A MEASURE OF ROBUST ROTATABILITY FOR SECOND ORDER RESPONSE SURFACE DESIGNS†

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ABSTRACT

In Response Surface Methodology (RSM), rotatability is a natural and highly desirable property. For second order general correlated regression model, the concept of robust rotatability was introduced by Das (1997). In this paper a new measure of robust rotatability for second order response surface designs with correlated errors is developed and illustrated with an example. A comparison is made between the newly developed measure with the previously suggested measure by Das (1999).

1. INTRODUCTION

Response Surface Methodology (RSM) is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable. The independent variables are often called the input or explanatory variables and the dependent variable is often called the response variable. Rotatability is one of the desirable characteristics of RSM. This was formally developed by Box and Hunter (1957), assuming the errors in the observations are uncorrelated and homoscedastic. Rotatability of different orders in connection with response surface designs have been studied extensively by a host of authors beginning with Box and Hunter (1957) in the

context of uncorrelated and homoscedastic errors. A literature survey of RSM is given by Myers *et al.* (1989).

In recent years, interests in RSM have been increased and books on this subject have been written by some authors such as Box and Draper (2007), Khuri and Cornell (1996), Pukelsheim (1993), Myers and Montgomery (2002), *etc.* Analogous to rotatability, the concept of slope-rotatability has been advanced by Hader and Park (1978) and Park (1987). Kim and Park (2006) studied slope rotatability of icosahedron and dodecahedron designs. All the literature for RSM was given assuming the errors in the observations are uncorrelated and homoscedastic.

So far all the authors studied rotatable designs and slope rotatable designs assuming errors to be uncorrelated and homoscedastic. However, it is not uncommon to come across practical situations when the errors are correlated, violating the usual assumptions. Several authors such as Kiefer and Wynn (1981, 1984), Gennings *et al.* (1989), Bischoff (1992, 1995), Panda and Das (1994), Das (2003b, 2004), Das and Park (2006), *etc.,* mentioned some references where errors are correlated. In Panda and Das (1994), a study of rotatable designs with correlated errors was initiated and a systematic study of first order rotatable designs was attempted for various correlated structures of the errors. In Das (1997), a study of Robust Second Order Rotatable Designs (RSORD) was introduced. Herein, rotatability conditions for second order regression designs were derived for general correlated error structure. Robust rotatable designs were derived for different correlation structures of errors by Das (2003b, 2004). Analogous to robust rotatability as introduced by Das (1997), the concept of robust slope-rotatability has been introduced by Das (2003a). Mukhopadhyay *et al.* (2002) used robust designs for improving the quality of a system in the reliability theory. Das and Park (2006) introduced slope rotatability over all directions for correlated errors.

In RSM, a natural and desirable property is that of rotatability, which requires that the variance of a predicted response at a point remains constant at all such points that are equidistant from the design center. To achieve stability in prediction variance, this important property of rotatability was evolved. If circumstances are such that exact rotatability is unattainable – because of more cost and time, and more important restrictions such as orthogonal blocking, it is still a good idea to make the design as rotatable as possible. Thus it is important to measure the extent of deviation from rotatability. The measure of rotatability for a design under ordinary (*i.e.,* with uncorrelated and homoscedastic errors) regression model was assessed by Draper and Guttman (1988), Khuri (1988), Draper and Pukelsheim (1990), Park *et al.* (1993), and a measure for stability of

slope estimation is suggested by Park *et al.* (2003).

In case of uncorrelated regression model the rotatability property remains unaltered if the design points of a rotatable design are permuted (*i.e.*, the order of experiments) in any manner. But in case of correlated regression model, rotatability may be distorted for some permutation of design points. Again the permutation of design points may be required to reduce the cost and time of the experiment. All these notions are explained with an illustrative example in Sections 2.4 and 3.2. For correlated regression model, the measure of robust rotatability of a first order regression design was developed by Panda and Das (1994) and second order regression design was suggested by Das (1999). For ready reference we have given some of our earlier results which are related in this paper in Sections 2.3, 2.4 and 2.5 following Das (1997) and Das (1999), respectively.

In this paper, we have developed a new measure of robust rotatability of a second order regression design under a fixed pattern of correlation structure of observations. This measure has been developed by following Park *et al.* (1993). Robustness of usual Second Order Rotatable Designs (SORD) can be examined with this new measure. A comparison is made between the newly developed measure with the previously suggested measure.

2. SECOND ORDER REGRESSION MODEL WITH CORRELATED ERROR

2.1. Model

Suppose there are k factors $\mathbf{x} = (x_1, x_2, \ldots, x_k)'$ which yield a response of y_u on the study variable y when $\mathbf{x} = \mathbf{x}_{0u} = (x_{1u}, x_{2u}, \ldots, x_{ku})', 1 \leq u \leq N$. Assuming that the response surface is of second order, we adopt the model:

$$y_u = \beta_0 + \sum_{i=1}^{k} \beta_i x_{iu} + \sum \sum_{i \leq j=1}^{k} \beta_{ij} x_{iu} x_{ju} + e_u, \ 1 \leq u \leq N$$

or

$$\mathbf{Y} = X\boldsymbol{\beta} + \mathbf{e}, \tag{2.1}$$

where $\mathbf{Y}$ is the vector of recorded observations on the study variable y,

$$\boldsymbol{\beta} = (\beta_0, \ \beta_{11}, \ \beta_{22}, \ldots, \beta_{kk}, \ \beta_1, \ \beta_2, \ldots, \beta_k, \ \beta_{12}, \ \beta_{13}, \ldots, \beta_{(k-1)k})'$$

is the vector of regression coefficients of order $\binom{k+2}{2} \times 1$, $X = (\mathbf{1} : Z)$ is the design matrix,

$$Z = (\mathbf{x}_1 \otimes_1 \mathbf{x}_1, \ldots, \mathbf{x}_k \otimes_1 \mathbf{x}_k, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_1 \otimes_1 \mathbf{x}_2, \ldots, \mathbf{x}_{k-1} \otimes_1 \mathbf{x}_k),$$

$$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iN})', \ 1 \le i \le k,$$

$$\mathbf{x}_i \otimes_1 \mathbf{x}_j = (x_{i1}x_{j1}, x_{i2}x_{j2}, \ldots, x_{iN}x_{jN})', \ 1 \le i, \ j \le k.$$

Here $\otimes_1$ denotes the Hadamard product of two matrices of the same order and it is defined as follows. Let $L_0 = ((a_{ij}))$ and $M_0 = ((b_{ij}))$ be two matrices of the same order, say $p \times q$. The Hadamard product $L_0 \otimes_1 M_0$ of L_0 and M_0 is a matrix H, of order $p \times q$, where $H = ((a_{ij}b_{ij}))$.

