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INFERENCE FOR RELIABILITY OF A BURR DISTRIBUTION†

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ABSTRACT

The distributional properties of $\Pr(Y < X)$ and $X/(X+Y)$ and related estimation procedures are derived when X and Y are independent and identically distributed according to the Burr distribution. An application of the results is provided to drought data from Nebraska.

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1. INTRODUCTION

The Burr distribution is one of the most versatile distributions in statistics. As shown by Rodriguez (1977) and Tadikamalla (1980), the Burr distribution contains the shape characteristics of the normal, log-normal, gamma, logistic and exponential distributions as well as a significant portion of the Pearson type I, II, V, VII, IX and XII families. It has received applications in life testing and many other areas (see, Wingo, 1983, 1993; Surles and Padgett, 1998; Zimmer *et al.*, 1998; Moore and Papadopoulos, 2000; Surles and Padgett, 2001; Soliman, 2002; Upadhyay *et al.*, 2004; Raqab and Kundu, 2005).

There are many practical situations that give rise to probabilities of the form $\Pr(Y < X)$ and ratios of the kind $X/(X+Y)$. In stress-strength modeling, $\Pr(Y < X)$ is a measure of component reliability when the component has a random strength X and is subjected to random stress Y. For example, if Y

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represents the maximum chamber pressure generated by ignition of a solid propellant and X represents the strength of the rocket chamber then $\Pr(Y < X)$ is the probability of successful firing of the engine. In electrical and electronic engineering, ratios of the form $X/(X+Y)$ are used to define outage probabilities.

In this paper, we derive estimation procedures for studying $\Pr(Y < X)$ and $X/(X+Y)$ when X and Y are independent random variables and come from the Burr distribution. The results are organized as follows. Section 2 notes some basic properties of the Burr distributions. The exact form of $\Pr(Y < X)$, its estimators, confidence interval and associated tests of hypotheses are provided in Sections 3–6. Section 7 derives the exact distribution of $X/(X+Y)$ and it moments. Finally, in Section 8, drought data from Nebraska are used to describe an application of these results.

2. BURR DISTRIBUTION

The probability density function (*pdf*) and the cumulative distribution functions (*cdf*) of the Burr distribution are given by

$$f(x) = \alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-\beta-1} \tag{2.1}$$

and

$$F(x) = 1-(1+x^{\alpha})^{-\beta},$$

respectively, for $x > 0$, $\alpha > 0$ and $\beta > 0$. The two parameters α and β both control the shape of the distribution. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from (2.1). If the shape parameter α is known then the maximum likelihood estimator (MLE) of β (see, Woo and Ali, 1999) is given by

$$\hat{\beta} = n\left\{\sum_{i=1}^{n}\log\left(1+X_i^{\alpha}\right)\right\}^{-1}. \tag{2.2}$$

The first two moments of (2.2) are

$$E\left(\hat{\beta}\right) = \frac{n}{n-1}\beta \tag{2.3}$$

and

$$E\left(\hat{\beta}^2\right) = \frac{n^2}{(n-1)(n-2)}\beta^2. \tag{2.4}$$

Let

$$\bar{\beta} = (n-1)\left\{\sum_{i=1}^{n} \log\left(1+X_i^{\alpha}\right)\right\}^{-1}. \tag{2.5}$$

Then, $\bar{\beta}$ is a uniform minimum variance unbiased estimator (UMVUE) of β with the variance (Woo and Ali, 1999)

$$\text{Var}\left(\bar{\beta}\right) = \frac{\beta^2}{n-2}. \tag{2.6}$$

Furthermore,

$$2\beta \sum_{i=1}^{n} \log\left(1+X_i^{\alpha}\right) \sim \chi^2_{2n}. \tag{2.7}$$

3. RELIABILITY $R = \Pr(Y < X)$

Let X and Y be independent random variables having the *pdf* (2.1) with parameters (α, β_1) and (α, β_2), respectively. Then, the form of $R = \Pr(Y < X)$ can be expressed as

$$R = \frac{\rho}{1+\rho}, \tag{3.1}$$

where $\rho = \beta_2/\beta_1$. Clearly, R is a monotone function of ρ and so inference about it is equivalent to inference on ρ (McCool, 1991).

