/m^{2o}C$]
h_e : fin tip heat transfer coefficient [$W/m^{2o}C$]
h_f : heat transfer coefficient in the inside wall [$W/m^{2o}C$]
k : thermal conductivity [W/m^oC]
l_b : fin base length [m]
L_b : dimensionless fin base length, l_b/l_c
l_c : characteristic length [m]
l_e : fin tip length [m]
L_e : dimensionless fin tip length, l_e/l_c
l_h : one half fin height [m]
L_h : dimensionless one half fin height, l_e/l_c
l_w : fin width [m]
M : convection characteristic number over the fin surface, $hl_c/k=\{(h\, l_h)/k\}(l_c/l_h)=Bi/L_h$
M_e : convection characteristic number at the fin tip, $(h_e\, l_c)/k$
M_f : fluid convection characteristic number in the inside wall, $(h_f\, l_c)/k$
q : heat loss from the fin [W]
Q : dimensionless heat loss from the fin, $q/(k\Phi_f\, l_w)$
q_w : heat loss from the bare wall [W]
Q_w : dimensionless heat loss from the bare wall, $q_w/(k\Phi_f\, l_w)$
T : fin temperature [°C]
T_b : fin base temperature [°C]
T_f : fluid temperature in the inside wall [°C]
T_∞ : ambient temperature [°C]
U^* : symbol for optimum heat loss or effectiveness
v : fin volume [m^3]
V : dimensionless fin volume, $v/(l_c^2\, l_w)$
W^* : symbol for optimum fin tip length or fin height
x : length directional variable [m]
X : dimensionless length directional variable, x/l_c
y : height directional variable [m]
Y : dimensionless height directional variable, y/l_c
β : ratio of convection characteristic numbers, M_e/M
ε : fin effectiveness
θ : dimensionless temperature, $(T-T_\infty)/(T_f-T_\infty)$
λ_n : eigenvalues (n=1, 2, 3, ···)
Φ_f : adjusted inside fluid temperature [°C], (T_f-T_∞)

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1. INTRODUCTION

Heat transfer is one of the important phenomena in the automotive technology. For studies about this heat transfer, the influence of gases thermodynamic properties on gross heat release has been esteemed (Lanzafame and Messina, 2003). The surface heat flux and combustion characteristics of premixed propane mixture in a constant volume chamber are investigated (Park, 2005).

The extended surfaces or fins are well known to be a simple and effective means of increasing heat dissipation in many engineering and industrial applications such as the cooling of combustion engines, electronic equipments, many kinds of heat exchangers, and so on. Various shapes of fins have been used and the profile of the rectangular fin is one of the widely used profiles in actual industrial areas. Performances of the rectangular profile

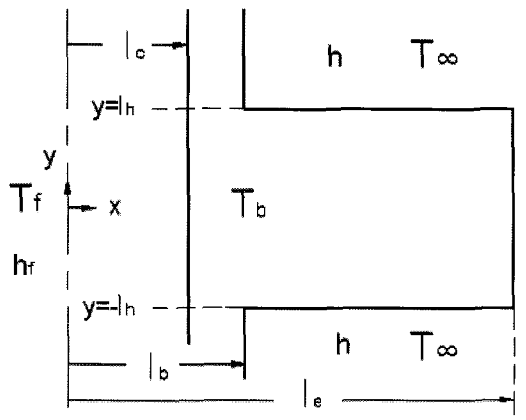

Figure 1. Geometry of a rectangular fin.

fin with many different conditions have been studied. For example, the two-dimensional analysis for a straight fin was carried out (Avrami and Little, 1942). The conduction problem for the rectangular plate fin simultaneously considering the convective heat transfer problem for the flowing fluid is solved (Garg and Velusamy, 1986). A method that enables one to reduce the problem of transient conduction in a 2-D rectangular fin to that of a 1-D problem is developed (Ju *et al.*, 1989). A two-dimensional rectangular fin with an arbitrary variable heat transfer coefficient on the fin surface is investigated using a Fourier series approach (Ma *et al.*, 1991). Comparison between performance of a plate fin and that of a modified plate fin is investigated using a two-dimensional separation of variables method (Kim and Kang, 2001).