Further, $\mathbf{e}$ is the vector of errors (of order $N \times 1$) which are as assumed to be normally distributed with $E(\mathbf{e}) = \mathbf{0}$ and $D(\mathbf{e}) = W$ with rank $(W) = N$. The matrix W may represent any structure with correlated errors for example, $W = [\sigma^2 \{\rho^{|i-j|}\}_{1 \le i, j \le N}]$ which is known as autocorrelated error structure. In general, the matrix W is unknown but for all the calculations as usual, W is assumed to be known. In practice, however, W includes a number of parameters unknown, and in the calculations which follow, the expressions for W and W^{-1} are replaced by those obtained by replacing the unknown parameters by their suitable estimates or some assumed values.

2.2. Analysis

We will assume $X'W^{-1}X$ is positive definite, when W is known. The best linear unbiased estimator of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (X'W^{-1}X)^{-1}(X'W^{-1}Y)$ with

$$D(\hat{\boldsymbol{\beta}}) = (X'W^{-1}X)^{-1} = \begin{pmatrix} A & B & C \\ B' & P & Q \\ C' & Q' & R \end{pmatrix}^{-1}_{\binom{k+2}{2} \times \binom{k+2}{2}},$$

where A, P and R are symmetric matrices, given by

$$A_{(k+1)\times(k+1)} = \begin{pmatrix} v_{00} & v_{0.11} & & v_{0.22} & \cdots & v_{0.kk} \\ v_{11.0} & \vdots & ((v_{ii.jj}), 1 \le i,j \le k) & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ v_{kk.0} & \cdots & & \cdots & \cdots & v_{k.kk} \end{pmatrix},$$

$$v_{00} = \mathbf{1}'W^{-1}\mathbf{1}, \quad v_{0.jj} = \mathbf{1}'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_j), \quad 1 \le j \le k,$$

$$v_{ii.jj} = (\mathbf{x}_i \otimes_1 \mathbf{x}_i)'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_j), \ 1 \le i, \ j \le k,$$

$$B_{(k+1)\times k} = \begin{pmatrix} v_{0.1} & & v_{0.2} & \cdots & v_{0.k} \\ & ((v_{ii.j})_{k \times k}, 1 \le i, \ j \le k) & & & \end{pmatrix},$$

$$v_{0.j} = \mathbf{1}'W^{-1}\mathbf{x}_j \ ; \ 1 \le j \le k, \quad v_{ii.j} = (\mathbf{x}_i \otimes_1 \mathbf{x}_i)'W^{-1}\mathbf{x}_j, \ 1 \le i, j \le k,$$

$$C_{(k+1)\times\binom{k}{2}} = \begin{pmatrix} v_{0.12} & v_{0.13} & \cdots & v_{0.(k-1)k} \\ & ((v_{ii.jl})_{k\times\binom{k}{2}}, \ 1 \le i,j < l \le k) & \end{pmatrix},$$

$$v_{0.jl} = \mathbf{1}'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_l), \ 1 \le j < l \le k,$$

$$v_{ii.jl} = (\mathbf{x}_i \otimes_1 \mathbf{x}_i)'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_l), 1 \le i,j < l \le k,$$

$$P_{k\times k} = ((v_{i.j})), \ v_{i.j} = \mathbf{x}_i'W^{-1}\mathbf{x}_j, \ 1 \le i,j \le k,$$

$$Q_{k\times\binom{k}{2}} = ((v_{i.jl})), \ v_{i.jl} = \mathbf{x}_i'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_l), \ 1 \le i,j < l \le k,$$

$$R_{\binom{k}{2}\times\binom{k}{2}} = ((v_{ij.lt})), \ v_{ij.lt} = (\mathbf{x}_i \otimes_1 \mathbf{x}_j)'W^{-1}(\mathbf{x}_l \otimes_1 \mathbf{x}_t),$$
$$1 \le i,l < j,t \le k. \tag{2.2}$$

Note that $v_{0.j} = v_{j.0}$, $v_{0.jj} = v_{jj.0}$, $v_{0.jl} = v_{jl.0}$, $v_{i.j} = v_{j.i}$, $v_{ii.j} = v_{j.ii}$, $v_{i.jl} = v_{jl.i}$, $v_{ii.jj} = v_{jj.ii}$, $v_{ij.lt} = v_{lt.ij}$. In the inverse matrix $(X'W^{-1}X)^{-1}$, the elements corresponding to v_m in $X'W^{-1}X$ is denoted by v^m for all m included in the preceding expressions.

2.3. Conditions for robust second order rotatability and rsord under autocorrelated error

Below are given the conditions for rotatability in second order regression designs with correlated error model (2.1) in terms of the elements of the moment matrix, which are given in Das (1997).

(I) (i) $v_{0.j} = v_{0.jl} = 0, \ 1 \le j < l \le k,$

    (ii) $v_{i.j} = 0, \ 1 \le i,j \le k, \ i \ne j,$

    (iii) (1) $v_{ii.j} = 0, \ 1 \le i \le k, \ 1 \le j \le k,$

        (2) $v_{i.jl} = 0, \ 1 \le i \le k, \ 1 \le j < l \le k,$

        (3) $v_{ii.jl} = 0, \ 1 \le i \le k, \ 1 \le j < l \le k,$

        (4) $v_{ij.lt} = 0, \ 1 \le i < j \le k, \ 1 \le l < t \le k, \ (i,j) \ne (l,t),$

(II) (i) $v_{0.jj} = \text{constant} = a_0(> 0), \ 1 \le j \le k,$

    (ii) $v_{i.i} = \text{constant} = \frac{1}{e}(> 0), \ 1 \le i \le k,$

    (iii) $v_{ii.ii} = \text{constant} = (\frac{2}{c} + d)(> 0), \ 1 \le i \le k,$

(III) (i) $v_{ii.jj} = \text{constant} = d(> 0), \ 1 \le i, \ j \le k, \ i \ne j,$

    (ii) $v_{ij.ij} = \text{constant} = \frac{1}{c}(> 0), \ 1 \le i < j \le k,$

(IV) $v_{ii.ii} = 2v_{ij.ij} + v_{ii.jj}, \ 1 \le i < j \le k, \tag{2.3}$

where v's are as in (2.2).