4. ESTIMATORS FOR ρ

Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from (2.1) with parameters (α, β_1) and (α, β_2), respectively. Assume that the common shape parameter α is known. From the MLE and the UMVUE given by (2.2) and (2.5), respectively, one can define the following estimators of ρ

$$\hat{\rho} = \frac{\hat{\beta}_2}{\hat{\beta}_1} \quad \text{and} \quad \bar{\rho} = \frac{\bar{\beta}_2}{\bar{\beta}_1}.$$

Using equations (2.3)–(2.7), the first two moments of $\hat{\rho}$ and $\bar{\rho}$ can be obtained as

$$E\left(\hat{\rho}\right) = \frac{n}{n-1}\rho, \tag{4.1}$$

$$E\left(\bar{\rho}\right) = \frac{m}{m-1}\rho, \tag{4.2}$$

$$E\left(\hat{\rho}^2\right) = \frac{n^2(m+1)}{(n-1)(n-2)m}\rho^2 \tag{4.3}$$

TABLE 4.1 *Comparison of the MSEs of $\hat{\rho}$, $\bar{\rho}$ and $\tilde{\rho}$*

m	n	$\hat{\rho}$	$\bar{\rho}$	$\tilde{\rho}$
10	10	0.3056	0.3056	0.2375
10	20	0.1813	0.2083	0.1611
10	30	0.1502	0.1843	0.1393
20	10	0.2361	0.2037	0.1813
20	20	0.1229	0.1229	0.1083
20	30	0.0748	0.0998	0.0875
30	10	0.2130	0.1751	0.1625
30	20	0.1034	0.0983	0.0907
30	30	0.0763	0.0763	0.0702

and

$$E\left(\bar{\rho}^2\right) = \frac{(n-1)(m+1)m}{(n-2)(m-1)^2}\rho^2. \tag{4.4}$$

Using equations (4.1)–(4.4), one can obtain the unbiased estimator of ρ

$$\tilde{\rho} = \frac{n-1}{n}\hat{\rho} \quad \text{or} \quad \tilde{\rho} = \frac{m-1}{m}\bar{\rho}$$

with

$$E\left(\tilde{\rho}^2\right) = \frac{(n-1)(m+1)}{(n-2)m}\rho^2.$$

Table 4.1 provides a comparison of the mean squared errors (MSE) of $\hat{\rho}$, $\bar{\rho}$ and $\tilde{\rho}$. It is evident that the estimator $\tilde{\rho}$ has the smallest MSE. The other estimators have the same MSE when the sample sizes m and n are equal.

5. CONFIDENCE INTERVAL FOR ρ

Using equation (2.7), it is easy to see that $B = 1/(1+T)$, where

$$T = \rho\frac{\sum_{i=1}^{n}\log\left(1+Y_i^\alpha\right)}{\sum_{i=1}^{m}\log\left(1+X_i^\alpha\right)},$$

has the beta distribution with parameters m and n. Let $0 < \gamma < 1$ and define $b_{\gamma/2}$ by

$$\frac{\gamma}{2} = \frac{1}{B(m,n)}\int_0^{b_{\gamma/2}} t^{m-1}(1-t)^{n-1}dt = I_{b_{\gamma/2}}(m,n). \tag{5.1}$$

Since $I_{b_{\gamma/2}}(m,n) = 1 - I_{1-b_{\gamma/2}}(n,m)$, a $100(1-\gamma)\%$ confidence interval for ρ can be obtained as

$$\left[\left(\frac{1}{b^*_{\gamma/2}} - 1\right)\frac{\sum_{i=1}^{m}\log\left(1+X_i^{\alpha}\right)}{\sum_{i=1}^{n}\log\left(1+Y_i^{\alpha}\right)}, \left(\frac{1}{b_{\gamma/2}} - 1\right)\frac{\sum_{i=1}^{m}\log\left(1+X_i^{\alpha}\right)}{\sum_{i=1}^{n}\log\left(1+Y_i^{\alpha}\right)}\right],$$

where $b^*_{\gamma/2} = 1 - b_{\gamma/2}$.

6. Test of Hypotheses for ρ

Consider testing $H_0 : \rho = 1$ *vs.* $H_1 : \rho \neq 1$ when $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from (2.1) with parameters (α, β_1) and (α, β_2), respectively. Under H_0, the maximum likelihood estimators of α and the common β are the simultaneous solutions of the equations

$$\left(\tilde{\beta}+1\right)\left\{\sum_{i=1}^{m}\frac{X_i^{\tilde{\alpha}}\log X_i}{1+X_i^{\tilde{\alpha}}} + \sum_{i=1}^{n}\frac{Y_i^{\tilde{\alpha}}\log Y_i}{1+Y_i^{\tilde{\alpha}}}\right\}$$
$$= \sum_{i=1}^{m}\log X_i + \sum_{i=1}^{n}\log Y_i + \frac{m+n}{\tilde{\alpha}}$$

and

$$\sum_{i=1}^{m}\log\left(1+X_i^{\tilde{\alpha}}\right) + \sum_{i=1}^{n}\log\left(1+Y_i^{\tilde{\alpha}}\right) = \frac{m+n}{\tilde{\beta}}.$$

