■ 論 文 ■

혼잡현상을 갖는 교통체계의 비용함수

Cost Function of Congestion-Prone Transportation Systems

문 동 주
(한양대학교 도시공학과 BK21사업단 계약교수)

김 홍 배
(한양대학교 도시공학과 교수)

목 차

Key Words : 대기행렬시스템, 통행시간가치의 이질성, 사회비용, 공급자비용, 한계혼잡비용
queuing system, heterogeneity of value-of-travel-times, social cost, supplier cost, marginal congestion cost

요 약

이 논문은 혼잡현상을 갖는 교통체계의 사회비용함수를 사회비용 최소화문제로부터 도출하여 분석하였다. 이 논문은 이 분야의 기존 연구에서 다루지 않았던 다음의 두 가지를 중점적으로 분석하였다. 하나는 이용자들의 시간가치가 다를 경우에 비용함수의 구조가 어떻게 달라지는지를 검토하는 것이고, 다른 하나는 사회비용함수를 구성하는 공급자 비용함수의 구조를 파악하는 것이었다.

분석의 결과는 다음과 같이 요약될 수 있다. 첫째, 한계사회비용은 특정한 시간가치를 가진 고객이 소비한 시간가치비용과 추가 고객의 처리에 수반되는 시스템 전체의 서비스시간 증가에 따른 한계혼잡비용으로 구성된다. 둘째, 한계혼잡비용은 공급자의 보상한계비용과 같은 바, 후자는 공급자가 추가의 고객을 가장 경제적으로 처리함에 필요한 용량의 변경에 의한 서비스시간의 변화 양에 대한 이용자 전체의 시간가치를보상해준다는 전제아래서의 공급자 한계비용을 지칭한다. 셋째, 보상한계비용은 서비스시간함수가 산출과 용량에 대해 동차함수일 경우 한계용량비용에 시스템 이용률의 역수를 곱한 값과 같다.

This paper analyzed the social cost function of a congestion-prone service system, which is developed from the social cost minimization problem. The analysis focused on the following two issues that have not been explicitly explored in the previous studies: the effect of the heterogeneity of value-of-travel-times among customers on the structure of cost functions; and the structure of the supplier cost function constituting the social cost function.

The analysis gave a number of findings that could be summarized as follows. First, the social marginal cost for one unit increase in system output having a certain value-of-travel-time is the sum of the service time cost for that value-of-travel-time and the marginal congestion cost for the average value-of-service-time of all the system outputs. Second, the marginal congestion cost equals the marginal supplier cost of system output under the condition that supplier compensates the customers for the changed service time costs which is incurred by the marginal capacity increase necessary for economically facilitating an additional system output. Third, the compensated marginal cost is the multiple of the marginal capacity cost and the inverse of system utilization ratio, if the service time function is homogeneous of degree zero in its inputs.

Ⅰ. Introduction

Congestion pricing is a special kind of marginal cost pricing, which deals with the public service exhibits congestion causing economic losses to its customers, due to the limited service capacity. Congestion pricing estimates the optimal pricing and investment rule for the congestion prone public services. The existing studies for congestion pricing have mainly targeted highways being the typical public service systems having congestion phenomena. Recently the application of this pricing is extended to the other types of congestion prone transportation systems which have the service time function different from that of highways in Moon and Park (2002b).

However all the previous studies for congestion pricing has a serious shortcoming in that its pricing rule does not carry any specific information about the supplier costs. The optimal price under congestion pricing is expressed by the multiple of the following three factors: value of service time (or value-of-travel-time); system output; and partial derivative of service time function with respective to system output. However this well known formula estimating the optimal price has no explicit linkage with the supplier cost, which might be a more important concern of the supplier.

Another shortcoming is the specification of the value-of-service-time in the pricing rule. This term in the existing studies is developed from the social welfare maximization problem constructed under the assumption that all the users have the identical value-of-service-time which implicitly refers to the average of value-of-service-times. However no attempt has been made to explicitly validate such a specification can be applicable to the case when the congestion-prone transportation system facilitates users with value-of-service-time heterogeneous one another.

The optimal price under congestion pricing equates the total cost experienced by each user to the social marginal cost. Therefore one simple way to identify the relationship between the optimal price and the supplier cost could be to analyze the social cost function developed from the social cost minimization problem of congestion-prone service system. Such an approach is similar to that which develops the cost function of the neoclassical firm from the cost minimization problem. However the detail of the former is significantly different from the latter.

Specifically, the congestion-prone transportation system considered in this study is a synonym to a queuing system which facilitates the random arrivals of customers. The queuing system with a given capacity can facilitate customers up to the level such that the mean arrival rate of demands, called the system output, does not exceed the capacity determining the upper limit of system outputs yielded. Also the queuing system generally exhibits the average service time per customer, which is monotonically increasing in the mean arrival rate of customers, and monotonically decreasing in the capacity of the system.

Therefore it is certain that the queuing system has the cost structure different from that of the neoclassical production system. The cost function of the neoclassical firm estimates the cost for the independent variable of final outputs. Also the cost function of the firm can readily be developed from the cost minimization problem which is the dual to the profit maximization problem of the neoclassical firm. The application of such an approach to the queuing system calls for the use of the cost minimization problem which has to accommodate the service process, and therefore gives the cost function different from that of the neoclassical firm.

The objective of this paper is to develop the social and supplier cost function of a congestion-prone transportation system with respect to system output. The cost function is developed from the social cost minimization problem under the assumption that the transportation system facilitates the random arrivals of demands. Also the consumers of the transportation system are assumed to have the value-of-travel-time different one another.

The social cost minimization problem of the queuing system is expressed by the non-linear programming problem with constraints, as will be shown in Section 2. The minimization problem is used to find the capacity and its inputs, which minimizes the total social cost incurred in facilitating a given system output. The choice of inputs to capacity has to satisfy the constraint for the production possibility range of capacity with respect to its inputs available in markets; whereas, the choice of capacity has to comprehend its effect on the service time being the function of capacity and system output. Also the total social cost is expressed by the sum of the supplier cost for the inputs to the capacity and the user cost for the service time of the queuing system.

Subsequently, in Section 3, the various cost functions for the independent variable of system output are developed from the social cost minimization problem. Such a cost analysis puts emphasis on estimating the marginal social and supplier cost functions for each system output endowed with its peculiar value-of-service-time. The marginal social cost is developed in the manner analogous to that which estimates the cost in the previous studies for the case when all the customers have the identical value-of-service-time. Also the marginal supplier cost is developed from the social cost minimization problem in a fashion similar to that which estimates the marginal social cost.

It is followed in Section 4 by the analysis to illustrate the structure of the social and supplier marginal cost functions with the two specific examples of service time functions. One example illustrates the various cost functions of the congestion-prone transportation system which has the service time function being homogeneous of degree zero in capacity and system output. Another example analyzes the various cost functions of the scheduled transportation service which has the non-homogeneous service time function. The final section offers the summary and concluding remarks.

Ⅱ. Social Cost Minimization Problem

1. Service time function

The service time function of a queuing system has the functional structure that estimates the service time being sensitive to capacity and system output. The service time function is supposed to estimate the expected value of the service time from its commencement to its completion, which is com-monto every customer of the queuing system. The service time is the sum of the net service time and the delay time due to the congestion.

The service time function of a queuing system depends upon the characteristics of the service procedure, termed *the service technology*, such as service priority, number of service channels, etc. The service time function is generally increasing in the mean arrival rate; that is, the net service time is usually constant irrespective of the mean arrival rate, but the delay time exhibits such a property. The delay time is taken place, even in the case when the mean arrival rate is smaller the capacity, due to the randomness of the service time and the time gaps between arrivals.

There are many queuing systems analyzed in the available literature of queuing theory dealing with the estimation of the expected service time or the expected value of queue. However, only a few queuing systems can have the closed-form solutions of the expected service time. Nonetheless, it can be confirmed that every queuing system shares the common property that the service time is increasing in the mean arrival rate, but decreasing in the capacity. Based on such findings of the previous studies, we postulate the service time function of various queuing systems below.

Assumption 1: The service time function of a transportation system, denoted by T, has the following structure.

(1) The system, which facilitates customers on the first-in-first-out basis, has *the service time function homogeneous of degree zero* in system output during the analysis period, denoted by s, capacity during the same period, expressed by c, such that

$$t = T(s/c) = t^0 + T^d(s/c)$$

where t is the expected service time per customer, t^o is the net service time, and T^d the delay time function.

(2) The system, which serves customers with the vehicles operated under a schedule controllable by the supplier, has the non-homogeneous service time function such that

$$T(s,c) = t^0 + \frac{1}{2y} T_1^d(s/c) + T_2^d(s/c)$$

where T_1^d and T_2^d are delay time functions homogeneous of degree zero, c the service frequency of the systemanalyzed, and y is the total service frequency being the sum of the frequencies of all the suppliers in competitions, including the system being the target of the analysis.

(3) The functions T^d, T_1^d and T_2^d are positive, convex and differentiable in s and c. They are also monotonically increasing in s, but monotonically decreasing in c.

Prior to the main discussion, we explain the difference between $T(s/c)$ and $T(s/c)$, in asso ci-ation with the expression of service time functions in the forthcoming discussion. The service time function $T(s/c)$ is the expression specific to the homogeneous function only. By the same token, the delay time function $T^d(s/c)$ is used in order to clarify that the delay time function is homogeneous. On the other hand, the function $T(s/c)$ is an inclusive expression of all kinds of service time functions. This expression is used when it is not necessary to make a distinction whether it is homogenous or not.