Following (2.3), the conditions for second order rotatability under the auto-correlated variance-covariance structure given in Section 2.1 simplify to:

(I)   (i)

$$v_{0.j} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{ju} - \rho \sum_{u=2}^{N-1} x_{ju} = 0,\ 1 \le j \le k,$$

$$v_{0.jl} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{ju} x_{lu} - \rho \sum_{u=2}^{N-1} x_{ju} x_{lu} = 0,\ 1 \le j < l \le k,$$

(ii)

$$v_{i.j} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{iu} x_{ju} + \rho^2 \sum_{u=2}^{N-1} x_{iu} x_{ju}$$
$$- \rho \left(\sum_{u=1}^{N-1} x_{iu} x_{j(u+1)} + \sum_{u=1}^{N-1} x_{i(u+1)} x_{ju} \right) = 0,$$
$$1 \le i,\ j \le k,\ \ i \ne j,$$

(iii) (1)

$$v_{ii.j} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{iu}^2 x_{ju} + \rho^2 \sum_{u=2}^{N-1} x_{iu}^2 x_{ju}$$
$$- \rho \left(\sum_{u=1}^{N-1} x_{iu}^2 x_{j(u+1)} + \sum_{u=1}^{N-1} x_{i(u+1)}^2 x_{ju} \right) = 0,$$
$$1 \le i \le k,\ \ 1 \le j \le k,$$

(2)

$$v_{ij.l} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{iu} x_{ju} x_{lu} + \rho^2 \sum_{u=2}^{N-1} x_{iu} x_{ju} x_{lu}$$
$$- \rho \left(\sum_{u=1}^{N-1} x_{iu} x_{ju} x_{l(u+1)} + \sum_{u=1}^{N-1} x_{i(u+1)} x_{j(u+1)} x_{lu} \right) = 0,$$
$$1 \le i < j \le k,\ \ 1 \le l \le k,$$

(3)

$$v_{ii.jl} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{iu}^2 x_{ju} x_{lu} + \rho^2 \sum_{u=2}^{N-1} x_{iu}^2 x_{ju} x_{lu}$$

$$-\rho \left(\sum_{u=1}^{N-1} x_{iu}^2 x_{j(u+1)} x_{l(u+1)} + \sum_{u=1}^{N-1} x_{i(u+1)}^2 x_{ju} x_{lu} \right) = 0,$$

$$1 \le i \le k, \quad 1 \le j < l \le k,$$

(4)

$$v_{ij.lt} = 0 \Leftrightarrow \sum_{u=1}^{N} x_{iu} x_{ju} x_{lu} x_{tu} + \rho^2 \sum_{u=2}^{N-1} x_{iu} x_{ju} x_{lu} x_{tu}$$

$$-\rho \left(\sum_{u=1}^{N-1} x_{iu} x_{ju} x_{l(u+1)} x_{t(u+1)} \right.$$

$$\left. + \sum_{u=1}^{N-1} x_{i(u+1)} x_{j(u+1)} x_{lu} x_{tu} \right) = 0,$$

$$1 \le i < j \le k, \ 1 \le l < t \le k, \quad (i,j) \neq (l,t),$$

(II) (i)

$$v_{0.jj} = \text{constant} \Leftrightarrow \{\sigma^2(1-\rho^2)\}^{-1}(1-\rho) \left(\sum_{u=1}^{N} x_{ju}^2 - \rho \sum_{u=2}^{N-1} x_{ju}^2 \right)$$

$$= a_0(> 0), \ 1 \le j \le k,$$

(ii)

$$v_{i.i} = \text{constant} \Leftrightarrow \{\sigma^2(1-\rho^2)\}^{-1} \left(\sum_{u=1}^{N} x_{iu}^2 + \rho^2 \sum_{u=2}^{N-1} x_{iu}^2 \right.$$

$$\left. -2\rho \sum_{u=1}^{N-1} x_{iu} x_{i(u+1)} \right) = \frac{1}{e}(> 0), \ 1 \le i \le k,$$

(iii)

$$v_{ii.ii} = \text{constant} \Leftrightarrow \{\sigma^2(1-\rho^2)\}^{-1} \left(\sum_{u=1}^{N} x_{iu}^4 + \rho^2 \sum_{u=2}^{N-1} x_{iu}^4 \right.$$

$$\left. -2\rho \sum_{u=1}^{N-1} x_{iu}^2 x_{i(u+1)}^2 \right) = \left(\frac{2}{c} + d \right)(> 0), \ 1 \le i \le k,$$

(III) (i)

$$v_{ii.jj} = \text{constant} \Leftrightarrow \{\sigma^2(1-\rho^2)\}^{-1} \left\{ \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 + \rho^2 \sum_{u=2}^{N-1} x_{iu}^2 x_{ju}^2 \right.$$
$$\left. - \rho \left(\sum_{u=1}^{N-1} x_{iu}^2 x_{j(u+1)}^2 + \sum_{u=1}^{N-1} x_{i(u+1)}^2 x_{ju}^2 \right) \right\} = d(> 0),$$
$$1 \le i, j \le k, \quad i \ne j,$$

(ii)

$$v_{ij.ij} = \text{constant} \Leftrightarrow \{\sigma^2(1-\rho^2)\}^{-1} \left\{ \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 + \rho^2 \sum_{u=2}^{N-1} x_{iu}^2 x_{ju}^2 \right.$$
$$\left. - 2\rho \sum_{u=1}^{N-1} x_{iu} x_{ju} x_{i(u+1)} x_{j(u+1)} \right\} = \frac{1}{c}(> 0),$$
$$1 \le i < j \le k,$$

(IV)

$$v_{ii.ii} = 2v_{ij.ij} + v_{ii.jj} ; \quad 1 \le i < j \le k, \tag{2.4}$$

where $v_{ii.ii}$, $v_{ii.jj}$ and $v_{ij.ij}$ are as in (II) (iii) and (III) (i), (ii) of (2.4).
The condition for non-singularity is given by

(V)

$$\frac{2}{c} + k \left(d - \frac{a_0^2}{v_{00}} \right) > 0, \tag{2.5}$$

where $v_{00} = \{N - (N-2)\rho\}/(\sigma^2(1+\rho))$ and a_0, d, $1/c$ are as in (2.4).