Under H_1, the maximum likelihood estimators of α, β_1 and β_2 are the simultaneous solutions of the equations

$$\left(\hat{\beta}_1+1\right)\sum_{i=1}^{m}\frac{X_i^{\hat{\alpha}}\log X_i}{1+X_i^{\hat{\alpha}}} + \left(\hat{\beta}_2+1\right)\sum_{i=1}^{n}\frac{Y_i^{\hat{\alpha}}\log Y_i}{1+Y_i^{\hat{\alpha}}}$$
$$= \sum_{i=1}^{m}\log X_i + \sum_{i=1}^{n}\log Y_i + \frac{m+n}{\hat{\alpha}},$$

$$\sum_{i=1}^{m}\log\left(1+X_i^{\hat{\alpha}}\right) = \frac{m}{\hat{\beta}_1}, \quad \text{and} \quad \sum_{i=1}^{n}\log\left(1+Y_i^{\hat{\alpha}}\right) = \frac{n}{\hat{\beta}_2}.$$

Thus, the likelihood ratio test (LRT) statistic for testing $H_0 : \rho = 1$ is

$$\begin{aligned}
-2\log\Lambda_1 &= 2(m+n)\log\hat{\alpha} - 2(m+n)\log\tilde{\alpha} + 2m\log\hat{\beta}_1 + 2n\log\hat{\beta}_2 \\
&\quad -2(m+n)\log\tilde{\beta} \\
&\quad +2\left(\hat{\alpha}-\tilde{\alpha}\right)\left\{\sum_{i=1}^{m}\log X_i + \sum_{i=1}^{n}\log Y_i\right\} \\
&\quad +2\left(1+\tilde{\beta}\right)\left\{\sum_{i=1}^{m}\log\left(1+X_i^{\tilde{\alpha}}\right) + \sum_{i=1}^{n}\log\left(1+Y_i^{\tilde{\alpha}}\right)\right\} \\
&\quad -2\left(1+\hat{\beta}_1\right)\sum_{i=1}^{m}\log\left(1+X_i^{\hat{\alpha}}\right) - 2\left(1+\hat{\beta}_2\right)\sum_{i=1}^{n}\log\left(1+Y_i^{\hat{\alpha}}\right).
\end{aligned}$$

Under H_0, the asymptotic distribution of $-2\log\Lambda_1$ is chi-squared with one degree of freedom. Hence, one would reject $H_0 : \rho = 1$ with significance level γ if $-2\log\Lambda_1 > \chi^2_{1,\gamma}$.

7. DISTRIBUTION OF THE RATIO $X/(X+Y)$

Let X and Y be independent random variables having the *pdf* (2.1) with parameters (α, β_1) and (α, β_2), respectively. Let $V = X/(X+Y)$ and $W = X+Y$. The joint *pdf* of V and W is

$$\begin{aligned}
f_{V,W}(v,w) &= w\alpha\beta_1(wv)^{\alpha-1}\left[1+(wv)^{\alpha}\right]^{-\beta_1-1} \\
&\quad \times\alpha\beta_2\left\{w(1-v)\right\}^{\alpha-1}\left[1+\left\{w(1-v)\right\}^{\alpha}\right]^{-\beta_2-1}. \qquad (7.1)
\end{aligned}$$

Integrating (7.1) with respect to w by using formula (2.29) in Oberhettinger (1974), one can obtain the marginal *pdf* of the ratio $V = X/(X+Y)$ as

$$f_V(v) = \begin{cases}
\alpha\beta_1\beta_2 B\left(2,\beta_1+\beta_2\right)v^{\alpha-1}(1-v)^{-1-\alpha} \\
\quad \times\ {}_2F_1\left(\beta_1+1,2;\beta_1+\beta_2+2;1-v^{\alpha}(1-v)^{-\alpha}\right), \\
\quad \text{if } 0 < v < 1/2, \\
\alpha\beta_1\beta_2 B\left(2,\beta_1+\beta_2\right)(1-v)^{\alpha-1}v^{-1-\alpha} \\
\quad \times\ {}_2F_1\left(\beta_2+1,2;\beta_1+\beta_2+2;1-v^{-\alpha}(1-v)^{\alpha}\right), \\
\quad \text{if } 1/2 \le v < 1,
\end{cases} \qquad (7.2)$$