Many papers also dealt with the optimization for the straight rectangular fin. For example, an extended integral approach is presented to determine the optimum dimensions for the rectangular longitudinal fins and pin fins by incorporating traverse heat conduction (Chung and Iyer, 1993). The minimum mass design of a straight rectangular fin array extending from a plane wall considering fin-to-fin and fin-to-base radiation interactions is determined (Krishnaprakas, 1996). An optimization of thermally and geometrically asymmetric trapezoidal fins (including a rectangular fin) is presented (Kang and Look, 2004). The optimal dimensions of a one-dimensional longitudinal rectangular fin and a cylindrical pin fin is determined (Yeh, 1996). The optimum design of single longitudinal fins with constant thickness considering different uniform heat transfer coefficients on the fin faces and on the tip is investigated by means of an accurate mathematical method yielding the solution of constrained minimization (maximization) problems (Casarosa and Franco, 2001).

In all these papers, the fin base temperature is given as a constant for the fin base boundary condition. In this study, for a straight rectangular fin optimization in the case of a fixed fin volume, the fluid temperature in the inside wall is given and the influences of the fluid convection characteristic number in the inside wall and the fin base thickness are considered.

2. 2-D ANALYTICAL METHOD

For a rectangular fin with fluid in the inside wall, the schematic diagram is presented in Figure 1. For this schematic diagram, the two-dimensional governing differential equation under steady state is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 . \tag{1}$$

The dimensionless governing differential equation is then given by

$$\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}=0 . \tag{2}$$

Four boundary conditions are required to solve the governing differential equation and these conditions are shown as Equations (3)~(6).

$$h_f(T_f - T|_{x=l_c})=-k\frac{T_b - T|_{x=l_c}}{l_b - l_c}=-k\frac{\partial T}{\partial x}\bigg|_{x=l_b} \tag{3}$$

$$\frac{\partial T}{\partial y}\bigg|_{x=0}=0 \tag{4}$$

$$\int_0^{l_h} -k\frac{\partial T}{\partial x}\bigg|_{x=l_e} l_w dy=\int_0^{l_h} h_e(T|_{x=l_e} - T_\infty)l_w dy \tag{5}$$

$$\int_{l_b}^{l_e} -k\frac{\partial T}{\partial y}\bigg|_{y=l_h} l_w dy=\int_{l_b}^{l_e} h(T|_{y=l_h} - T_\infty)l_w dx \tag{6}$$

Four Equations (3)~(6) can be transformed into the dimensionless forms and denoted by Equations (7)~(10).

$$-\frac{\partial \theta}{\partial X}\bigg|_{X=L_b}=\frac{1-\theta|_{X=L_b}}{1/M_f+L_b-1} \tag{7}$$

$$\frac{\partial \theta}{\partial Y}\bigg|_{Y=0}=0 \tag{8}$$

$$\frac{\partial \theta}{\partial X}\bigg|_{X=L_e}+M_e\cdot\theta|_{X=L_e}=0 \tag{9}$$

$$\frac{\partial \theta}{\partial Y}\bigg|_{Y=L_h}+M\cdot\theta|_{Y=L_h}=0 \tag{10}$$

The solution for the temperature distribution $\theta(X, Y)$ within the rectangular fin obtained using the separation of variables method with Equations (2), (7), (8) and (9) is

$$\theta(X,Y)=\sum_{n=1}^{\infty}\frac{g_1(\lambda_n)\cdot f(X)\cdot\cos(\lambda_n Y)}{g_2(\lambda_n)+g_3(\lambda_n)} \tag{11}$$

where,

$$f(X)=\cosh(\lambda_n X)+g_4(\lambda_n)\cdot\sinh(\lambda_n X) \quad (12)$$

$$g_1(\lambda_n)=\frac{4\sin(\lambda_n L_h)}{2\lambda_n L_h+\sin(2\lambda_n L_h)} \quad (13)$$

$$g_2(\lambda_n)=\cosh(\lambda_n L_b)-R_w\cdot\lambda_n\cdot\sinh(\lambda_n L_b) \quad (14)$$

$$g_3(\lambda_n)=g_4(\lambda_n)\cdot\{\sinh(\lambda_n L_b)-R_w\cdot\lambda_n\cdot\cosh(\lambda_n L_b)\} \quad (15)$$

$$g_4(\lambda_n)=-\frac{\lambda_n\cdot\tanh(\lambda_n L_e)+M_e}{\lambda_n+M_e\cdot\tanh(\lambda_n L_e)} \quad (16)$$

$$R_w=1/M_f+L_b-1 \quad (17)$$

The eigenvalues λ_n can be calculated using Equation (18), which is arranged from Equation (10).