2.4. Method of construction of RSORD under the autocorrelated structure

In this subsection we discuss one method of construction of a RSORD under the autocorrelated structure of the errors. RSORDs are those designs which remain second order rotatable for all the variance-covariance matrices belonging to a well-defined class, possessing the errors in observations. These designs depend on the definite pattern of correlation structure of errors in observations but free of correlation parameter or parameters involved in it. The designs obtainable by this method satisfy the moment conditions (2.4) and (2.5).

TABELS 2.1 *A RSORD with 2 factors and 17 design points*

d_0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
x_1	0	−1	0	1	0	−1	0	1	0	−1.414	0	1.414	0	0	0	0	0
x_2	0	−1	0	−1	0	1	0	1	0	0	0	0	0	−1.414	0	1.414	0

2.4.1. Description of the method. We start with a usual SORD (*i.e.*, in case of uncorrelated errors) having n non-central design points involving k factors. The set of n design points can be extended to $(2n + 1)$ points by incorporating $(n+1)$ central points in the following way. One central point is placed in between each pair of non-central design points in the sequence, utilizing thus $(n - 1)$ such central points. Of the other two central points one is placed at the beginning and one at the end. The number of central points of the usual SORD with which we started may be different from the number of central points required in the design so constructed in the autocorrelated error structure situation.

Following the above method, a RSORD under the autocorrelated structure with 2 factors and 17 design points is given in Table 2.1. We start with a usual SORD having 8 non-central design points and 9 central points involving 2 factors. The design (RSORD, denoted by d_0) is displayed in Table 2.1 (column being runs).

REMARK 2.1. The RSORD under autocorrelated structure thus constructed above is not invariant under some permutations (*i.e.*, the order of experiments) of the design points with respect to robust rotatability.

2.5. Measure of robust rotatability based on moment matrix

In this subsection we have given our earlier measure of second order robust rotatability based on moment matrix following Draper and Pukelsheim (1990), for comparing our new measure which is developed as in Section 3.

The traditional representation of a second order model is such that a row of the design matrix (X) consists of the terms

$$(1; x_1, x_2, \ldots, x_k, x_1^2, x_2^2, \ldots, x_k^2, x_1 x_2, \ldots, x_{k-1} x_k). \tag{2.6}$$

Here we are interested in a special type of representation of design matrix (X). The notation, which we shall use here, is the following. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)'$ and $\mathbf{x}_i = (x_{i1}, x_{i2}, x_{i3}, \ldots, x_{iN})'$, $1 \leq i \leq k$. We shall denote the terms in the

second order model by a matrix $Z(\mathbf{x})$ of order $N \times (1 + k + k^2)$ which is given by

$$Z(\mathbf{x}) = (\mathbf{1} : \mathbf{x}' : \mathbf{x}' \otimes \mathbf{x}'), \tag{2.7}$$

where the symbol $\otimes$ denotes the Kronecker product.

Thus there are $(1 + k + k^2)$ terms in a row, say the ith row looks like

$$(1; x_{1i}, x_{2i}, \ldots, x_{ki}; x_{1i}^2, x_{1i}x_{2i}, \ldots, x_{1i}x_{ki};$$

$$x_{2i}x_{1i}, x_{2i}^2, \ldots, x_{2i}x_{ki}; \cdots ; x_{ki}x_{1i}, x_{ki}x_{2i}, \ldots, x_{ki}^2), \ 1 \le i \le N. \tag{2.8}$$

An obvious disadvantage of (2.8) is that all cross product terms occur twice, so the corresponding $Z'(\mathbf{x})Z(\mathbf{x})$ matrix is singular. A suitable generalized inverse is obvious, however, and this notation is very easily extended to higher orders.

Let us consider any second order robust rotatable design d_0 under the correlated regression model (2.1). The normed moment matrix of d_0 is V of order $(1 + k + k^2) \times (1 + k + k^2)$ where

$$V = \frac{Z(\mathbf{x})'W^{-1}Z(\mathbf{x})}{\mathbf{1}'W^{-1}\mathbf{1}}. \tag{2.9}$$

Normed moment matrix V in (2.9) of order $(1 + k + k^2) \times (1 + k + k^2)$ can be written in the form

$$V = V_0 + \left(\frac{a_0}{v_{00}}\right)(2k)^{1/2}V_1 + \left(\frac{\frac{1}{e}}{v_{00}}\right)(k)^{1/2}V_2 + \left(\frac{d}{v_{00}}\right)\{k(k-1)\}^{1/2}V_3$$

$$+ \left\{\frac{(\frac{2}{c} + d)}{v_{00}}\right\}(k)^{1/2}V_4 + \left(\frac{\frac{1}{c}}{v_{00}}\right)\{2k(k-1)\}^{1/2}V_5, \tag{2.10}$$

where $v_{00} = \mathbf{1}'W^{-1}\mathbf{1}$, a_0, d, $1/c$, $1/e$ and $(2/c + d)$ are as in (2.3), and each V_i is of order $(1 + k + k^2) \times (1 + k + k^2)$, $0 \le i \le 5$.

In the above, V_0 consists of a one in $(1, 1)$ position and zeros elsewhere; V_1 consists of $(2k)^{-1/2}$ in each of the $2k$ positions viz. $(1, j(k + 1) + 1)$ and $(j(k + 1) + 1, 1)$; $1 \le j \le k$ and zeros elsewhere; V_2 consists of $(k)^{-1/2}$ in each of the k diagonal positions viz. (i, i), $2 \le i \le (k + 1)$ and zeros elsewhere; V_3 consists of $\{k(k - 1)\}^{-1/2}$ in each of the $k(k - 1)$ positions corresponding to mixed even fourth-order moments, i.e., $(\mathbf{x}_i \otimes_1 \mathbf{x}_i)'W^{-1}(\mathbf{x}_j \otimes_1 \mathbf{x}_j)$, $1 \le i \ne j \le k$ in V, and zeros elsewhere; V_4 consists of $(k)^{-1/2}$ in each of the k positions corresponding to pure fourth-order moments, i.e., $(\mathbf{x}_i \otimes_1 \mathbf{x}_i)'W^{-1}(\mathbf{x}_i \otimes_1 \mathbf{x}_i)$, $1 \le i \le k$ in V, and zeros elsewhere, and finally V_5 consists of $\{2k(k - 1)\}^{-1/2}$ in each of the

$2k(k-1)$ positions corresponding to twisted mixed even fourth-order moments, i.e., $(\mathbf{x}_i \otimes_1 \mathbf{x}_j)'W^{-1}(\mathbf{x}_i \otimes_1 \mathbf{x}_j), 1 \leq i \neq j \leq k$ in V, and zeros elsewhere. Here $\otimes_1$ denotes the Hadamard product is defined as in Section 2.1.