The key feature of the conditions in Assumption 1 is that the delay time function T^d is homogeneous of degree zero. This function satisfies the condition such that

$$T^d(s/c) = T^d(\alpha s/\alpha c) \text{ or } \frac{\partial T^d(s/c)}{\partial s} = \frac{c}{s}\frac{\partial T^d(s/c)}{\partial c} \quad (1)$$

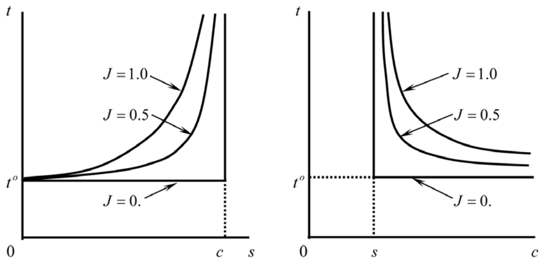
where α is a positive constant. This condition i-mplies that the delay time depends only upon the system utilization ratio, estimated by s/c, irrespective of system output and capacity. Number of examples satisfying this condition is presented below.

A large group of transportation systems facilitates the steady flow of demands on the first-in-first-out basis introduced in Assumption 1(1). This group has the service time function which usually satisfies the degree-zero homogeneity[1)]. One approximation of such service time functions, which is widely applied to this group, is

$$T(s/c) = t^o + t^d \frac{Js/c}{1 - s/c} \quad (2)$$

where $0 < J < 1$ is a parameter characterizing the queuing systems[2)]. This formula for the case of $J=1$ becomes the exact solution to the queuing system

1) The examples of service time function for the first-in-first-out queuing systems are available in many literatures dealing with stochastic process and queuing theory, such as Cinlar (1975).
2) The approximation formula is called the Davidson formula available in many books dealing with transportation planning such as Manheim (1980).



〈Figure 1〉 Configuration of homogeneous service time function

with a single channel. Also the formula has a smaller value of J as the system with a larger number of channels[3]. The configurationof the formula isillustrated in 〈Figure 1〉.

The transportation system, which facilitates the peaking demand on the first-in-first-out basis, should have the service time function that can estimate the service time for the mean flow rate exceeding the capacity at the peak period. Such a service time function can be estimated either by the empirical study or by the approximation metho d[4] One example of the empirical study is the highway service time function such that

$$T(s/c) = t^o + t^d\left(\frac{s}{c}\right)^{\alpha}, \tag{3}$$

where 4.0≤α≤6.0 (Highway Capacity Manual, 1999).

Subsequently, the service time function of scheduled transportation services considered in Assumption 1(2) generally does not have the closed-form solution[5]. One possible approximation of the service time function could be the expression of $T(s,c)$ in the assumption. The first part $(2y)^{-1}\,T_1^d(s/c)$ estimates the mean waiting time before finding the seat available. The second part $T_2^d(s/c)$ represents the delay time experienced at terminal and/or vehicle.

One simple example of such an approximation is the service time function of urban public transit which offers the service in a monopolistic position, such that

$$T(s/c) = t^o + \frac{1}{2c}\left(1 + \frac{J\,s/\tau c}{1 - s/\tau c}\right) \tag{4}$$

where and τ is the seat capacity per vehicle being a constant. Here the term $1/2c$ estimates the mean waiting time under the condition that the passenger can board the first vehicle that arrives, and the remaining term in the parenthesis represents the waiting time caused by not boarding the full vehicles[6].

2. Specification of social cost minimization problem

Here we present the specification for the social cost minimization problem of a queuing system facilitating the steady flow of random arrivals. The social cost minimization problem has the structure almost identical to that of the problems considered in the previous studies, which analyze the social cost of highway services under the assumption that

3) The fact that J has a smaller value, as the number of service channel increases implies that J is a function decreasing in capacity. However, the specification that J is a decreasing function in the capacity results in the non-homogeneous service time function.

4) One approximation formula is

$$T^d(s/c) \cong T_o^d + \frac{\delta}{2}\left(\left(\frac{c}{s}-1\right) + \frac{c}{s}\left(\frac{c}{s}-1\right)^2\right),$$

where T_o^d is the delay caused by the queue at the moment when the peaking period starts, and δ is the ratio of the peak period to the total analysis period. This equation is estimated by reorganizing the delay time function for the approximation model of the queuing systemserving the peaking demand in Parzen (1962).

5) The queue length of this queuing system is estimated in Boudreau, Griffin, and Mark Kac (1962). Note that the estimated queue length is not a closed-form solution.

6) This approximation formula is available in many books dealing with transportation planning such as Manheim (1980).

the mean flow rate is constant throughout the analysis period. However the former has one distinct difference from the latter in that the value-of-service-time differs by customers demanding the service.

The queuing system being the target of the social cost analysis is supposed to facilitates the given system output, expressed by a vector $\bar{s}=(s^1,...,s^1)$, where s^i is the demand of consumer $i \in <1, I>$ for the service. It is assumed that the value-of-service-time perceived by consumer i is a deterministic term v^i. It is also assumed that this system output $\bar{s}$ forms the steady flow of random arrivals to the service system with a mean rate s estimated by $\sum_i S^i$.

The supplier of the congestion-prone service is supposed to minimize the total social cost which is the sum of supplier and user costs. The supplier cost estimates the cost of inputs necessary for the construction of an arbitrary capacity; whereas, the user cost is the value of service times incurred in facilitating the system output $\bar{s}$. The supplier is also supposed to have two different kinds of choice variables the capacity and its inputs. Such a social cost minimization problem of the supplier is specified below.

Assumption 2: The transportation system, which facilitates the given system output $\bar{s}$ of random arrivals forming the steady flow, satisfies the following conditions.

(1) The expected service time per random arrival, denoted by t, is a function of the *aggregated system output* $s=\sum_i s^i$, and is identical across all the arrivals. This expected service time and is estimated by

$$t = T(s,c).$$

For this expected service time, the user cost is

$$User\ Cost = t\sum_i v^i\ s^i = t\,\bar{v}\,s,$$

where $\bar{v}$ is the average value-of-service-time estimated by $\bar{v}=\sum_i v^i\ s^i/s$.

(2) The production of capacity c satisfies the condition such that

$$c \le F(x)$$

where $x=(x_1,...,x_J)$ is the vector of inputted goods and services , and F is the production function which is increasing and differentiable in x. This production incurs the cost to the supplier, estimated by

$$Supplier\ Cost = \sum_j p_j x_j,$$

where p_j is the price of input j.

The social cost minimization problem specified above that the variable cost of supplier for system output is zero. That is, the supplier cost defined in the assumption consists only of the capacity cost which estimates the cost of inputs consumed in constructing the certain capacity of the queuing system. This assumption neglects that most of transportation services spend the variable cost constituting the large portion of the total supplier costs. Such an unrealistic assumption is adopted only to simplify the expression associated with the inclusion of the variable cost, which does not cause any basic change in the forthcoming analysis to estimate the various types of social and supplier costs.

Finally, Assumption 2 does not specify the functional form of the production technology, expressed by the function F. The production function F can be either a convex function or a concave function, each of which leads to the different returns-to-scale of the congestion-prone transportation system in capacity. Also the reason why not to specify the returns-to-scale of the production function is that the forthcoming cost analysis is free from the returns-to-scale.

3. Optimality conditions for social cost minimization problem

Under Assumptions 1 and 2, the social cost minimization problem of a congestion-prone transportation system can be expressed by the Lagrangian, denoted by Z^o, such that

$$Z^o(x,c,t,\varphi,\mu) = \min\left\{\sum_j p_j p_j + t\sum_i v_i s_i\right\} + \varphi(c-F(x)) + \mu(T(s,c)-t), \quad (5)$$

where $\varphi > 0$ and $\mu > 0$ are Lagrange coefficients. Arranging the Kuhn-Tucker conditions for the problem Z^o gives the optimal investment rule presented below.

Proposition 1: The optimum solution to the problem Z^o for the independent variable of s, denoted by $Z^o(\overline{x}, \overline{c}, \overline{t}, \overline{\varphi}, \overline{\mu})$, satisfies the following two conditions.

(1) One is the investment rule for the inputs to system capacity, such that

$$MKC(\bar{c}) \equiv \frac{\partial KC(\bar{c})}{\partial c} = \overline{\varphi} = p_j / \frac{\partial F(\bar{x})}{\partial x_j}, \forall j,$$

where MKC is the marginal capacity cost estimating the marginal increase in the social cost for one unit increase in c. Here the terms $\bar{c}$ and $\bar{x}$ are the abbreviations of functions $\bar{c}(s)$ and $\dot{x}(s)$ respectively, and estimates the optimal value of c and x for an arbitrary s, respectively.

(2) The other one is the investment rule for capacity, such that

$$MKC(\bar{c}) = -\bar{v}s\frac{\partial T(s,\bar{c})}{\partial c}$$

Proof. (1) The equality that $\partial KC(\bar{c})/\partial c + \overline{\varphi}$ follows from the fact that $\overline{\varphi}$ equals the marginal supplier cost for the increase in one unit of c. The second equality is none other than the first order conditions for Z^o with respect to x.

(2) Substituting the first order condition of Z^o with respect to t into the first order condition with respect to c gives the investment rule in Proposition 1(2). Q.E.D.

Proposition 1 introduces the two investment rules which are sufficient to determineall the unknowns in the social cost minimization problem. These two conditions are identical to the optimality conditions for the two sub-optimization problems of the original optimization problem in Eq. (5). Using these two sub-optimization problems, the economic implications of the two optimality conditions are considered below.