$$\lambda_n\cdot\tanh(\lambda_n)=M \quad (18)$$

The heat loss conducted into the fin through the fin base is calculated by

$$q=\int_{-l_h}^{l_h}-k\frac{\partial T}{\partial x}\bigg|_{x=l_b} l_w dy. \quad (19)$$

Dimensionless heat loss is written as

$$Q=\frac{q}{k\varphi_f l_w}=-\sum_{n=1}^{\infty}\frac{8g_5(\lambda_n)\cdot\sin^2(\lambda_n)}{g_2(\lambda_n)+g_3(\lambda_n)}. \quad (20)$$

where,

$$g_5(\lambda_n)=\sinh(\lambda_n L_b)+g_4(\lambda_n)\cdot\cosh(\lambda_n L_b) \quad (21)$$

The fin effectiveness ε is defined as the ratio of the heat loss from the fin Q to the heat loss from the bare wall Q_w and can be written by Equation (22).

$$\varepsilon=Q/Q_w \quad (22)$$

Assuming that the heat transfer coefficient around the bare wall is h, the heat loss from the bare wall q_w can be calculated by Equation (23).

$$q_w=2\int_0^{l_h} h(T|_{x=l_b}-T_\infty)l_w dy \quad (23)$$

Solving Equation (23), the resulting expression is divided by $k\Phi_f l_w$; the dimensionless heat loss from the bare wall Q_w is given by Equation (24).

$$Q_w=\frac{q_w}{k\phi_f l_w}=\frac{2L_h}{1/M_f+(L_b-1)+1/M} \quad (24)$$

The rectangular fin volume, as shown in Figure 1, can be calculated by Equation (25).

$$v=\int_{-l_h}^{l_h}(l_e-l_b)l_w dy \quad (25)$$

The dimensionless fin volume is expressed as Equation (26).

$$V=\frac{v}{l_c^2 l_w}=2L_h(L_e-L_b) \quad (26)$$

3. RESULTS AND DISCUSSION

The dimensionless heat loss as a function of the fin tip length for different values of the dimensionless fin volume is shown in Figure 2. It is observed that the heat loss increases rapidly as the fin tip length approaches the fin base thickness (i.e. very short fin). It is because the fin height increases as the fin tip length decreases for the fixed fin volume. Obviously, the design in this case is impractical, although the heat dissipation is large. Another important phenomenon shown in Figure 2 is that the maximum heat loss does not exist when the fin volume is beyond a certain value. For example, the maximum heat loss exists at about L_e=1.94 for V=0.3, at about L_e=1.99 for V=0.4, and at about L_e=2.01 for V=0.5 while it does not exist for V=0.6. The maximum heat loss will be referred to as the optimum heat loss and the fin tip length at which the heat loss becomes the maximum is referred

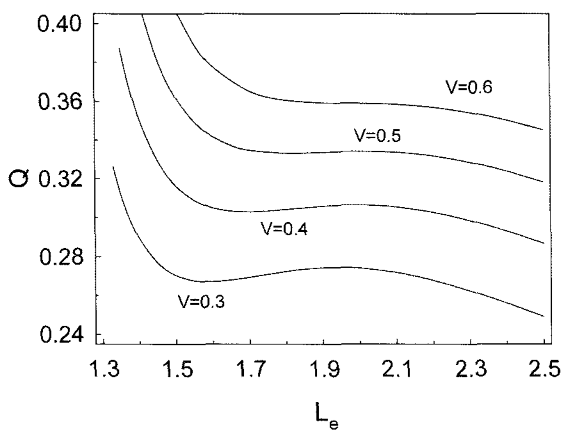

Figure 2. Dimensionless heat loss as a function of fin tip length (L_b=1.1, M=0.2, β=1, M_f=1000).