where ${}_2F_1(a,b;c;x)$ denotes the Gauss hypergeometric function defined by

$${}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Using formulas (1.112.1) and (1.112.2) in Gradshteyn and Ryzhik (2000) and formula (15.3) in Oberhettinger (1974), the first two moments of V can be obtained as

$$\begin{aligned} E(V) = \beta_1\beta_2 B(2, \beta_1+\beta_2) \sum_{k=1}^{\infty} (-1)^{k-1} \Bigg[& \left(\frac{k}{\alpha}+1\right)^{-1} \\ & \times {}_3F_2\left(\beta_1+1, 2, 1; \beta_1+\beta_2+2, \frac{k}{\alpha}+2; 1\right) \\ & + \left(\frac{k-1}{\alpha}+1\right)^{-1} \\ & \times {}_3F_2\left(\beta_2+1, 2, 1; \beta_1+\beta_2+2, \frac{k-1}{\alpha}+2; 1\right)\Bigg] \end{aligned}$$

and

$$\begin{aligned} E\left(V^2\right) = \beta_1\beta_2 B(2, \beta_1+\beta_2) \sum_{k=1}^{\infty} k(-1)^{k-1} \Bigg[& \left(\frac{k+1}{\alpha}+1\right)^{-1} \\ & \times {}_3F_2\left(\beta_1+1, 2, 1; \beta_1+\beta_2+2, \frac{k+1}{\alpha}+2; 1\right) \\ & + \left(\frac{k-1}{\alpha}+1\right)^{-1} \\ & \times {}_3F_2\left(\beta_2+1, 2, 1; \beta_1+\beta_2+2, \frac{k-1}{\alpha}+2; 1\right)\Bigg], \end{aligned}$$

where ${}_3F_2(a, b, c; d, e; x)$ denotes the hypergeometric function defined by

$${}_3F_2(a, b, c; d, e; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{x^k}{k!}.$$

8. APPLICATION

Here, we illustrate application of the results in Sections 2–7 to drought data from the State of Nebraska, freely downloadable from the web-site:

http://lwf.ncdc.noaa.gov/oa/climate/onlineprod/drought/xmgrg3.html.

The data comprises of the monthly modified Palmer Drought Severity Index (PDSI) from the period from January 1895 to December 2004. A drought is said

TABLE 8.1 *Basic drought statistics for Nebraska PDSI data*

Climate division	Number of droughts	Drought frequency (number/year)	Mean drought duration (months)	Std. of drought duration (months)
1	83	0.75	6.0	8.0
2	66	0.60	8.6	12.0
3	89	0.81	6.3	9.7
5	81	0.74	6.3	10.5
6	90	0.82	6.3	10.1
7	81	0.74	6.1	9.7
8	76	0.69	6.5	13.4
9	74	0.67	7.5	10.9

to have happened when PDSI is below 0 and is defined by the theory of runs (Yevjevich, 1967). The State of Nebraska is divided into eight climate divisions numbered 1, 2, 3, 5, 6, 7, 8 and 9 — there is no climate division 4 for Nebraska. Some statistics of the observed drought for the eight climatic divisions are summarized in Table 8.1. The real drought data set for the 83 drought events in climate division 1 is illustrated in Table 8.2.

Using the PDSI data, data on drought duration (X) and non-drought duration (Y) were obtained for each climate division. The primary questions would be: Are the drought periods longer than the non-drought periods (or *vice versa*)? If so, by what proportion? We address these questions by using the results derived in Sections 2–7.

Firstly, we fitted the Burr distribution given by (2.1) to the observed values of X and Y. The method of maximum likelihood was used: if $X_1, X_2, \ldots, X_m$ is a random sample of X then we solved the equations

$$(\bar{\beta}_1 + 1) \sum_{i=1}^{m} \frac{X_i^{\bar{\alpha}_1} \log X_i}{1 + X_i^{\bar{\alpha}_1}} - \sum_{i=1}^{m} \log X_i = \frac{m}{\bar{\alpha}_1} \quad \text{and} \quad \sum_{i=1}^{m} \log\left(1 + X_i^{\bar{\alpha}_1}\right) = \frac{m}{\bar{\beta}_1}$$