The first sub-optimization problem is the cost minimization problem to find the optimum solution of inputs x necessary for the production of an arbitrary capacity c. This optimization problem, denoted by Z^1, is

$$Z^1(x, \varphi) = \min\left\{\sum_j p_j x_j\right\} + \varphi(c - F(x)) \quad (6)$$

This sub-optimization problem has the first order condition with respect to x_j, which is identical to the optimality condition in Proposition 1(1).

The value of the Lagrangian Z^1 for the optimum solution, expressed by $Z^1(\hat{x}, \hat{\varphi})$, estimates the minimum supplier cost necessary for constructing a certain capacity c. Hence the capacity cost function, denoted by $KC(c)$, is

$$KC(c) = Z^1(\hat{x}, \hat{\varphi}) = \sum_j P_j \hat{x}_j + \hat{\varphi}(c - F(\hat{x})) \quad (7)$$

where $\hat{x}$ and $\hat{\varphi}$ are functions estimating the optimum solutions of x and φ for a given capacity c, respectively.

Substituting the above capacity cost function into Eq. (5) gives the second sub-optimization problem, denoted by Z^2, such that

$$Z^2(c, t, \mu) = \min\left\{KC(c) + t\sum_j v^i\ s^i\right\} + \mu(T(s,\ c) - t) \tag{8}$$

This minimization problem Z^2 has the first order conditions with respect to c and t, which are identical to the conditions for the original problem Z^o. Hence, the optimum solution of $(c,\ t)$ to the problem Z^2 is identical to the solution to the problem Z^o.

The fact that social cost minimization problems Z^o and Z^2 have the identical optimum solution of c implies the following:

$$\bar{x}(s) = \hat{x}(\bar{c}(s)) \tag{9}$$

Here the function $\bar{x}$ estimates the solution of x to the Lagrangian Z^o for variable s. On the other hand, the function $\hat{x}$ estimates the solution of x to the Lagrangian Z^1 for variable c. Also the term $\bar{c}(s)$ is the optimal capacity of the problem Z^2, which is identical to the solution of c to the problem Z^o.

Ⅲ. Marginal Cost Of System Output

1. Social marginal cost function: Uniform value-of-service-times

The social marginal cost of system outputs refers to the additional minimum social cost necessary for increasing one additional system output. Thissocial marginal cost can be estimated from the social cost minimization problem specified in Assumption 2. Such an analysis to estimate the social marginal cost is illustrated with a simplified version of the social cost minimization problem below.

Suppose that the all the random arrivals to a transportation system have the identical value-of-service-time, denoted by v. Reflecting this assumption to the social cost minimization problem Z^o gives the amended version, denoted by Z^3, such that

$$Z^3(x,\ c,\ t, \varphi, \mu) = \min\left\{\sum_j p_j x_j + tvs\right\} + \varphi(c - F(x)) + \mu(T(s,\ c)t) \tag{10}$$

For this minimization problem, the total social cost function, denoted by *TSC*, is

$$TSC(s) = Z^3\ (\overline{x},\ \bar{c},\ \bar{t}, \overline{\varphi}, \hat{\mu}) \tag{11}$$

Here, the term $\bar{x}$ is the function estimating the optimal value of x for the varying value of s, and the other terms with a bar has the same property with $\bar{x}$.

Hence the social marginal cost function, denoted by *SMC*, is estimated by

$$SMC(s) = \frac{\partial Z^3(\overline{x},\ \overline{\ },\ \bar{t},\ \overline{\varphi},\ \overline{\mu})}{\partial s} \tag{12}$$

This definition of *SMC* gives

$$SMC(s) = \frac{\partial}{\partial s} vs\,T(s;\bar{c}) = vT(s,\ \bar{c}) + vs\frac{\partial T(s;\bar{c})}{\partial s} \tag{13}$$

where $T(s;\ \bar{c})$ is an abbreviation of $T(s;\ c)|_{c=\bar{c}(s)}$ in which c is a constant being equal to $\bar{c}(s)$.

To prove Eq. (13), we first develop the alternative expressions of $\overline{\varphi}$ and $\overline{\mu}$ from the Kuhn-Tucker conditions for the Lagrangian Z^3, such that

$$\overline{\varphi} = p_j / \frac{\partial F(\bar{x})}{\partial x_j},\ \ \forall J\ ,\ \ \overline{\varphi} = -vs\frac{\partial\, T(s,\ \bar{c})}{\partial c},\ \text{and}\ \overline{\mu} = vs \tag{14}$$

Subsequently, i) differentiating $Z^3\ (\bar{x},\ \bar{c}, \bar{t}, \overline{\varphi}, \overline{\mu})$ with respect to s, ii) replacing the Lagrange coefficients in the result of the previous step with the alternative expressions in Eq. (14), and iii) simplifying the result of the second step leads to Eq. (13)

The function *SMC* in Eq. (13) estimates the additional increase in the total social cost for an increase in system outputs from s to $s+1$. This function is composed of two terms. Those two terms in common estimate the effect of the marginal increase in system outputs on the user cost at the optimal capacity, but reflect the different kinds of the effects.

The first term $vT(s, \bar{c})$ is called *the marginal user cost*, and is denoted by

$$MC(s) = vT(s, \bar{c}) > 0. \quad (15)$$

The function MUC estimates the social marginal cost of the service system, under the assumption that the facilitation of the additional output would not change the service time. The function MUC is the product of the value-of-service-time v and the expected service time $T(s, \bar{c}(s))$, both of which are common to all the system outputs. This estimate of the marginal user cost equals the average user cost which is the average time cost per user.

The second term $vs\partial T(s, \bar{c})/\partial s$ is called the marginal congestion cost, and is expressed by

$$MCC(s) = vs\frac{\partial T(s; \bar{c})}{\partial s} > 0 \quad (16)$$

The function MCC represents the additional social cost for the marginal increase in service time estimated by $\partial T(s, \bar{c})/\partial s$. The facilitation of an additional system output increases the service time of the existing consumers as well as the additional consumer by the margin $\partial T(s, \bar{c})/\partial s$ which is common to all the demands. Therefore the social marginal cost for this effect is estimated by multiplying $\partial T(s, \bar{c})/\partial s$ to vs.

2. Social marginal cost function: Heterogeneous value-of-service-times

In the above, we estimated the social marginal cost function of system output under a restrictive assumption that all the random arrivals have the identical value-of-service-time. However, in reality, the congestion-prone service systemof a service option usually facilitates its consumers who have value-of-service-times different one another, as specified in Assumption 2. We therefore extend the previous analysis for the special case to a more realistic case specified in Assumption 2.

The optimum solution of (x, c, t) to Z^o can be expressed by the function of aggregated system output $s = \sum_i s^i$, instead of $\bar{s} = (s^1, \ldots, s^1)$, as in the case of Z^3. Hence the function TSC for the case of Z^o can be expressed by

$$TSC(s) = Z^o(\bar{x}, \bar{c}, \bar{t}, \bar{\varphi}, \bar{\mu}) \quad (17)$$

Differentiating $Z^o(\bar{x}, \bar{c}, \bar{t}, \bar{\varphi}, \bar{\mu})$ with respect to s^i gives the social marginal cost specific to consumer i, as shown below.

Proposition 2: The social marginal cost specific to consumer $i$who has the value-of-service-time v^i, denoted by $SMC^i(s)$, has the expression such that

$$SMC^i(s) = \frac{\partial}{\partial s^i}\sum_i v^i s^i T(s; \bar{c}) = \mu c^i(s) + MCC(s)$$

where

$$MUC^i(s) = v^i T(s; \bar{c}), \text{ and } MCC(s) = \bar{v}s\frac{\partial T(s; \bar{c})}{\partial s}.$$

Proof. Differentiating Z^o with respect to s^i, and arranging the result of the previous step a manner analogous to that which led to Eq. (14) gives the above expression of $SMC^i(s)$. Q.E.D.

The function $SMC^i(s)$ estimates the marginal social cost increase incurred in the process to facilitate one more unit of system output s^ihaving the value-of-service-time v^i. The function $SMC^i(s)$ has the expression identical to that of the special case in Eq. (13), except for differences associated with the heterogeneous value-of-service-times among customers. Focusing on the difference between them, we examine the economic implications of $SMC^i(s)$ below.

The first term $v^i T(s,\bar{c})$ represents the marginal user cost for the demand of customer i who has the value-of-service-time v^i. This marginal user cost estimates the additional user cost incurred in facilitating one more unit of system outputs demanded by customer i, under the assumption that the facilitation of the marginal output would not change the service time. This social marginal cost $v^i T(s,\bar{c})$ equals the multiple of the value-of-service-time v^i specific to consumer i and the expected service time $T(s,\bar{c})$ common to all the system outputs.

The second term $\bar{v}s\partial T(s,\bar{c})/\partial s$ is the marginal congestion cost for the demand of consumer i. This marginal cost estimates the additional social cost which estimates the value of the marginal service time increase by one unit increase in the demand of consumer i. This marginal congestion cost is identical, irrespective of the value-of-service-time of marginal system outputs. This result is the consequence of the following: the facilitation of an additional system output increases the service time of the existing consumers as well as the additional consumer by the margin $\partial T(s,\bar{c})/\partial s$ which is common to all the demands.