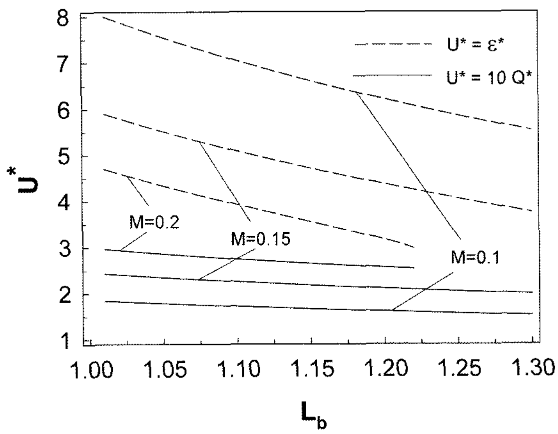

Figure 3. Optimum heat loss and fin effectiveness versus the fin base thickness (M_f=1000, β=1, V=0.3).

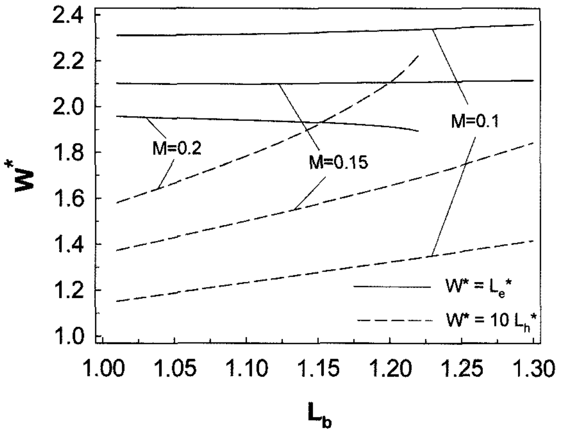

Figure 4. Optimum fin tip length and fin height versus the fin base thickness (M_f=1000, β=1, V=0.3).

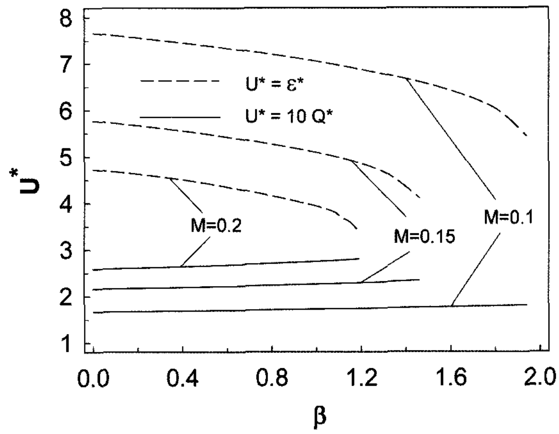

Figure 5. Optimum heat loss and fin effectiveness versus convection characteristic numbers ratio (M_f=1000, L_b=1.1, V=0.3).

to as the optimum fin tip length. The superscript 'asterisk' means the optimum in this study.

Figure 3 presents the variation of the optimum heat loss and fin effectiveness as a function of the fin base thickness for a rectangular fin when the fin volume is fixed. The optimum effectiveness means the effectiveness when the heat loss becomes the maximum heat loss for given conditions. This figure indicates that both the optimum heat loss and the corresponding optimum fin effectiveness decreases almost linearly as the fin base thickness increases. Note that the optimum heat loss increases while the corresponding optimum effectiveness decreases as the convection characteristic number increases for the same value of the fin base thickness.

Figure 4 represents the variation of the optimum fin tip length and height for the same condition as given in Figure 3. The variation of the fin tip length is not much while the fin height increases remarkably with the increase of fin base thickness Physically, it means that the actual optimum fin length becomes shorter due to the increase of the base thickness and the fin shape becomes fatter. It also shows that the optimum fin height increases while the optimum fin tip length decreases as the convection characteristic number increases for the same value of fin base thickness. Now, to illustrate how to use this figure practically, L_e^*=2.32 and L_h^*=0.12 for L_b=1.1 and M=0.1 are considered. The value of h is arbitrarily chosen to be 200 W/m^2°C for a forced convection of some gases and k is given as 60 W/m°C for carbon steel. Then, l_c becomes 3 cm, the practical optimum fin tip length is 2.32×3 cm = 6.94 cm, and the optimum half fin height is 0.12×3 cm = 0.36 cm.