to obtain the parameter estimates $\bar{\beta}_1$ and $\bar{\alpha}_1$; similarly, if $Y_1, Y_2, \ldots, Y_n$ is a random sample of Y then we solved the equations

$$(\bar{\beta}_2 + 1) \sum_{i=1}^{n} \frac{Y_i^{\bar{\alpha}_2} \log Y_i}{1 + Y_i^{\bar{\alpha}_2}} - \sum_{i=1}^{n} \log Y_i = \frac{n}{\bar{\alpha}_2} \quad \text{and} \quad \sum_{i=1}^{n} \log\left(1 + Y_i^{\bar{\alpha}_2}\right) = \frac{n}{\bar{\beta}_2}$$

to obtain the parameter estimates $\bar{\beta}_2$ and $\bar{\alpha}_2$. The goodness of fit was examined by means of probability plots. A probability plot is where the observed probability is plotted against the probability predicted by the fitted model. Thus, we plotted

TABLE 8.2 *Drought data for Nebraska climate division 1*

Case	Drought Duration	Drought Intensity	Non-drought Duration	Case	Drought Duration	Drought Intensity	Non-drought Duration
1	5	2.66	58	43	1	0.11	7
2	11	14.34	1	44	5	8.05	3
3	1	0.91	1	45	20	53.04	1
4	2	3.59	1	46	3	4.92	11
5	8	6.82	2	47	6	6.23	4
6	1	1.24	1	48	2	1.54	9
7	1	0.40	2	49	7	9.32	10
8	2	0.18	13	50	1	0.26	4
9	18	36.54	73	51	1	0.19	2
10	1	0.69	80	52	2	2.55	2
11	5	3.40	10	53	4	0.83	1
12	5	6.71	32	54	5	5.87	11
13	1	0.44	30	55	1	0.45	1
14	4	3.29	10	56	1	0.38	1
15	11	25.45	57	57	11	13.39	19
16	11	26.24	1	58	6	9.21	2
17	2	3.67	2	59	3	1.23	1
18	4	3.49	2	60	10	18.84	1
19	13	51.50	1	61	2	0.23	2
20	31	93.51	6	62	5	3.15	1
21	1	0.06	1	63	2	1.36	1
22	1	0.61	3	64	1	0.55	1
23	1	0.10	1	65	3	2.04	2
24	27	79.85	3	66	1	0.64	5
25	4	4.80	26	67	24	47.60	12
26	1	1.16	27	68	8	19.22	33
27	2	2.76	2	69	1	0.20	4
28	2	0.51	6	70	32	63.07	30
29	6	4.75	33	71	8	10.68	1
30	6	5.52	4	72	4	6.46	1
31	2	1.19	15	73	2	0.04	1
32	2	0.38	1	74	1	0.72	6
33	1	0.26	1	75	5	12.96	11
34	2	0.49	1	76	2	1.34	43
35	6	4.80	2	77	1	0.22	2
36	37	77.38	1	78	1	0.25	5
37	2	1.52	26	79	3	2.64	15
38	2	1.01	3	80	1	0.62	5
39	14	23.01	1	81	2	2.02	1
40	1	0.77	1	82	3	2.13	1
41	1	0.64	1	83	34	116.12	1
42	5	3.53	3				