Note that V_i's, $0 \leq i \leq 5$ are symmetric and orthogonal so that $V_i V_j = 0$, and also each V_i has norm $|| V_i || = [tr(V_i V_i)]^{1/2} = 1$.

Let A_d be the moment matrix of a second order design d. We regress A_d on V_0, V_1, V_2, V_3, V_4 and V_5 to yield the fitted equation

$$\bar{A}_d = \sum_{i=0}^{5} \alpha_i V_i \tag{2.11}$$

with regression coefficients $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5. These coefficients are determined by multiplying the equation (2.11) in turn by $V_0, V_1, V_2, V_3, V_4, V_5$ and taking traces

$$\alpha_0 = tr(A_d V_0) = 1,$$
$$\alpha_i = tr(A_d V_i), \ 1 \leq i \leq 5.$$

Hence we obtain the fitted regression equation as

$$\bar{A}_d = V_0 + \sum_{i=1}^{5} V_i tr(A_d V_i) \tag{2.13}$$

and $\bar{A}_d$ is called the rotatable component of A_d.

Two measures based on A_d and $\bar{A}_d$ are defined as follows:

(a) the measure of robust rotatability:

$$Q_k(d) = || \bar{A}_d - V_0 ||^2 \ / \ || A_d - V_0 ||^2$$
$$= \{tr(\bar{A}_d - V_0)^2\}/\{tr(A_d - V_0)^2\} \tag{2.14}$$

($Q_k(d) \leq 1$, with equality if and only if A_d is second order robust rotatable), and

(b) the distance between A_d and $\bar{A}_d$:

$$\delta = || A_d - \bar{A}_d || = \{tr(A_d - \bar{A}_d)^2\}^{1/2} \tag{2.15}$$

with smaller δ meaning more robust rotatable.

Then the following holds:

$$Q_k(d) = \frac{|| \bar{A}_d - V_0 ||^2}{\delta^2 + || \bar{A}_d - V_0 ||^2}. \tag{2.16}$$

3. PROPOSED MEASURE FOR ROBUST SECOND ORDER ROTATABLE DESIGNS

In this section we introduce a new measure to assess the degree of robust rotatability of a second order design under the correlated model (2.1). We first consider the following definitions and terminology used in measuring the degree of robust rotatability.

DEFINITION 3.1. (Robust Second Order Rotatable Design). *A design D on k factors under the correlated model (2.1) which remains second order rotatable for all the variance covariance matrices belonging to a well defined class $W_0 = \{W$ positive definite: $W_{N \times N}$ defined by a particular correlation structure possessing a definite pattern\} is called a Robust Second Order Rotatable Design (RSORD), with reference to the variance-covariance class W_0.*

DEFINITION 3.2. (Derived Design). *A design which is obtained by any permutation of design points of a robust rotatable design is called a derived design.*

The permutation of design points is recommended from practical point of view according to the desired cost and time of the experiment. The class of all derived designs obtained by suitable permutation of design points of a robust rotatable design d_0 is denoted by $D(d_0)$. Note that all derived designs are not necessarily robust rotatable.

DEFINITION 3.3. (Weakly Robust Rotatable Design). *When ρ (correlation parameter involved in W) $\neq 0$, a derived design which is not robust rotatable but is very near to robust rotatable one (in some sense) for a certain range of correlation parameter $\rho \in W$, under a fixed pattern of correlation structure W, is called a Weakly Robust Rotatable Design (WRRD), under that correlation structure class W_0 as in Definition 3.1.*

3.1. Measure of robust second order rotatability

In this subsection we introduce a measure of robust second order rotatability. The second order response surface model with correlated error is given in Subsection 2.1. Following (2.1), we can write

$$y(\mathbf{x}) = \eta(\mathbf{x}) + \mathbf{e}$$

or

$$y_u(\mathbf{x}) = \eta(\mathbf{x}_u) + e_u,$$

where

$$\eta(\mathbf{x}_u) = \beta_0 + \sum_{i=1}^{k} \beta_i x_{iu} + \sum \sum_{i \leq j=1}^{k} \beta_{ij} x_{iu} x_{ju}, \qquad (3.1)$$

which may be written in matrix notation as

$$\eta(\mathbf{x}) = \mathbf{x}_s' \boldsymbol{\beta}, \qquad (3.2)$$

in which the $1 \times m$ vector $\mathbf{x}_s' = (1, x_1^2, x_2^2, \ldots, x_k^2, x_1, x_2, \ldots, x_k, x_1 x_2, x_1 x_3, \ldots, x_{k-1} x_k)$ and $\boldsymbol{\beta}$ is the $m \times 1$ column vector of unknown regression coefficients given in Subsection 2.1 and $m = \binom{k+2}{2}$.

For a known variance-covariance matrix W of errors, the best linear unbiased estimate of $\boldsymbol{\beta}$ assuming $(X'W^{-1}X)$ is positive definite, is

$$\hat{\boldsymbol{\beta}} = (X'W^{-1}X)^{-1}(X'W^{-1}Y).$$

Therefore, the fitted response at $\mathbf{x}_s$ is

$$\hat{y}(\mathbf{x}) = \mathbf{x}_s' \hat{\boldsymbol{\beta}}.$$

When the fitted response $\hat{y}(\mathbf{x}) = \mathbf{x}_s' \hat{\boldsymbol{\beta}}$ is to be used to estimate $\eta(\mathbf{x})$, it is well known that

$$\text{Var}\,[\hat{y}(\mathbf{x})] = \mathbf{x}_s'(X'W^{-1}X)^{-1}\mathbf{x}_s = V(\mathbf{x}). \qquad (3.3)$$

$\text{Var}[\hat{y}(\mathbf{x})]$ thus depends on the particular values of the independent variables through the vectors $\mathbf{x}_s'$. It also depends on the design and the correlation parameter or parameters involved in W through the matrix $(X'W^{-1}X)^{-1}$.