Table 1 lists the dimensionless temperature of the fin with the variation of fin base thickness, the fin height coordinate, and the fluid convection characteristic number in the inside wall for M=0.2, β=1, L_e=1.8 and L_h=0.15. This table illustrates that the base temperature decreases as the fin base thickness increases or the fluid convection characteristic number in the inside wall decreases. It can be guessed that the fin base temperature approaches the fluid temperature in the inside wall as L_b approaches 1 and M_f approaches infinity. As expected, this table also

Table 1. Dimensionless fin temperature with the variations of Y, L_b and M_f. (M=0.2, β=1, L_e=1.8, L_h=0.15).

M_f	θ(X, Y)			
	X=L_b, Y=0		L_b=1.2, X=L_e	
	L_b=1.05	L_b=1.2	Y=0	Y=L_h
5	0.821	0.760	0.552	0.545
10	0.885	0.809	0.587	0.580
1000	0.959	0.864	0.627	0.621

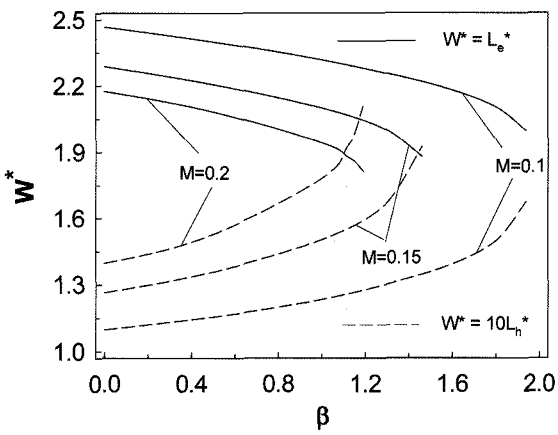

Figure 6. Optimum fin tip length and fin height versus convection characteristic numbers ratio (M_f=1000, L_b= V=0.3).

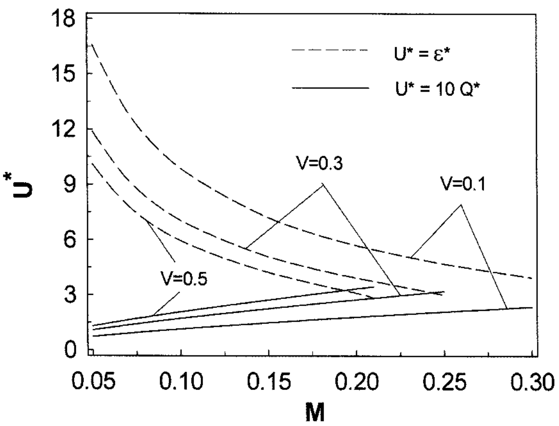

Figure 7. Optimum heat loss and fin effectiveness versus the convection characteristic number (M_f=1000, β=1, L_b=1.1).

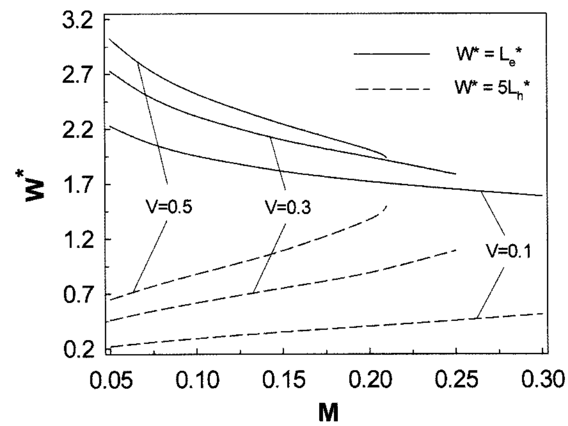

Figure 8. Optimum fin tip length and fin height versus the convection characteristic number (M_f=1000, β=1, L_b= 1.1).

shows that the temperature at the fin tip decreases as Y increases.