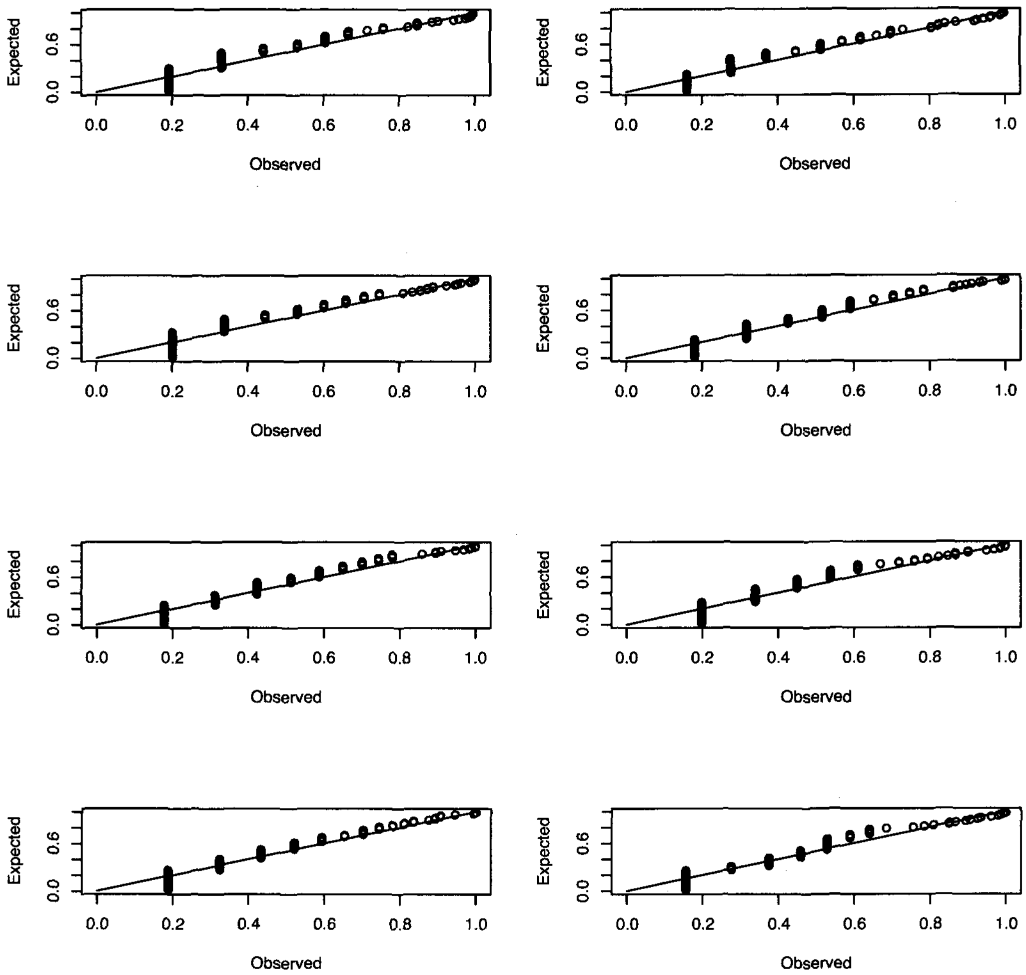

FIGURE 8.1 *Probability plots of the fit of (2.1) for drought duration data from the eight climate divisions.*

$F_X(x_{(i)})$ *vs.* $(i - 0.375)/(m + 0.25)$ and $F_Y(y_{(i)})$ *vs.* $(i - 0.375)/(n + 0.25)$, as recommended by Blom (1958) and Chambers *et al.* (1983), where

$$F_X(x) = 1 - \left(1 + x^{\bar{\alpha}_1}\right)^{-\bar{\beta}_1} \quad \text{and} \quad F_Y(y) = 1 - \left(1 + y^{\bar{\alpha}_2}\right)^{-\bar{\beta}_2}$$

and $x_{(i)}$ and $y_{(i)}$ are the sorted values, in the ascending order, of the observed drought duration and the observed non-drought duration, respectively. The plots are shown in Figure 8.1 and 8.2. Both figures suggest that the fit of the Burr distribution is reasonable especially in the upper tails.