In the k-dimensional space $(k \geq 2)$, $V(\mathbf{x})$ can be expressed in terms of spherical coordinates of $(r, \phi_1, \phi_2, \ldots, \phi_{k-2}, \theta)$, where

$$x_1 = r \cos \phi_1,$$
$$x_2 = r \sin \phi_1 \cos \phi_2,$$
$$\vdots$$
$$x_{k-1} = r \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{k-2} \cos \theta,$$
$$x_k = r \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{k-2} \sin \theta \qquad (3.4)$$

and $r \geq 0$, $0 \leq \phi_1, \phi_2, \ldots, \phi_{k-2} \leq \pi$, $0 \leq \theta \leq 2\pi$, (Fleming, 1977, p. 218).

The absolute value of the Jacobian of this transformation is

$$|J| = r^{k-1} \sin^{k-2} \phi_1 \sin^{k-3} \phi_2 \cdots \sin^2 \phi_{k-3} \sin \phi_{k-2}.$$

If we substitute (3.4) into (3.3), then (3.3) will be expressed as a function of $r, \phi_1, \phi_2, \ldots, \phi_{k-2}$, θ and correlation parameter ρ or parameters (ρ_i's; $i = 1, 2, \ldots, s$) involved in W, i.e.,

$$V(\mathbf{x}) = \omega(r, \phi_1, \phi_2, \ldots, \phi_{k-2}, \theta, \rho \text{ or } \rho_i\text{'s}) = \omega. \qquad (3.5)$$

Let

$$\bar{\omega}(r) = \frac{1}{T_k} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \omega(r, \phi_1, \phi_2, \ldots, \phi_{k-2}, \theta, \rho \text{ or } \rho_i\text{'s}) d\Omega, \quad (3.6)$$

where $d\Omega = \sin^{k-2} \phi_1 \sin^{k-3} \phi_2 \cdots \sin^2 \phi_{k-3} \sin \phi_{k-2} \, d\phi_1 \, d\phi_2 \cdots d\phi_{k-2} \, d\theta$ and

$$T_k = \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi d\Omega = \frac{2\pi^{k/2}}{\Gamma(k/2)}.$$

$\bar{\omega}(r)$ means the averaged value of $V(\mathbf{x})$ over all the points on the hyper sphere of radius r centered at the origin. To be robust rotatable, $\omega(r, \phi_1, \phi_2, \ldots, \phi_{k-2}, \theta, \rho \text{ or } \rho_i\text{'s}) = \bar{\omega}(r, \rho \text{ or } \rho_i\text{'s}) = \bar{\omega}$, say for all r, ϕ_i, θ, ρ or ρ_i's.

For a given design d, the discrepancy from rotatability at r can be expressed as

$$h_d(r) = \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi (\omega_d - \bar{\omega})^2 \, d\Omega, \qquad (3.7)$$

where ω_d is the value of $V(\mathbf{x}) = \omega$ as in (3.5) for the design d. If the region of interest is $0 \leq r \leq 1$, the proposed measure of robust rotatability for a design d, will be

$$P_k(d) = \frac{1}{1 + G_k(d)}, \qquad (3.8)$$

where

$$G_k(d) = \frac{1}{S_k} \int_0^1 r^{k-1} h_d(r) dr \qquad (3.9)$$

and S_k is a positive constant depending only on k.

Let us take S_k to be

$$S_k = \int_0^1 r^{k-1} T_k dr = \frac{T_k}{k}$$

for convenience. By this way, $G_k(d)$ represents the average of $(\omega - \bar{\omega})^2$ over the region of integration.

Note that $P_k(d) \leq 1$, with equality if and only if the design d is robust rotatable, and it is smaller than one for a non-rotatable design. Also note that $P_k(d)$ is invariant with respect to the rotation of the co-ordinate axes, since $\bar{\omega}$, $h_d(r)$ and $G_k(d)$ are invariance with respect to the rotation of the co-ordinate axes.

DEFINITION 3.4. (Weakly Robust Second Order Rotatable Design). *Any design d or a derived design $d \in D(d_0)$ is said to be Weakly Robust Second Order Rotatable Design* (WRSORD) *of strength ν if*

$$P_k(d) \geq \nu. \tag{3.10}$$

Note that $P_k(d)$ involves the correlation parameter $\rho \in W$ and as such, $P_k(d) \geq \nu$ for all ρ is too strong to be met with. On the other hand, for a given ν, we can possibly find the range of values of ρ for which $P_k(d) \geq \nu$. We will call this range as the Weak Rotatability Region (WRR), $(R_{d(\nu)}(\rho))$ of the design d. Naturally, the desirability of using d will rest on the wide nature of WRR, $(R_{d(\nu)}(\rho))$ along with its strength ν. Generally, we would require ν to be very high say, around 0.95.

Now, we introduce the following fact which is useful for evaluating our measure.

(1)
$$\int d\Omega = \frac{2\pi^{k/2}}{\Gamma(k/2)},$$

(2)
$$\int x_i^2 \, d\Omega = r^2 \frac{T_k}{k},$$

(3)
$$\int x_i^2 x_j^2 \, d\Omega = \frac{1}{3} \int x_i^4 \, d\Omega = r^4 \frac{T_k}{k(k+2)}, \; i \neq j,$$

(4)

$$\int x_i^2 \ x_j^2 \ x_l^2 \ d\Omega = \frac{1}{3} \int x_i^4 \ x_j^2 d\Omega = \frac{1}{15} \int x_i^6 \ d\Omega$$

$$= r^6 \frac{T_k}{k(k+2)(k+4)}, \ i \neq j \neq l,$$

(5)

$$\int x_i^2 \ x_j^2 \ x_l^2 \ x_m^2 \ d\Omega = \frac{1}{3} \int x_i^4 \ x_j^2 \ x_l^2 d\Omega = \frac{1}{9} \int x_i^4 x_j^4 \ d\Omega = \frac{1}{15} \int x_i^6 x_j^2 \ d\Omega$$

$$= \frac{1}{105} \int x_i^8 \ d\Omega = r^8 \frac{T_k}{k(k+2)(k+4)(k+6)},$$

$$i \neq j \neq l \neq m,$$

where i, j, l, m could be $1, 2, \ldots, k$ and $\int$ means $\int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi}$ over $\phi_1, \phi_2, \ldots,$ ϕ_{k-2}, θ. The values of other integrals where at least one x_i has an odd exponent are all zeros.