The effects of the convection characteristic numbers ratio β on the optimum performance and design are shown in Figures 5~6. Physically, a larger β represents stronger convection at the fin tip, and hence, dissipates more heat from the fin tip. For the effectiveness, the surrounding convection characteristic number over the bare wall is considered as M instead of M_e. Figure 5 shows that the optimum heat loss increases while the corresponding optimum effectiveness decreases as the convection characteristic numbers ratio increases. It is because that the optimum fin tip length decreases with the increase of convection characteristic numbers ratio and the optimum fin height increases due to the fixed fin volume as shown in Figure 6.

Figure 7 represents the variation of the optimum heat loss and fin effectiveness as a function of the convection characteristic number for several values of the fixed fin volume. It shows that the optimum heat loss increases linearly while the corresponding optimum effectiveness decreases parabolically with the increase of the convection characteristic number.

The optimum fin tip length and fin height as a function of the convection characteristic number for the same condition as given in Figure 7 is shown in Figure 8. The optimum fin tip length decreases as the convection characteristic number increases and the optimum fin height increases due to the fixed fin volume. It can be known that both the optimum fin tip length and the fin height increase with the increase of the fin volume for fixed convection characteristic number.

To validate the present analytical approach and results, a comparison of the increasing rate of the optimum heat loss for a fixed fin height between the present work and a reference (Kang, 2006) is made in Table 2. In that

Table 2. Comparison of increasing rate of the optimum heat loss between the present results and those of a reference (Kang, 2006) (L_b=1.1, L_h=1, β=1, M_f=1000).

M=Bi	I. R. of Q^* (%)	
	a reference (Kang, 2006)	the present work
0.01 → 0.03	71.44	71.34
0.03 → 0.05	28.09	28.00
0.05 → 0.1	39.19	38.86
0.1 → 0.15	20.85	20.84

reference, the fluid inside is not considered and the dimensionless half fin height is fixed to be 1. For almost the same condition in the present work, the values of the variables are given as L_b=1.1, M_f =1000, β=1, and the dimensionless half fin height L_h is fixed to be 1 to make M=Bi. The fin volume is not fixed for this comparison. As can be seen, very good agreement is found between the two works.

The optimum performance and dimensions versus fluid convection characteristic number in the inside wall are presented in Figures 9~10. The optimum heat loss, the corresponding fin effectiveness, and the fin tip length increase as the fluid convection characteristic number in the inside wall increases and the fin height decreases due to the fixed fin volume. It also shows that the variations of all optimum values are remarkable until the fluid convection characteristic number in the inside wall increases to about 10. It can be noted that the optimum value does not exist when fluid convection characteristic number in the inside wall is smaller than certain values while the optimum value does not exist when other variables are larger than certain values.

The dimensionless fin volume, V, was arbitrarily selected to be 0.3 in the previous discussion. The

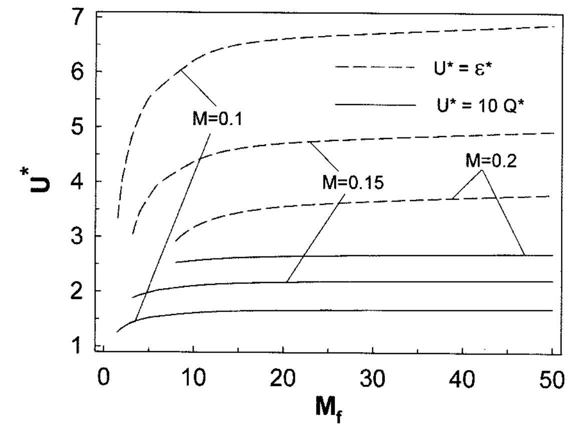

Figure 9. Optimum heat loss and fin effectiveness vs. fluid convection characteristic number in the inside wall (L_b=1.1, β=1, V=0.3).