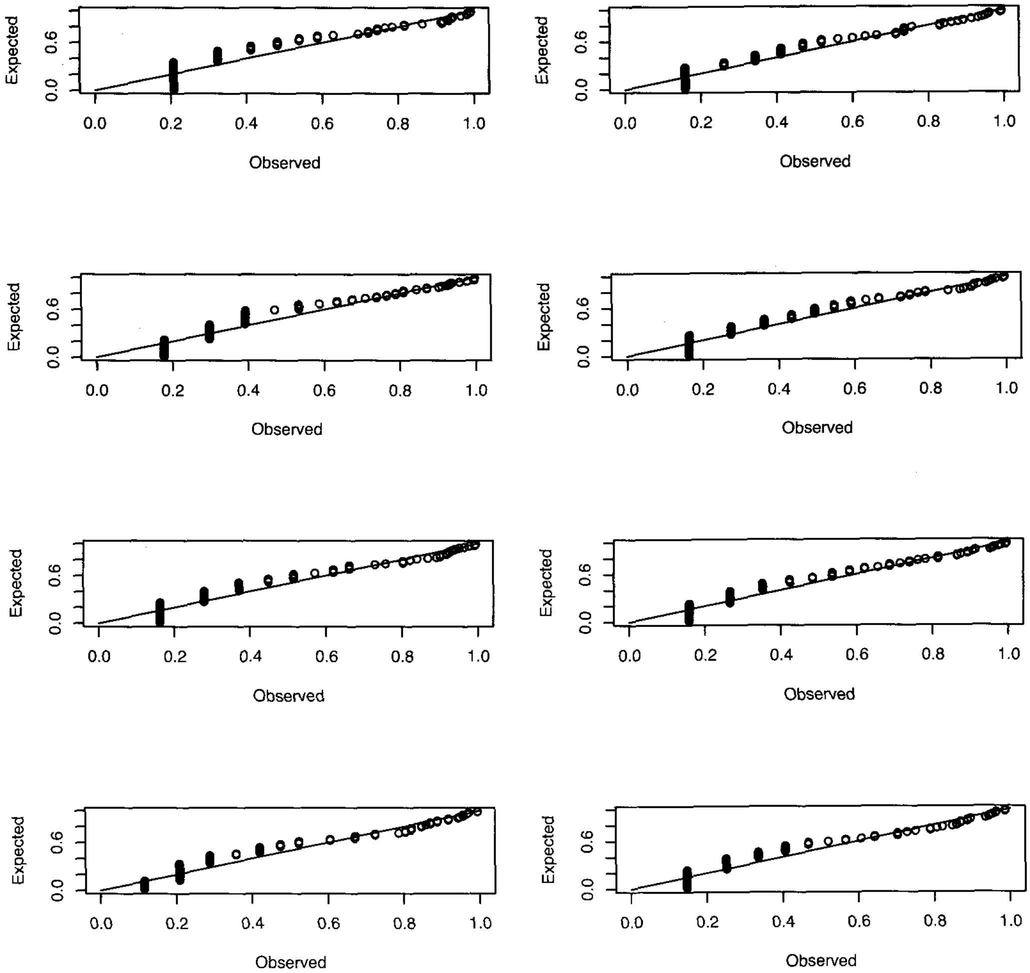

FIGURE 8.2 *Probability plots of the fit of (2.1) for non-drought duration data from the eight climate divisions.*

Next, we examined to see whether the Burr distribution model with two parameters for each X and Y (four parameters in total) can be reduced to a simpler model. We fitted the following variations:

Model 1: $\alpha_1 = \alpha_2 = \alpha$ (3 parameter model),

Model 2: $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ (2 parameter model).

Both models were fitted by the method of maximum likelihood. We used the LRT to see whether models 1 or 2 provided a simplification. The LRT statistic

TABLE 8.3 *Likelihood ratio test statistics*

Climate Division	$-2\log\Lambda_2$	$-2\log\Lambda_1$
1	3.5	4.1
2	0.5	1.5
3	0.1	2.0
4	1.3	5.4
5	0.4	2.8
6	0.4	6.6
7	0.0	8.7
8	0.5	2.3

for testing $\alpha_1 = \alpha_2$ is

$$\begin{aligned} -2\log\Lambda_2 = {} & 2m\log\bar{\alpha}_1 + 2n\log\bar{\alpha}_2 - 2(m+n)\log\hat{\alpha} + 2m\log\bar{\beta}_1 \\ & -2m\log\hat{\beta}_1 + 2n\log\bar{\beta}_2 - 2n\log\hat{\beta}_2 \\ & +2\left(\bar{\alpha}_1 - \hat{\alpha}\right)\sum_{i=1}^{m}\log X_i + 2\left(\bar{\alpha}_2 - \hat{\alpha}\right)\sum_{i=1}^{n}\log Y_i \\ & +2\left(1+\hat{\beta}_1\right)\sum_{i=1}^{m}\log\left(1+X_i^{\hat{\alpha}}\right) + 2\left(1+\hat{\beta}_2\right)\sum_{i=1}^{n}\log\left(1+Y_i^{\hat{\alpha}}\right) \\ & -2\left(1+\bar{\beta}_1\right)\sum_{i=1}^{m}\log\left(1+X_i^{\bar{\alpha}_1}\right) - 2\left(1+\bar{\beta}_2\right)\sum_{i=1}^{n}\log\left(1+Y_i^{\bar{\alpha}_2}\right), \end{aligned}$$

where $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are as defined in Section 6. The LRT statistic for testing Model 2 *vs.* Model 1 is $-2\log\Lambda_1$ described in Section 6. The values of $-2\log\Lambda_1$ and $-2\log\Lambda_2$ for the eight climate divisions are shown in Table 8.3. It follows by the LRT that Model 1 provides as good a fit as the full four parameter model for all of the eight climate divisions. However, Model 1 cannot be simplified further since the values of $-2\log\Lambda_1$ are significant. Hence, one can say that drought duration and non-drought duration are Burr distributed with the same α but different β's. The parameter estimates $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ of Model 1 and their standard errors (obtained by inverting the observed information matrix) are shown in Table 8.4.