3.2. Illustration of the proposed measure

Herein we have considered an example of a nearly robust second order rotatable design with autocorrelated error. A general method of construction of robust second order rotatable designs with autocorrelated error is given in subsection 2.4.

A derived design d which is obtained by permutation of design points of d_0 (as given in Table 2.1), where the derived design $d \in D(d_0)$ (class of all derived designs obtained from d_0), is displayed in Table 3.1 (column being runs).

Here the RSORD d_0 is constructed under the autocorrelated structure is given by

$$W_0 = \left\{ D(\mathbf{e}) = \left[\sigma^2 \{ \rho^{|i-j|} \}_{1 \leq i, j \leq N} \right] = W_{N \times N}(\rho) \right\}. \qquad (3.11)$$

Note that,

$$W_{N \times N}^{-1}(\rho) = \{ \sigma^2(1-\rho^2) \}^{-1} \left[(1+\rho^2)I_N - \rho^2 A_0 - \rho B_0 \right], \qquad (3.12)$$

where I_N is the $N \times N$ identity matrix, A_0 is the $N \times N$ matrix with elements $a_{11} = a_{NN} = 1$ and all other elements 0 (zeros), and B_0 is the $N \times N$ matrix with $b_{ij} = 1$ for $|i-j| = 1$ and all other elements 0 (zeros).

TABLE 3.1 *A nearly RSORD with 2 factors and 17 design points*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
x_1	0	0	−1	0	1	0	−1	0	1	0	−1.414	0	1.414	0	0	0	0
x_2	0	0	−1	0	−1	0	1	0	1	0	0	0	0	0	−1.414	0	1.414

For the above design d, the moment matrix $X'W^{-1}X$ is given by

$$
\begin{pmatrix}
v_{00} & 0 & 1.414\rho(1-\rho) & 7.998(1-\rho)^2 & v_{0.22} & 0 \\
0 & 7.998(1+\rho^2) & 0 & 0 & 0 & 0 \\
1.414\rho(1-\rho) & 0 & (5.999\rho^2+7.998) & 0 & -2.827\rho^2 & 0 \\
7.998(1-\rho)^2 & 0 & 0 & 11.992(1+\rho^2) & 4(1+\rho^2) & 0 \\
v_{0.22} & 0 & -2.827\rho^2 & 4(1+\rho^2) & 7.996\rho^2+11.992 & 0 \\
0 & 0 & 0 & 0 & 0 & 4(1+\rho^2)
\end{pmatrix},
$$

where $v_{00} = (15\rho^2 - 32\rho + 17)$ and $v_{0.22} = (5.999\rho^2 - 13.997\rho + 7.998)$.

Now it is very difficult to find $D(\hat{\boldsymbol{\beta}}) = \{\sigma^2(1-\rho^2)\}^{-1}(X'W^{-1}X)^{-1}$ explicitly in terms of ρ. Without loss of generality, we will assume that $\sigma^2 = 1$ for notational convenience. So, we have computed numerically $(1 - \rho^2)^{-1}(X'W^{-1}X)^{-1}$ for different values of $\rho = -0.9, -0.8, \ldots, 0.0, 0.1, \ldots, 0.9$. It is seen that the inverse has the following form:

$$
\begin{pmatrix}
v^{00} & v^{0.1}=0 & v^{0.2}\simeq 0 & v^{0.11} & v^{0.22} & v^{0.12}=0 \\
v^{1.0}=0 & v^{1.1} & v^{1.2}=0 & v^{1.11}=0 & v^{1.22}=0 & v^{1.12}=0 \\
v^{2.0}\simeq 0 & v^{2.1}=0 & v^{2.2} & v^{2.11}\simeq 0 & v^{2.22}\simeq 0 & v^{2.12}=0 \\
v^{11.0} & v^{11.1}=0 & v^{11.2}\simeq 0 & v^{11.11} & v^{11.22} & v^{11.12}=0 \\
v^{22.0} & v^{22.1}=0 & v^{22.2}\simeq 0 & v^{22.11} & v^{22.22} & v^{22.12}=0 \\
v^{12.0}=0 & v^{12.1}=0 & v^{12.2}=0 & v^{12.11}=0 & v^{12.22}=0 & v^{12.12}
\end{pmatrix},
$$

where v^m's are the corresponding elements of the inverse matrix as in (2.2).

A comparison between $v^{1.1}$ and $v^{2.2}$, and also between $v^{11.11}$ and $v^{22.22}$, is given in Table 3.2. From Table 3.2, it is clear that $v^{1.1} \simeq v^{2.2}$ and $v^{11.11} \simeq v^{22.22}$ and also from the variance-covariance matrix of $D(\hat{\boldsymbol{\beta}})$ as the form is given above, it is seen that some covariance components are exactly 0 (zero) and some are approximately 0 (zero). We delete those components from the variance function. Therefore, from (3.5) $V(\hat{y}_x)$ is given below.

$$
\begin{aligned}
V(\mathbf{x}) = V(\hat{y}_x) &= Var(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_{11} x_1^2 + \hat{\beta}_{22} x_2^2 + \hat{\beta}_{12} x_1 x_2) \\
&= v^{00} + v^{1.1} x_1^2 + v^{2.2} x_2^2 + v^{11.11} x_1^4 + v^{22.22} x_2^4 + v^{12.12} x_1^2 x_2^2
\end{aligned}
$$

TABLE 3.2 *A comparison table for $v^{1.1}$ and $v^{2.2}$, $v^{11.11}$ and $v^{22.22}$, and values of $P_k(d)$ and $Q_k(d)$ for different values of ρ*