Subsequently, we consider the amended version of the cost minimization problem Z^o, such that the user cost is expressed by $t\bar{v}s$, instead of $t\sum_i v^i s^i$. This cost minimization problem, denoted by Z^4, is

$$Z^4(x, c, t, \varphi, \mu) = \min\left\{\sum_j p_j x_j + t\bar{v}s\right\} + \varphi(c - F(x)) + \mu(T(s,c) - t) \quad (18)$$

This optimization problem has the identical structure with the Lagrangian Z^2 in Eq. (11), except for one difference that the value-of-service-time is expressed by the average value $\bar{v}$. Therefore, it is immediate from the previous analysis for the Lagrangian Z^2 that the Lagrangian Z^4 gives the social marginal cost introduced below.

Proposition 3: The social cost minimization problem Z^4 has the social marginal cost function which satisfies the following equality:

$$SMC(s) = \frac{1}{s}\sum_i s^i\, SMC^1(s) = MUC(s) + MCC(s),$$

where

$$MUC(s) = \bar{v}T(s, \bar{c}) \text{ and } MCC(s) = \bar{v}s\frac{\partial T(s;\bar{c})}{\partial s}.$$

Proof. The above results directly follow from Eq. (13). Q.E.D.

Proposition 3 shows that the function $SMC(S)$ of Z^4 equals the arithmetic average of the functions $SMC^I(S)$ of Z^3 for every i. Specifically, the function $MUC(s)$ estimated by $\bar{v}\,T(s, \bar{c})$ is the arithmetic average of $MUC^i(s)$ estimated by $v^i\,T(s, \bar{c})$ for all the system outputs amounting to $\sum_i s^i$. On the other hand, the function $MCC(s)$ of the problem Z^4 has the expression identical with that of problem Z^3.

3. Marginal private cost function

The social cost minimization problem of a queuing system, which is specified in Assumption 2, includes the supplier cost in its objective function. However, the expression of the social cost function in Proposition 2 does not carry any tangible information about how the social cost function is related to the supplier cost. We therefore here introduce one way to construct the supplier cost functions, and then identify the relationship with the marginal capacity cost function.

Picking up the portion of the supplier cost from the total social cost function for the Lagrangian Z^o gives the supplier cost function, called the total cost function and denoted by TC, such that

$$TC(s) = Z^o(\bar{x}, \bar{c}, \bar{t}, \bar{\varphi}, \bar{\mu}) - \sum_i \bar{t}\, v^i s^i \quad (19)$$

$$\sum_j p_j \bar{x}_j + \bar{\varphi}(\bar{c} - F(\bar{x})) + \bar{\mu}(T(\bar{s}, \bar{c}) - \bar{t})$$

The function TC is identical with the function TSC, except for one difference that the former does not include the term $\sum_i \bar{t}\, v^i s^i$ estimating the user cost. It means that, for example, the function $\bar{x}$ is identical to the function estimating the solution of x to the Lagrangian Z^o.

Differentiating TC with respect to system output s gives the marginal cost of aggregated system output s, called the marginal private cost function and denoted by MPC, such that

$$MPC(s) = \frac{\partial TC(s)}{\partial s} \quad (20)$$

This marginal private cost function satisfies the relationship with the marginal capacity cost function identified below.

Proposition 4: The marginal private cost function MPC satisfies the following equalities:

$$MPC(s) = \frac{\partial TC(s)}{\partial s^i} = MKC(\bar{c})\frac{\partial \bar{c}}{\partial s}, \quad \forall \quad i.$$

Proof. The first equation is immediate, since $s = \sum_i s^i$. The second one is developed by arranging Eq. (20), as shown in Appendix A. Q.E.D.

The marginal private cost is equivalent to the marginal cost of a neoclassical firm. This marginal cost is common to every random arrival, irrespective of its value-of-service-time, as shown in the first equation of the proposition. The marginal cost has the relationship with the marginal capacity cost shown in the second equation of the proposition. This expression of the marginal private cost however does not give any tangible information about the marginal congestion cost constituting the social marginal cost estimated in Proposition 2. For this reason, we search for another version of the supplier cost function.

4. Compensated marginal cost function

Another way to construct the supplier cost function is to pick up the portion of the supplier cost from the total social cost function, under the condition that the service time $\bar{t}$ is fixed to the optimum solution of t to the original social cost minimization problem Z^o. This supplier cost function, called the compensated total cost function and denoted by CTC, can be expressed as follows:

$$CTC(s) = Z^o(\bar{x}, \bar{c}, \bar{\varphi}, \bar{\mu}; \bar{t}) - \sum_i \bar{t}\, v^i s^i \quad (21)$$

$$\sum_j p_j \bar{x}_j + \bar{\varphi}(\bar{c} - F(\bar{x})) + \bar{\mu}(T(\bar{s}, \bar{c}) - t)|_{t=\bar{t}}$$

The function CTC has the expression identical to the function TC, except for one difference that the term $\bar{t}$ is not a function of s but a constant. Also the term 'compensated' is introduced so as to reflect the condition that the service time is fixed.

Differentiating CTC with respect to system output s gives the marginal cost of system output, called the compensated marginal cost function and denoted by CMC, such that

$$CMC(s) = \frac{\partial CTC(s)}{\partial s} \quad (22)$$

This compensated marginal cost function has the relationships with the other marginal costs shown below.

Proposition 5: The compensated marginal cost function CMC satisfies the following equalities:

$$CMC(s) = \frac{\partial CTC(s)}{\partial s^i}$$

$$= MCC(s) = MPC(s) + MEC(s), \forall i$$

where MEC stands for the marginal external cost function, defined by

$$MEC(s) = \bar{v}\, s \frac{\partial T(s,\ \bar{c})}{\partial s}$$

$$= \bar{v}\, s \frac{\partial T(s;\bar{c})}{\partial s} + \bar{v} s \frac{\partial T(s,\bar{c})}{\partial c} \frac{\partial \bar{c}}{\partial s}$$

Proof. See Appendix B.

The compensated marginal cost function is a kind of the marginal supplier cost function. This marginal cost of system output is common to every random arrival, irrespective of its value-of-service-time, as in the case the marginal private cost. The compensated marginal cost is equal to the marginal congestion cost constituting the marginal supplier cost, as shown in the second equation of the proposition. Also the compensated marginal cost is the sum of the marginal private cost and the marginal external cost, estimated by $MEC(s)$, and this relationship can be interpreted as below.

The marginal private cost, defined by $\partial TC/\partial s$, refers to the marginal supplier cost under the condition that supplier can choose the service time minimizing the total social cost without constraint. Whereas, the marginal external cost, estimated by $\bar{v} s \partial T/\partial s$, is the marginal increase in user cost, which is triggered by the supplier's action to adjust the capacity in response to the marginal increase in system output so as to minimize the total social cost. Hence it can be said that the compensated marginal cost equals the marginal supplier cost under the condition that supplier compensates the customers for the change in service time costs, which is incurred by the marginal capacity change necessary for facilitating an additional system output.

Proposition 5 shows that the marginal compensated cost equals the marginal congestion cost. However, the marginal congestion cost estimated in Proposition 2 is expressed using the service time function which has no direct relationship with the supplier cost. For this reason, we examine the possibility to identify the relationship between the compensated marginal cost and the marginal capacity cost.

Proposition 6: The functions CMC and MKC satisfy the following relationship.

(1) In the case of the transportation system with the homogeneous service technology defined in Assumption 1(1), the relationship is

$$CMC(s) = \frac{\bar{c}}{s} MKC(\bar{c}).$$

(2) In the case of the system with non-homogeneous service technology defined in Assumption 1(2), the relationship becomes

$$CMC(s) = \frac{\bar{c}}{s}(MKC(\bar{c}) - MWT(\bar{c}))$$

where

$$MWT(c) = \frac{\bar{v} s}{2\bar{y}^2} T_1^d (s/\bar{c}),$$

and $\bar{y}(s)$ is the value of y, in which the capacity of the system analyzed is $\bar{c}(s)$.

(3) The cost functions $CMC(s)$ estimated above satisfy the second equality of Proposition 5.

Proof. See Appendix C.

The above expressions of CMC in Propositions 5(1) and 5(2) have the following two advantages. First, the functions are expressed using the terms all of which are directly observable or statistically estimable. Second, the economic implication of the expressions is clear. For example, the expression for homogeneous service technology shows that the compensated marginal cost equals the multiple of the marginal capacity cost and the inverse of system utilization ratio, estimated by $\bar{c}/s$.

5. Long- versus short-run social marginal cost functions

The capacity in the social cost minimization is a long-run choice variable. Therefore the function SMC estimates in nature a long-run cost i that is, the function estimates the social marginal cost of system output under the condition that the supplier adjusts the capacity optimally to the change in system output. On the hand, the short-run social marginal cost function, denoted by $SRSMC$, estimates the social marginal cost of system output under the condition that the capacity being a long-run variable is fixed. We examine the relationship between SMC and $SRSMC$ with the social cost minimization problem for aggregated system output Z^4 below.