The probability $R = \Pr(Y < X)$ will tell us whether the drought periods are longer than the non-drought periods or not. By equation (3.1), computing R amounts to computing ρ. Table 8.5 provides various estimates of ρ — both point and interval estimates — obtained using the results in Sections 4 and 5. It seems that, on average, the 95% confidence interval for ρ ranges from 0.5 to 1. Thus,

TABLE 8.4 *Parameter estimates and standard errors for Model 1*

Climate Division	$\hat{\alpha}$ (s.e.)	$\hat{\beta}_1$ (s.e.)	$\hat{\beta}_2$ (s.e.)
1	0.840 (0.043)	12.664 (1.882)	9.196 (1.204)
2	0.860 (0.050)	10.216 (1.594)	8.280 (1.207)
3	0.877 (0.043)	13.317 (1.889)	10.770 (1.430)
4	0.897 (0.046)	14.066 (2.128)	9.743 (1.316)
5	0.923 (0.046)	14.795 (2.155)	11.505 (1.557)
6	0.885 (0.046)	14.090 (2.124)	9.375 (1.264)
7	0.933 (0.050)	15.111 (2.364)	9.309 (1.309)
8	0.906 (0.051)	12.125 (1.882)	9.440 (1.353)

TABLE 8.5 *Estimates of ρ and its confidence interval*

Climate Division	$\hat{\rho}$	$\bar{\rho}$	$\tilde{\rho}$	95% CI
1	0.726	0.726	0.717	(0.536, 0.984)
2	0.810	0.810	0.798	(0.577, 1.139)
3	0.809	0.809	0.800	(0.589, 1.085)
4	0.693	0.693	0.684	(0.509, 0.942)
5	0.778	0.778	0.769	(0.568, 1.042)
6	0.665	0.665	0.657	(0.489, 0.905)
7	0.616	0.616	0.608	(0.437, 0.847)
8	0.779	0.779	0.768	(0.549, 1.075)

the 95% confidence interval for R ranges from 0.33 to 0.50.

The ratio $V = X/(X+Y)$ will give us some measure of the relative proportion of droughts. Using the results in Section 7, we plotted the fitted density of V for all of the eight climate divisions, see Figure 8.3. The densities are skewed to the left, meaning that there is a higher probability that non-drought periods are longer than drought periods. The similarities of the densities suggest that there is little difference between the climate divisions. This is what one would expect given the geography of the State of Nebraska.

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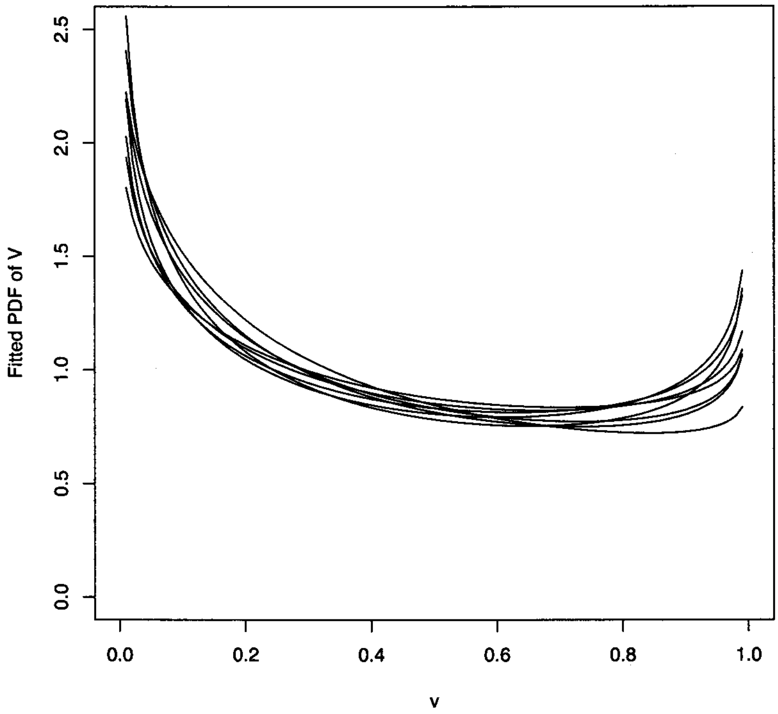

FIGURE 8.3 *Fitted densities of (7.2) for the eight climate divisions. Parameter estimates from Model 1 are used.*

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