ρ	$v^{2.2} - v^{1.1}$	$v^{22.22} - v^{11.11}$	$P_k(d)$	$Q_k(d)$
−0.9	0.002	0.003	0.9999999	.9913*
−0.8	0.004	0.002	0.9999987	.9928
−0.7	0.004	0.005	0.9999954	.9942
−0.6	0.005	0.009	0.9999916	.9956
−0.5	0.004	0.010	0.9999894	.9968
−0.4	0.004	0.010	0.9999897	.9979
−0.3	0.004	0.009	0.9999927	.9988
−0.2	0.001	0.005	0.9999958	.9994
−0.1	0.0003	0.004	0.9999988	.9998
0.0	0.000	0.000	1.0000000	1.0000
0.1	0.0003	0.003	0.9999985	.9998
0.2	0.001	0.005	0.9999940	.9991
0.3	0.002	0.006	0.9999868	.9977
0.4	0.006	0.007	0.9999780	.9954
0.5	0.004	0.007	0.9999691	.9921
0.6	0.005	0.006	0.9999622	.9878*
0.7	0.005	0.005	0.9999596	.9828
0.8	0.004	0.004	0.9999642	.9772
0.9	0.002	0.002	0.9999795	.9713

$$+ 2v^{0.11}x_1^2 + 2v^{0.22}x_2^2 + 2v^{11.22}x_1^2 x_2^2$$
$$= v^{00} + 2(v^{1.1} + v^{0.11})r^2 + v^{11.11}r^4$$
$$+ \{v^{12.12} - d(\rho)\}x_1^2 x_2^2 + 2g(\rho)x_2^2, \tag{3.13}$$

where $d(\rho) = 2(v^{11.11} - v^{11.22})$ and $g(\rho) = v^{0.22} - v^{0.11}$. Therefore, from (3.6),

$$\bar{\omega}(r) = \frac{1}{T_k} \int V(\mathbf{x})d\Omega$$

$$= v^{00} + 2(v^{1.1} + v^{0.11})r^2 + v^{11.11}r^4 + \{v^{12.12} - d(\rho)\}\frac{r^4}{8} + 2g(\rho)\frac{r^2}{2}. \tag{3.14}$$

Therefore,

$$[\omega_d - \bar{\omega}(r)]^2 = \left[\{v^{12.12} - d(\rho)\}\left(x_1^2 x_2^2 - \frac{r^4}{8}\right) + 2g(\rho)\left(x_2^2 - \frac{r^2}{2}\right)\right]^2$$

$$= \{v^{12.12} - d(\rho)\}^2 \left(x_1^4 x_2^4 + \frac{r^8}{64} - 2\frac{r^4}{8}x_1^2 x_2^2\right)$$

$$+ 4(g(\rho))^2 \left(x_2^4 + \frac{r^4}{4} - r^2 x_2^2 \right)$$

$$+ 4\ g(\rho)\ \{v^{12.12} - d(\rho)\}$$

$$\times \left(x_1^2 x_2^4 - \frac{r^2}{2}\ x_1^2 x_2^2 - \frac{r^4}{8} x_2^2 + \frac{r^6}{16} \right).$$

Therefore, from (3.7)

$$h_d(r) = \left[\{v^{12.12} - d(\rho)\}^2 \frac{r^8}{128} + 4(g(\rho))^2 \frac{r^4}{8} \right.$$

$$\left. + 4g(\rho)\ \{v^{12.12} - d(\rho)\}\ \frac{r^6}{48} \right] T_k. \qquad (3.15)$$

Therefore, from (3.9)

$$G_k(d) = \frac{2}{T_k} \left[\frac{\{v^{12.12} - d(\rho)\}^2}{128 \times 9} + 4\ \frac{(g(\rho))^2}{8 \times 5} + 4\ \frac{g(\rho)\ \{v^{12.12} - d(\rho)\}}{48 \times 7} \right] T_k$$

$$= \left[\frac{\{v^{12.12} - d(\rho)\}^2}{576} + \frac{(g(\rho))^2}{5} + \frac{g(\rho)\ \{v^{12.12} - d(\rho)\}}{42} \right].$$

Therefore, from (3.8)

$$P_k(d) = \frac{1}{1 + \left[\frac{\{v^{12.12} - d(\rho)\}^2}{576} + \frac{(g(\rho))^2}{5} + \frac{g(\rho)\ \{v^{12.12} - d(\rho)\}}{42} \right]}, \qquad (3.16)$$

where $d(\rho)$ and $g(\rho)$ as in (3.13).

Following (2.16), the earlier measure of robust rotatability of the above design d is given by

$$Q_k(d) = \frac{(480 - 960\rho + 1764\rho^2 - 840\rho^3 + 390\rho^4)}{(8\rho^2 - 8\rho^3 + 30\rho^4) + (480 - 960\rho + 1764\rho^2 - 840\rho^3 + 390\rho^4)}. \qquad (3.17)$$

The values of $P_k(d)$ and $Q_k(d)$ for different values of ρ are given in Table 3.2. From Table 3.2, Weak Rotatability Region (WRR) based on $P_k(d)$ is $R_{d(.99)}(\rho) = (-0.9, 0.9)$, and based on $Q_k(d)$ are $R^*_{d(.99)}(\rho) = (-0.9, 0.6)$, $R^*_{d(.95)}(\rho) = (-0.9, 0.9)$. Therefore, $P_k(d)$ gives wider range of variation of ρ than $Q_k(d)$.

4. COMPARISON OF ROBUST ROTATABILITY MEASURES

In this section we are interested to compare our proposed measure of robust rotatability $P_k(d)$ with the measure $Q_k(d)$. The measure $P_k(d)$ is used only for

TABLE 4.1 *Comparison of two measures of robust rotatability*

Criteria	$Q_k(d)$	$P_k(d)$
Applicability to asymmetric design	Yes	Yes
Invariance *w.r.t.* the design rotation	Yes	Yes
Information about variance contour shape	No	No
Range	0 to 1	0 to 1
Order of the model to which the measure applies	$d \geq 1$	$d = 2$
Weak Rotatability Region of ρ	Smaller	Greater

second ($d = 2$) order model but the measure $Q_k(d)$ can be used for any model ($d \geq 1$). However, both the measures do not provide information about variance contour shape. For the usefulness of our proposed measure, we want to mention the following facts which are given in Table 4.1.

5. CONCLUDING REMARKS

In this article we have developed a new measure ($P_k(d)$) of robust second order rotatability. One can easily compute the variance contour $V(\mathbf{x})$ from the equations (3.3) and (3.5). The measure $P_k(d)$ is illustrated with an example which is very near robust second order rotatable design with autocorrelated errors. This measure is also compared with the measure $Q_k(d)$. This measure $P_k(d)$ gives the wider range of ρ than the measure $Q_k(d)$. With the help of this measure we can examine the robust rotatability of a second order design with respect to any variance-covariance structure of errors.

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