To begin with, we consider the social cost minimization problem under the condition that the capacity is a fixed term c^O such that $C^O = \bar{c}(s^o)$. This short-run social cost minimization problem, denoted by SRZ^4, is

$$SRZ^4(t,\mu;c^o) = \min KC(c^o) + \bar{v}ts + \mu(T(s,c^o)-1) \quad (23)$$

For this cost minimization problem, the function $SRSMC(s;c^o)$ is expressed by

$$SRSMC(s;c^o) = \frac{\partial SRZ^4(\bar{t},\bar{\mu};c^o)}{\partial s} \quad (24)$$

$$SRMUC(s;c^o) + SRMCC(s;c^o)$$

where

$$SRMUC(s;c^o) = \bar{v}\,T(s;c^o), \text{ and}$$

$$SRMCC(s;c^o) = \bar{v}s\frac{\partial T(s;c^o)}{\partial s}$$

The function $SRSMC(s;c^o)$ is composed of two terms, $SRSMC(s;c^o)$ and $SRMCC(s;c^o)$. The function $SRMUC(s;c^o)$ estimates the short-run marginal user cost of the queuing system with the capacity c^o for varying value of s. This function is positive and monotonically increasing in s, since T is monotonically increasing and convex in s, as specified in Assumption 5.1(3). On the other hand, the function $SRMUC(s;c^o)$ estimates the marginal congestion cost of the service system with the capacity c^o for varying value of s. This function, expressed by $\bar{v}s\partial T(s;c^o)/\partial s$, is also positive and monotonically increasing in s, since T is monotonically increasing and convex in s.

Subsequently, we consider the relationship between the long- and short-run social marginal cost functions. By the condition that $c^o = \bar{c}(s^o)$, it is certain that the long-run social marginal cost at system output s^o, denoted by $SMC(s^o)$, equals $SRSMC(s^o,c^o)$:

$$SMC(s^o) = SRSMC(s^o;c^{o)} \quad (25)$$

$$= \bar{v}\,T(s^o,\ c^o) + \bar{v}s^o\frac{partila\,T(s^o;c^o)}{\partial s}$$

This implies that

$$SRMUC(s^o;\ c^o) = MUC(s^o) \text{ and}$$

$$SRMCC(s^o;c^o) = MCC(s^o) \quad (26)$$

The above results show that that the long- and short-run social marginal cost functions have the identical value at the point $(s^o,\ c^o)$ These two relationship can be seen in the literature dealing with the congestion pricing of highway service, such as Wohl (1972) and Moon and Park (2002).

Ⅴ. Specific Examples

1. An example of homogeneous service time functions

The service time function in Eq. (2) is an approximation which could be applicable to most of transportation systems offering services on the first-in-first-out basis. Using this service time function, we here illustrate the procedure to estimate the optimal capacity and the various cost functions from the social cost minimization problem for the aggregated system output in Eq. (18).

Substituting this service time function in Eq. (2) into the social cost minimization problem in Eq. (18) gives

$$Z^4(c, t, \varphi, \mu) = \min\{KC(c) + \bar{v}ts\} + \mu\left(t^o + t^d \frac{Js/c}{1-s/c} - tRIGHT\right) \quad (27)$$

For the above minimization problem, the optimality investment rule in Proposition 1(2) has the expression such that

$$MKC(\hat{c}) = \frac{\partial KC(\hat{c})}{\partial c} = \bar{v}J\left(\frac{s/\bar{c}}{1-s/\bar{c}}\right) \quad (28)$$

From this investment rule, we develop the specific expression of the function to estimate the optimal capacity for the varying value of s below.

Let $\theta(s)$ denote that

$$\theta(s) = \frac{MKC(\bar{c})}{\bar{v}J} \quad (29)$$

Then Eq. (28) can be rearranged as follows:

$$\bar{c}(s) = \left(1 + \frac{1}{\sqrt{\theta(s)}}\right)s. \quad (30)$$

This equation depicts that the optimal capacity $\bar{c}(s)$ should be larger then the system output s by the ratio $(1+1/\sqrt{\theta})$.

The formula (30) could be interpreted as follows. First, the ratio $(1+1/\sqrt{\theta})$ becomes smaller as the value of $MKC(\bar{c})$ is larger that is, the transportation system which has the larger value of $MKC(\bar{c})$ should have the optimal capacity larger then the system output S by a smaller margin. Second, the ratio $(1+1/\sqrt{\theta})$ becomes larger as the value of $\bar{v}$ is larger that is, the service system which serves the customers having larger value-of-service-time should offer the optimal capacity larger than the system output by a larger margin.

Subsequently, we find the specific expressions of the various cost functions. By Proposition 3, the function SMC satisfies the following relationship:

$$SMC(s) = MUC(S) + MCC(s) \quad (31)$$

Also substituting Eq. (30) into the function MUC in Proposition 3 gives

$$MUC(s) = \xi t^o + \bar{v}t^d J \frac{s/\bar{c}}{1-s/\bar{c}} \quad (32)$$

$$\bar{v}t^o + t^d \sqrt{\bar{v}JMKC(\bar{c})}$$

Subsequently, it follows from Proposition 5 that MCC is equal to CMC being a kind the marginal supplier cost function. By Proposition 6, the function CMC for homogeneous service time function in Eq. (2) satisfies the following relationship:

$$CMC(s) = \frac{\bar{c}}{s} MKC(\bar{c}). \quad (33)$$

Substituting Eq. (30) into the right side of Eq. (33) gives

$$CMC(s) = MKC(\bar{c}) + \sqrt{\bar{v}J\,MKC(\bar{c})}. \quad (34)$$

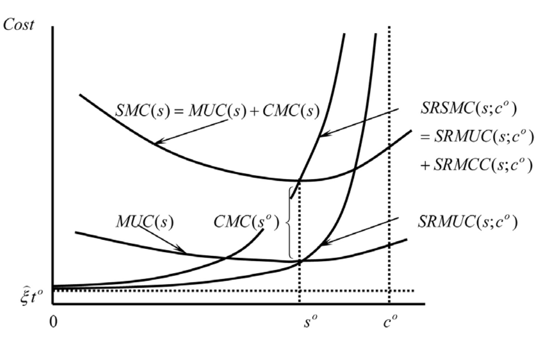
This shows that the function CMC can be expressed as the function of $MKC(\bar{c})$.

Finally, we depict the configurations of $SMC(s)$ under the following conditions: $MKC(\bar{c})$ is locally convex in s, and has a minimum at the point s^o. Then it is immediate that $\sqrt{MKC(\bar{c})}$ is also convex in s, and has the minimum at s^o. Hence it follows from Eqs. (32) and (34) that both of $MUC(s)$ and $CMC(s)$ have such properties. Therefore, it is clear that $SMC(s)$ being the sum of $MUC(s)$ and $CMC(s)$ shares the same properties. Such results are schematically illustrated in ⟨Figure 2⟩.

⟨Figure 2⟩ also illustrates the following relationships between long- and short-run marginal costs, which are depicted in Eqs. (25) and (26). First, the functions $SRMUC(s;c^o)$ and $SRMCC(s;c^o)$ are increasing in system output , as explained previously in connection with Eq. (24). Therefore the function $SRSMC(s;c^o)$ is also increasing in s. Second, the value of $SRMUC(s;c^o)$ at the point s^o is $\bar{v}T(s^o/c^o)$ which equals $MKC(s^o)$, as shown in Eq.(26). Third, the value of $SRSMC(s;c^o)$ at the point s^o equals the value of $CMC(s)$ at that point, as claimed in Eq. (25).

2. An example of non-homogeneous service time function

The service time function in Eq. (4) could be a plausible approximation of the urban public transit or shuttle service in a monopolistic position on a certain corridor. This service time function is not homogeneous of degree zero in its variables of s and , due to the presence of the term $1/2c$. For this service time function, we carry out a series



⟨Figure 2⟩ Social marginal cost function for homogeneous service technology

of analyses in the fashion analogous to the analyses for the homogeneous service time function in Eq. (2).

For that service time function, the social cost minimization problem for the aggregated system output can be expressed as follows:

$$Z^4(c,t,\varphi,\mu) = \{KC(c) + \bar{v}ts\} + \mu\left(t^o + \frac{1}{2c}(1 + T^d(s/\tau c)) - t\right) \tag{35}$$

Here the delay function T^d is defined by

$$T^d(s/\tau c) = \frac{Js/\tau c}{1 - s/\tau c} = \frac{J\rho}{1-\rho} \tag{36}$$

where $\rho \equiv s/\tau c$ is the system utilization ratio of the transit system considered here. It is also assumed that

$$KC(c) = \alpha + \beta c \tag{37}$$

where $\alpha \geq 0$, and $\beta \rangle 0$, in order to simplify the analysisto figure out the configuration of $SMC(S)$.

For the social cost minimization problem defined above, the optimal investment rule in Proposition 1(2) has the expression such that

$$\beta = \bar{v}s\left(\frac{1}{2\bar{c}^2}\left(1 + T^d(s/\tau\bar{c})\right) - \frac{1}{2\bar{c}}\frac{\partial T^d(s/\tau\bar{c})}{\partial c}\right) \tag{38}$$

Differentiating the above equation with respect to s, and estimating $\partial\bar{c}/\partial s$ from the result of the previous step gives

$$0 \langle \frac{s}{\bar{c}}\frac{\partial\bar{c}}{\partial s} \leq 1 \tag{39}$$

as shown in Appendix D. These inequalities implies that

$$\frac{\partial\bar{\rho}}{\partial s} = \frac{\partial}{\partial s}\left(\frac{s}{\tau\bar{c}}\right) = \frac{1}{\tau\bar{c}}\left(1 - \frac{s}{\bar{c}}\frac{\partial\bar{c}}{\partial s}\right) \leq 0 \tag{40}$$

Hence it can be concluded that the system utilization ratio $\bar{\rho}(s)$ is diseasing in s that is, the optimal capacity $\bar{c}(s)$ approaches to system output s, as output $s$increases.