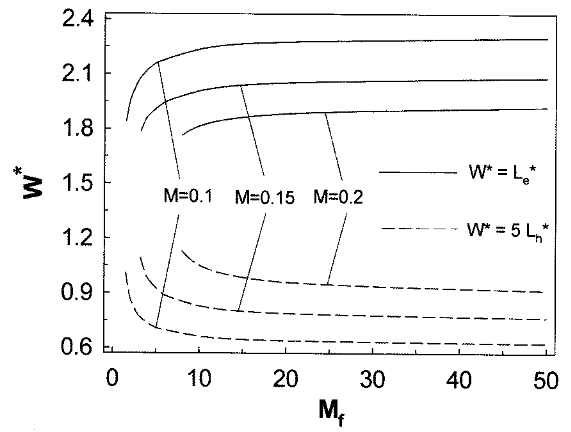

Figure 10. Optimum fin tip length and fin height vs. fluid convection characteristic number in the inside wall (L_b=1.1, β=1, V=0.3).

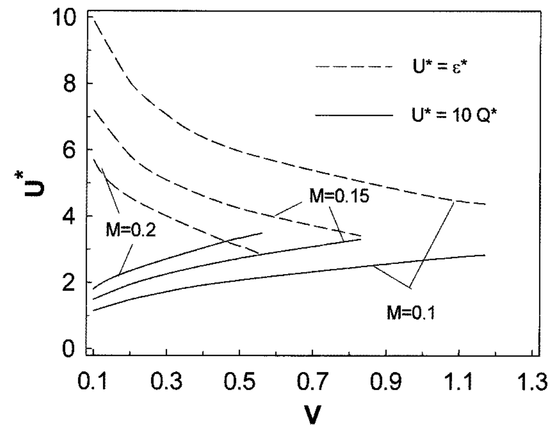

Figure 11. Optimum heat loss and fin effectiveness vs. dimensionless fin volume (L_b=1.1, β=1, M_f=1000).

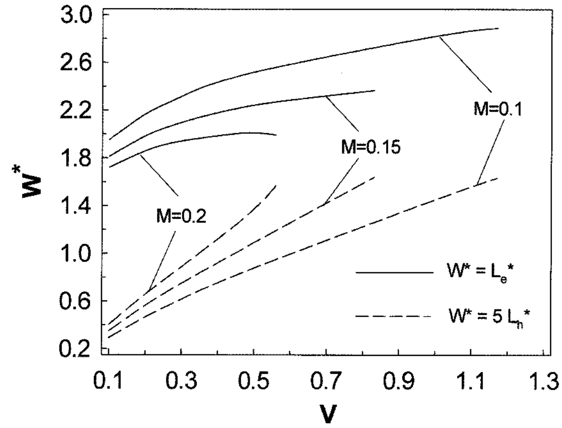

Figure 12. Optimum fin tip length and fin height vs. dimensionless fin volume (L_b=1.1, β=1, M_f=1000).

variations of the optimum heat loss and effectiveness as a function of V are shown in Figure 11. As expected, the increase of V enhances the optimum heat loss. The corresponding optimum effectiveness decreases remarkably with the increase of the fin volume because the increasing rate of the optimum fin height is larger than that of the optimum fin tip length with the increases of fin volume as will be shown in Figure 12.

Figure 12 presents the optimum dimensions versus the fin volume for the same condition as given in Figure 11. This figure shows the optimum fin tip length increases rapidly at first and then levels off as the fin volume increases. It can be seen that the optimum fin tip length increases as M decreases for the same fixed fin volume. Also, the optimum fin height increases almost linearly with the increase of total fin volume. Physically, the optimum straight rectangular profile fin becomes slightly 'fatter' with the increase of the fin volume.

4. CONCLUSIONS

From this two-dimensional analysis of a rectangular fin with a fluid in the inside wall, the following conclusions can be drawn:

(1) For a fixed fin volume, the maximum heat loss does not exists when the fluid convection characteristic number in the inside wall is less than a certain value while it does not exist when other variables (for example, fin base thickness, convection characteristic numbers ratio, etc.) are larger than certain values.

(2) Both the optimum heat loss and the corresponding fin effectiveness increase as the fin base thickness decreases and as the fluid convection characteristic number in the inside wall increases.

(3) Even though the optimum heat loss increases, the corresponding optimum fin effectiveness decreases

as the convection characteristic numbers ratio, fin volume and convection characteristic number over the fin increase.

It is because the optimum fin tip length decreases while the optimum fin height increases or the increasing rate of fin tip length is less than that of fin height with the increase of these variables.

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