Subsequently, we estimate the specific expressions of the various cost functions. By Proposition 3, it follows that

$$SMC(c) = MUC(s) + MCC(s) \tag{41}$$

Also, by Proposition 6(2), it holds that

$$MCC(s) = CMC(s) \le \frac{\beta}{\tau\bar{\rho}} \tag{42}$$

This implies that that the marginal supplier cost $SMC(S)$ is smaller than the average supplier cost of system output, estimated by $\beta/\tau\bar{\rho}(s)$. However, this inequalityis not sufficient to figure out the configuration of $SMC(S)$. We therefore carry out the two different kinds of analyses.

The first kind of the analysis to estimate the limiting values of MUC and CMC. Such an analysis shows that

$$\lim_{s\to+0} MUC(s) = \infty\ , \quad \lim_{s\to+0} CMC(s) = \frac{J\beta}{\tau} \tag{43}$$

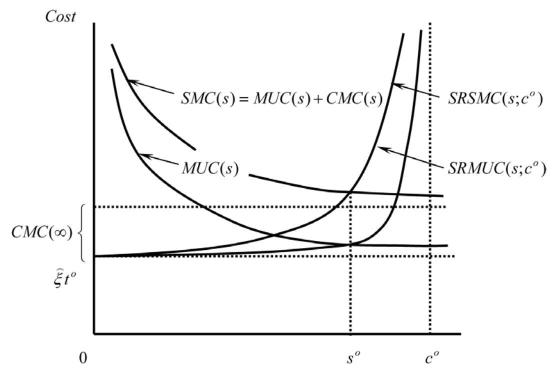
$$\lim_{s\to\infty} MUC(s) = t_o\ , \quad \lim_{s\to\infty} CMC(s) = \lim_{s\to\infty}\frac{1}{\bar{\rho}}MKC(\bar{c}) = \frac{\beta}{\tau} \tag{44}$$

as proved in Appendix E. The second kind of the analysis to estimate the slope of MUC and MMC. Differentiating these two functions with respect to s gives

$$\frac{\partial MUC(s)}{\partial s} \langle\ 0, \tag{45}$$

$$\frac{\partial CMC(s)}{\partial s} \rangle\ 0 \tag{46}$$

as shown in Appendix E. Using the above results,



〈Figure 3〉 Socialmarginal cost function for non-homogeneous service technology

the configurations of various cost functions are depicted in 〈Figure 3〉.In addition, the relationships between the long- and short-run marginal costs in Eqs. (25) and (26) are also illustrated in the figure.

Ⅴ. Summary And Concluding Remarks

This paper analyzed the various marginal cost functions of congestion-prone transportation systems, all of which are developed from the social cost minimization problem. The cost analysis of this paper for congestion-prone transportation systems has the following peculiar aspects differentiated from that of the previous studies. First, the service time function of the congestion-prone service system is formulated in a generalized format so that the analysis can be applicable to as many as possible number of congestion-prone systems. Second, the social cost minimization problem is constructed under the condition that each consumer has a value-of-service-time different from the others. Third, the relationship between the marginal congestion and marginal supplier costs are explored.

One important finding of the cost analysis is that the social marginal cost specific to individual i, denoted by $SMC^i(S)$, is expressed by

$$SMC^i(s) = v^i\,T(s,\bar{c}) + MCC(s)\ , \text{ and}$$

$$MCC = (\bar{v}s)\frac{\partial T(s;\bar{c})}{\partial s}$$

where s^i is the demand of consumer i, $s \equiv \Sigma_i s^i$ the aggregated system output, v^i the value-of-service-time perceived by consumer i, and $\bar{v} = \Sigma_i v^i s^i / s$ the average value-of-service-time of all the customers.

The right side of SMC^i is composed of the two terms, each of which estimates the different effect of the marginal increase in system output on the user cost. The first term, called the marginal user cost, estimates the social marginal user cost for an additional increase in the demand of consumer i, under the condition that the facilitation of an additional system output does not change the service time. On the other hand, the second term, called the marginal congestion cost, estimates the social marginal cost for the service time increase caused by the one unit increase in system output, under the conditionthat the capacity is not changed.

Another finding is the relationship between the marginal congestion cost and the two different kinds of the marginal supplier cost. One kind of the marginal supplier cost is the marginal private cost of the supplier, denoted by MPC, such that

$$MPC(s) = MKC(\bar{c})\frac{\partial \bar{c}}{\partial s},$$

where $MKC(c) = \partial KC(c)/\partial c$ is the marginal capacity cost function estimating the marginal supplier cost necessary for increasing one unit of capacity c. Another kind of the marginal supplier cost is the compensated marginal cost, denoted by , which satisfies the following relationship:

$$CMC(s) = MCC(s) = MPC(s) + \bar{v}s\frac{\partial T(s,\bar{c})}{\partial s}.$$

Here, the last term, called the marginal external cost function, estimates the marginal user cost increase caused by one unit growth in system output under the condition that the supplier adjusts the capacity optimally to a marginal increase in system output.

The marginal private cost MPC is equivalent to the marginal cost of the neoclassical firm. The expression of MPC however has a shortcoming in that it does not contain any information about SMC^i introduced previously. On the other hand, the compensated marginal cost is the marginal supplier cost under the condition that the supplier compensates the customers for the change in service time costs, which is incurred by the marginal capacity increase necessary for facilitating an additional system output. The marginal cost CMC equals to the marginal congestion cost MCC constituting the social marginal cost SMC^i.

The other finding is the relationship between CMC and MKC. This relationship for the service system with homogeneous service technology is

$$CMC(s) = \frac{1}{\bar{\rho}(s)}MKC(\bar{c}),$$

where $\bar{\rho}(s) \equiv s/\bar{c}(s)$ is the optimal system utilization ratio. On the other hand, the relationship for the scheduled transportation service with non-homogeneous service technology is

$$CMC(s) = \frac{1}{\bar{\rho}(s)}\left(MKC(\bar{c}) - MWT(s)\right).$$

Here, $\bar{\rho}^{-1}MWT(s)$ is the external benefit for the marginal waiting time decrease common to all the customers, which is accrued by the marginal capacity increase necessary for facilitating one more system output.

Subsequently, the analysis presented in this paper can readily be extended to the more generalized cases of congestion-prone transportation systems. One direction of the extension could be to accommodate the system which consumes the variable costs which is usually an increasing function of system output. This extension can be approached by simply adding the variable cost term to the supplier cost side of the social cost minimization problem. Another possible direction could be to extend the analysis for the single period problem to the multi-period problem which covers the congestion-prone transportation system serving the peaking demand. One way of the extension could be to expresses the user and supplier cost of the social cost minimization problem in the fashion analogous to that of Moon and Park (2002a).

Finally, we consider the utility of the analysis presented in this paper. The analysis has the significance in that it shows the more detailed structure of the various marginal cost functions, especially the relationship between the social marginal cost function and the compensated marginal cost function. Also the analysis can be incorporated into the social welfare maximization problem that estimates the optimal pricing and investment rule of the congestion-prone transportation systems operated by the public agency.

References

1. Boudreau P.E., Griffin J.S. and Mark Kac (1962) An elementary queuing problem. American Mathematics Monthly, Vol. 69: 713~724
2. Cinlar E. (1975) Introduction to Stochastic Processes. Prentice-Hall, Inc.
3. Hall R.E. (1973) The specification of technology with several kinds of outputs. Journal of Political Economy 85(1) 1-25
4. Keeler T.E. and Small K.A. (1977) Optimal peak-load pricing, investment, and service levels on urban expressways. Journal of Political Economy, 85(1): 1-25
5. Manheim M.L. (1982) Fundamentals of Transportation Systems Analysis. The MIT press, Cambridge MA
6. Meyer J.R., Kain J.F. and Wohl. M (1965) The Urban Transportation Problem. Harvard University Press, Cambridge Mass.
7. Mohring H (1970) The peak load problem with increasing returns and pricing constraints. American Economic Review 60:693-705
8. Mohring H. (1972) Optimization and scale economies in urban bus transportation. American Economic Review, 62: 591-604
9. Mohring H. and Harwitz M. (1962) Highway Benefits: An Analytical Framework. Northwestern University, Evanston Illinois
10. Moon D.J. and Park C.H. (2002a) Cost functions for multiple transportation modes in the presence of congestion. Annals of Regional Science, 36:315-346
11. Moon, D. J. and Park, C. H (2002b) Congestion-pricing of multi-modal transportation systems. The Annals of Regional Science, 36, 455-481
12. Parzen (1962) Stochastic Process. Holden Day Co., San Francisco
13. Small K.A. (1992) Urban Transportation Economics. Harwood Academic Publishers,
14. Transportation research board (1999) Highway Capacity m Manual, Special Report 209. National research council, Washington DC
15. Uzawa H. (1964) Duality principles in the theory of cost and production. International Economic Review, 5(2):216-220
16. Vickrey, W. S. (1963) Pricing in urban and suburban transport. American Economic Review, Papers and Proceedings, 53, 452-465
17. Williamson O.E. (1966) Peak-load pricing andoptimal capacity under indivisible constraints. American Economic Review, 56:810-827
18. Wohl M. (1972) Transportation Investment Planning. Lexington Books, Lexington Mass.

Appendix A: Proof of Proposition 4

The total cost function of the supplier, and denoted by TC, is defined as follows

$$TC(s) = \sum_j p_j \bar{x}_j + \bar{\varphi}\left(\bar{c} - F(\bar{x})\right) + \bar{\mu}\left(T(\bar{s},\bar{c}) - \bar{t}\right) \quad (A.1)$$

Here the optimal values of $\bar{\varphi}$ and $\bar{\mu}$ satisfies the following equality:

$$\bar{\varphi} = p_j \Big/ \frac{\partial F(\bar{x})}{\partial x_j}, \ \forall j, \quad \bar{\varphi} = -\bar{v}s\frac{\partial T(s,\bar{c})}{\partial c}, \text{ and } \bar{\mu} = \bar{v}s \quad (A.2)$$

as shown in Eq. (18) of the text.

Differentiating TC with respect to s, and substituting Eq. (A.2) into the result of the previous step gives the marginal private cost function MPC such that

$$\frac{\partial TC(s)}{\partial s} = MPC(s) = \sum_j p_j \frac{\partial \bar{x}_j}{\partial s} + \bar{\varphi}\frac{\partial \bar{c}}{\partial s} - \sum_j \bar{\varphi}\frac{\partial F(\bar{x})}{\partial x_j}\frac{\partial \bar{x}_j}{\partial s}$$
$$+ \bar{\mu}\frac{\partial T(s;\bar{c})}{\partial s} + \bar{\mu}\frac{\partial T(s,\bar{c})}{\partial c}\frac{\partial \bar{c}}{\partial s} - \bar{\mu}\frac{\partial T(s,\bar{c})}{\partial c} = \bar{\varphi}\frac{\partial \bar{c}}{\partial s} \quad (A.3)$$

Substituting (A.2) and Proposition 1(2) sequentially into (A.3) gives

$$\bar{\varphi}\frac{\partial \bar{c}}{\partial s} = -\bar{v}s\frac{\partial T(s,\bar{c})}{\partial c}\frac{\partial \bar{c}}{\partial s} = MKC(\bar{c})\frac{\partial \bar{c}}{\partial s} \quad (A.4)$$

Appendix B: Proof of Proposition 5

(1) Prove first that $CMC(s) = MPC(s) + MEC(s)$. The compensated total cost function and denoted by CTC, is defined as follows:

$$CTC(s) = \sum_j p_j \bar{x}_j + \bar{\varphi}\left(\bar{c} - F(\bar{x})\right) + \bar{\mu}\left(T(\bar{s},\bar{c}) - t\right)\Big|_{t=\bar{t}} \quad (B.1)$$

Differentiating CTC with respect to s, and arranging the result in the manner that led to (A.3) gives

$$\frac{\partial CTC(s)}{\partial s} = CMC(s) = \bar{\varphi}\frac{\partial \bar{c}}{\partial s} + \bar{\mu}\frac{\partial T(s;\bar{c})}{\partial s} + \bar{\mu}\frac{\partial T(s,\bar{c})}{\partial c}\frac{\partial \bar{c}}{\partial s}$$
$$= MPC(s) + \bar{\mu}\frac{\partial T(s;\bar{c})}{\partial s} + \bar{\mu}\frac{\partial T(s,\bar{c})}{\partial c}\frac{\partial \bar{c}}{\partial s} \quad (B.2)$$

Substituting the relationship $\bar{\mu} = \bar{v}s$ in (A.2) into the second and third terms of (B.2) respectively gives the term $MEC(s)$ in the proposition.

(2) Prove next that $MCC(s) = MPC(s) + MEC(s)$. Since the function $STC(s)$ defined in Eq. (17) has an additional term $\Sigma_i \bar{t}v^i s^i$ than the function $TC(s)$ in Eq. (19), it is immediate that

$$SMC^i(s) = MUC^i(s) + MCC(s)$$
$$= MUC^i(s) + \sum_i v^i s^i \frac{\partial T(s,c)}{\partial s} + MPC(s) \quad (B.3)$$

This equality implies the assertion.

Appendix C: Proof of Proposition 6

(1) By Proposition 5, it follows that

$$CMC(s) = \bar{v}\,s\frac{\partial T(s/c;c=\bar{c})}{\partial s} \quad (C.1)$$

Substituting Eq. (1) and Proposition 1(2) into the right side of (C.1) sequentially gives

$$CMC(s) = \bar{v}\frac{\bar{c}}{s}\left(-s\frac{\partial T(s/\bar{c})}{\partial c}\right) = \frac{\bar{c}}{s}\frac{\partial KC(\bar{c})}{\partial c} \quad (C.2)$$

(2) By the homogeneity of T_1^d and T_2^d, it follows that

$$\frac{\partial T(s,c)}{\partial s}=\frac{1}{2y}\frac{\partial T_1^d(s/c)}{\partial s}+\frac{\partial T_2^d(s/c)}{\partial s}$$

$$=-\frac{c}{s}\left(\frac{1}{2y}\frac{\partial T_1^d(s/c)}{\partial c}+\frac{\partial T_2^d(s/c)}{\partial c}\right) \quad (C.3)$$

Using the above relationship, the optimality condition in Proposition 1(2) can be rearranged as below:

$$MKC(\bar{c})=\bar{v}s\left(\frac{1}{2\bar{y}^2}T_1^d(s/\bar{c})-\frac{1}{2\bar{y}}\frac{\partial T_1^d(s/\bar{c})}{\partial c}-\frac{\partial T_2^d(s/\bar{c})}{\partial c}\right)$$

$$=\bar{v}\frac{s}{2\bar{y}^2}T_1^d(s/\bar{c})+\frac{s}{\bar{c}}\bar{v}s\left(\frac{1}{2\bar{y}}\frac{\partial T_1^d(s/c;c=\bar{c})}{\partial s}+\frac{\partial T_2^d(s/c;c=\bar{c})}{\partial s}\right)$$

$$=\frac{\bar{v}s}{2\bar{y}^2}T_1^d(s/\bar{c})+\frac{s}{\bar{c}}MCC(s) \quad (C.4)$$

(3) Prove first that the function CMC for the homogeneous service technology in Proposition 6(1) satisfies the relationship in Proposition 5. The first term of MEC in Proposition 5 equals CMC. Replacing this function CMC by the alternative expression of CMC in Proposition 6(1), and substituting Proposition 1(2) into the second term of MEC gives

$$MEC(s)=\left(\frac{\bar{c}}{s}-\frac{\partial\bar{c}}{\partial s}\right)MKC(\bar{c}) \quad (C.5)$$

Substituting MPC in Proposition 4 into (C.5), and substituting again the result of the previous step into Proposition 6(1) gives the equality such that $CMC$is the sum of MPC and MEC, as claimed in Proposition 5.

Prove next the case of non-homogeneous service technology. Substituting Proposition 6(2) into the first termof MEC in Proposition 5, and substituting Proposition 1(2) into the second term of MEC gives

$$MEC(s)=\left(\frac{\bar{c}}{s}-\frac{\partial\bar{c}}{\partial s}\right)MKC(\bar{c})-\frac{\bar{c}}{s}MWT(\bar{c}) \quad (C.6)$$

Here the term $(\bar{c}/s)MWT(\bar{c})$ represents the marginal external benefit which estimates the value for the waiting time decrease brought by an addition of one more service frequency. Substituting Proposition 4 into (C.6), and substitutingagain the result of the previous step into Proposition 6(2) gives the equality in Proposition 5.

Appendix D: Proof of Eq. (39)

(1) Outline of the proof: We first estimate $\partial\bar{c}/\partial s$ from the following optimality condition:

$$\beta=\bar{v}s\left(\frac{1}{2\bar{c}^2}\left(1+\bar{T}^d\right)-\frac{1}{2\bar{c}}\frac{\partial\bar{T}^d}{\partial c}\right) \quad (D.1)$$

where $\bar{T}^d=T^d(s/\bar{c})$. Differentiating (D.1) with respect to s yields

$$0=\bar{v}\frac{1+\bar{T}^d}{2\bar{c}^2}-\bar{v}s\frac{1+\bar{T}^d}{\bar{c}^3}\frac{\partial\bar{c}}{\partial s}+\bar{v}s\frac{1}{2\bar{c}^2}\frac{\partial\bar{T}^d}{\partial s}+\bar{v}s\frac{1}{2\bar{c}^2}\frac{\partial\bar{T}^d}{\partial c}\frac{\partial\bar{c}}{\partial s}$$

$$-\bar{v}\frac{1}{2\bar{c}}\frac{\partial\bar{T}^d}{\partial c}+\bar{v}s\frac{1}{2\bar{c}^2}\frac{\partial\bar{T}^d}{\partial c}\frac{\partial\bar{c}}{\partial s}-\bar{v}s\frac{1}{2\bar{c}}\frac{\partial^2\bar{T}^d}{\partial s\partial c}-\bar{v}s\frac{1}{2\bar{c}}\frac{\partial^2\bar{T}^d}{\partial c^2}\frac{\partial\bar{c}}{\partial s} \quad (D.2)$$

Estimating $\partial\bar{c}/\partial s$ from (D.2) gives

$$\frac{s}{\bar{c}}\frac{\partial\bar{c}}{\partial s}=\frac{A}{B} \quad (D.3)$$

where

$$A=\frac{1}{2\bar{c}^2}\left(1+\bar{T}^d\right)+\frac{s}{2\bar{c}^2}\frac{\partial\bar{T}^d}{\partial s}-\frac{1}{2\bar{c}}\frac{\partial\bar{T}^d}{\partial c}-\frac{s}{2\bar{c}}\frac{\partial^2\bar{T}^d}{\partial s\partial c} \quad (D.4)$$

$$B=\frac{1+\bar{T}^d}{\bar{c}^2}-\frac{1}{\bar{c}}\frac{\partial\bar{T}^d}{\partial c}+\frac{1}{2}\frac{\partial^2\bar{T}^d}{\partial c^2} \quad (D.5)$$

By rearranging terms in (A.4) and (A.5), it will be shown that

$$\frac{1}{2} < \frac{s}{\bar{c}}\frac{\partial \bar{c}}{\partial s} \le 1 \tag{D.6}$$

(2) Rearrangement of (D.4) and (D.5): Since the function T^d is homogeneous, the last term of (D.4) can be rearranged as follows:

$$-\frac{s}{2\bar{c}}\frac{\partial^2 \bar{T}^d}{\partial s \partial c} = \frac{s}{2\bar{c}}\frac{\partial}{\partial c}\left(-\frac{\partial \bar{T}^d}{\partial s}\right) = \frac{s}{2\bar{c}}\frac{\partial}{\partial c}\left(\frac{\bar{c}}{s}\frac{\partial \bar{T}^d}{\partial c}\right)$$

$$= \frac{1}{2\bar{c}}\frac{\partial \bar{T}^d}{\partial c} + \frac{1}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2}. \tag{D.7}$$

Subsequently, multiplying $\bar{v}s$ to A in (D.4), and substituting (D.1) and (D.7) into the result of the previous step gives

$$\bar{v}\,s\,A = \beta + \frac{\bar{v}s^2}{2\bar{c}^2}\frac{\partial \bar{T}^d}{\partial s} + \frac{\bar{v}s}{2c}\frac{\partial \bar{T}^d}{\partial c} + \frac{\bar{v}s}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2} \tag{D.8}$$

$$= \beta + \frac{\bar{v}s}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2} > 0 \tag{D.9}$$

Note that the (D.8) is simplified into (D.9) using the fact that the function T^d is homogeneous. Finally, multiplying $\bar{v}s$ to B in (D.5), and substituting (D.1) into the result of the previous step gives

$$\bar{v}\,s\,B = 2\beta + \frac{\bar{v}s}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2} > 0 \tag{D.10}$$

(3) The proof of the inequalities in (D.6): Substituting (D.9) and (D.10) into (D.3) gives

$$\frac{s}{\bar{c}}\frac{\partial \bar{c}}{\partial s} = \left(\beta + \frac{\bar{v}s}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2}\right) \Big/ \left(2\beta + \frac{\bar{v}s}{2}\frac{\partial^2 \bar{T}^d}{\partial c^2}\right) \tag{D.11}$$

Since the second term of the numerator and denominator is positive, it is clear that $(s/\bar{c})\partial\bar{c}/\partial s < 1/2$. On the other hand, when $\bar{c}$ approaches to s, it follows that

$$\lim_{\bar{c}\to s}\frac{s}{\bar{c}}\frac{\partial^2 \bar{T}^d}{\partial c^2} = \lim_{\bar{c}\to s}\frac{2Js^2\tau^2/(\tau\bar{c})^3}{\bar{c}(1-s/\tau\bar{c})^3} = \infty \tag{D.13}$$

It will be shown in Appendix E that, when s approaches to ∞, the term $\bar{c}(s)$ also approaches to s.

Appendix E: Proof of Eqs. (43)~(46)

(1) Proof of Eq. (43): The proof is worked out using the special solution to the differential equation in (D.1), such that

$$\bar{c}(s) = \left(\frac{\bar{v}J}{2\beta}\right)^{1/2} s^{1/2} \equiv \alpha s^{1/2} \tag{E.1}$$

Note that this special solution satisfies the equality in (D.1) under the condition that s approaches to 0 from the right.

Using the above special solution, the value of $\lim_{s\to\infty} MUC(s)$ is estimated below:

$$\lim_{s\to+0} MUC(s) = \lim_{s\to+0}\frac{\bar{v}}{2\bar{c}}\frac{Js/\bar{c}}{1-s/\bar{c}}$$

$$= \lim_{s\to+0}\frac{\bar{v}}{\alpha s^{1/2}}\frac{Js/\tau\alpha s^{1/2}}{1-s/\tau\alpha s^{1/2}}$$

$$= \lim_{s\to+0}\frac{\bar{v}Js}{\tau\alpha s^{1/2}(\tau\alpha s^{1/2}-s)} = \infty \tag{E.2}$$

Finally, the value of $\lim_{s\to+0} CMC(s)$ is

$$\lim_{s\to+0} CMC(s) = \lim_{s\to+0}\frac{\bar{v}s}{2\bar{c}}\frac{\partial \bar{T}^d}{\partial s} = -\lim_{s\to+0}\frac{\bar{v}}{2}\frac{\partial \bar{T}^d}{\partial c}$$

$$= \lim_{s\to+0}\frac{\bar{v}}{2}\frac{J\tau s}{(\tau\bar{c}-s)^2}$$

$$= \lim_{s\to 0}\frac{\bar{v}}{2}\frac{J\tau}{(\tau\alpha - s^{1/2})^2} = \frac{J\beta}{\tau} \tag{E.3}$$

(2) Proof of Eq. (44): The proof is worked out using the special solution to the differential equation in (D.1), such that

$$\breve{c}(s) = \frac{s}{\tau} + \left(\frac{\bar{v}J}{2\beta}\right)^{1/2} s^{1/2} \equiv \frac{s}{\tau} + \alpha s^{1/2} \quad \text{(E.4)}$$

Note that this special solution satisfies the equality in (D.1) under the condition that s approaches to ∞.

Using the above special solution, the value of $\lim_{s\to\infty} MUC(s)$ is estimated below:

$$\lim_{s\to\infty} MUC(s) = \lim_{s\to\infty} \frac{\bar{v}}{s/\tau + \alpha s^{1/2}} \frac{Js/\tau(s/\tau + \alpha s^{1/2})}{1 - s/\tau(s/\tau + \alpha s^{1/2})}$$

$$= \lim_{s\to\infty} \bar{v} \frac{Js}{\alpha\tau s^{1/2}(s/\tau + \alpha s^{1/2})} = 0 \quad \text{(E.5)}$$

On the other hand, the value of $\lim_{s\to\infty} CMC(s)$ is

$$\lim_{s\to\infty} MCC(s) = \lim_{s\to\infty} \frac{\bar{v}}{2} \frac{2\beta}{\bar{v}J} \frac{J\tau s}{\tau^2 s} = \frac{\beta}{\tau} \quad \text{(E.6)}$$

(3) Proof of Eq.(45): It is obvious from (E.2) and (E.5) that $MUC(s)$ is decreasing in s. However, it appears to difficult to prove this assertion by evaluating the sign of $\partial MUC(s)/\partial s$, as shown below:

$$\frac{\partial MUC(s)}{\partial s} = -\frac{\bar{v}}{2\bar{c}^2}(1 + \bar{T}^d)\frac{\partial \bar{c}}{\partial s} + \frac{\bar{v}}{2\bar{c}}\frac{\partial \bar{T}^d}{\partial c}\frac{\partial \breve{c}}{\partial s} + \frac{\bar{v}}{2\bar{c}}\frac{\partial \bar{T}^d}{\partial s}$$

$$= -\frac{1}{s}\left(\beta\frac{\partial \bar{c}}{\partial s} - CMC(s)\right) \quad \text{(E.7)}$$

$$= -\frac{1}{s}\left(\beta\frac{\partial \bar{c}}{\partial s} - \frac{\bar{c}}{s}\beta + \frac{\bar{v}}{2s}T^d\right)$$

$$= -\frac{1}{s}\frac{\bar{c}}{s}\left(\beta\left(\frac{s}{\bar{c}}\frac{\partial \bar{c}}{\partial s} - 1\right) + \frac{\bar{v}}{2\bar{c}}\bar{T}^d\right) \le 0 \quad \text{(E.8)}$$

The above result does not provide concrete information about the sign of $\partial MUC(s)/\partial s$. Specifically, the first term of (E.7) is positive; whereas, the second is negative. In contrast, the first term of (E.8) is negative; whereas, the second term is positive. Therefore it is difficult to find clear-cut information about the sign from the above results.

Nonetheless, we can infer from the above expressions information necessary for figuring out the configuration of $MUC(s)$, which is depicted in 〈Figure 3〉. First, if s has the value near to zero, the absolute value of the first term of (E.8) is significantly smaller than that of the second term being infinitely large. That is, the sign is negative when s is small. Second, it is certain that the absolute value of the first term of (E.7) approaches to the absolute value of the second term, as s grows to infinite. That is, the value of $\partial MUC(s)/\partial s$ approaches to zero, as s grows to infinite.

(4) Proof of Eq. (46):

$$\frac{\partial CMC(s)}{\partial s} = \frac{\bar{v}}{2\bar{c}}\frac{\partial \bar{T}^d}{\partial s}\left(1 - \frac{s}{\bar{c}}\frac{\partial \breve{c}}{\partial s}\right) + \frac{\bar{v}s}{2\bar{c}}\frac{\partial^2 \bar{T}^d}{\partial s^2}\left(1 - \frac{s}{\bar{c}}\frac{\partial \bar{c}}{\partial s}\right)$$

$$= \frac{\bar{v}}{2\bar{c}}\left(\frac{\partial \bar{T}^d}{\partial s} + s\frac{\partial^2 \bar{T}^d}{\partial s^2}\right)\left(1 - \frac{s}{\bar{c}}\frac{\partial \bar{c}}{\partial s}\right) \ge 0 \quad \text{by (D.6).}$$

♧ 주 작 성 자 : 문동주
♧ 교 신 저 자 : 문동주
♧ 논문투고일 : 2007. 10. 23
♧ 논문심사일 : 2007. 11. 29 (1차)
2007. 12. 21 (2차)
♧ 심사판정일 : 2007. 12. 21
♧ 반론접수기한 : 2008. 4. 30