Journal of the Korean Statistical Society (2007), **36**: *4, pp 501–522*

INFERENCE ON THE SEASONALLY COINTEGRATED MODEL WITH STRUCTURAL CHANGES†

DAEGUN SONG[1] AND SINSUP CHO[2]

ABSTRACT

We propose an estimation procedure that can be used for detecting structural changes in the seasonal cointegrated vector autoregressive model. The asymptotic properties of the estimates and the test statistics for the parameter change are provided. A simulation example is presented to illustrate this method and its concept.

AMS 2000 subject classifications. Primary 60G15; Secondary 91B24.
Keywords. Seasonal cointegration, seasonal cointegration rank test, seasonal error correction model, structural change.

1. INTRODUCTION

Detection of the change points in time series has been an important issue and has been studied by many authors. See Box and Tiao (1975) for the level change in the nonstationary time series, Wichern *et al.* (1976) for the variance change in the AR(1) model, Picard (1985) for change of mean and autocovariane, Kim *et al.* (2000) for the parameter change in GARCH(1,1) model, and Wang and Wang (2006) for the change in long memory parameters, among others.

Since Hylleberg *et al.* (1990) first analyzed the seasonal cointegration system, several approaches for detection of changes in the system have been developed. Seo (1998) proposed the Lagrange multiplier (LM) test for the detection of structural changes of the cointegrating vector and the adjustment vector. Inoue (1999) derived a rank test for cointegrated processes with a broken trend, and Hansen

Received December 2006; accepted July 2007.
†This research was supported by Korea Research Foundation Grant(KRF-2005-070-C00022) funded by the Korean Government (MOHEAD).
[1]Risk Management Division, Korea Exchange Bank, Seoul 100-793, Korea
[2]Corresponding author. Department of Statistics, Seoul National University, Seoul 151-747, Korea (email: sinsup@snu.ac.kr)

(2003) generalized the cointegrated vector autoregressive model (VAR) of Johansen (1988) to allow for structural changes when the time of the change points is known.

In this paper, we extend the method of Hansen (2003) and propose an estimation procedure of the structural change in the seasonally cointegrated VAR. We focus mainly on complex roots because the estimation and the test for a parameter change with the root at 1 is identical to that of Hansen (2003).

This paper is organized as follows. Section 2 contains the statistical formulation of structural changes in the seasonal cointegrated VAR. Estimation of parameters, asymptotic properties of estimators and test statistics for the parameter changes are provided in Section 3. Section 4 presents the simulation results and Section 5 presents the conclusion. The proofs are presented in Appendix.

2. The Statistical Model

For seasonal cointegration with constant parameters, Johansen and Schaumburg (1999) formulated the n-dimensional process of order k, VAR(k) for the quarterly data as follows:

$$\begin{aligned}\triangle_4 X_t &= \sum_{i=1}^{2} \alpha_i \beta_i' X_{i,t} + 2(\alpha_R \beta_R' + \alpha_I \beta_I') X_{R,t} + 2(\alpha_R \beta_I' - \alpha_I \beta_R') X_{I,t} \\ &\quad + \sum_{j=1}^{k-4} \Gamma_j \triangle_4 X_{t-j} + \phi D_t + \epsilon_t, \quad t = 1, \ldots, T, \end{aligned} \tag{2.1}$$

where $\epsilon_t \sim iid\,\mathrm{N}(0,\, \Omega)$, D_t consists of deterministic terms, and

$$\begin{aligned} X_{1,t} &= \frac{1}{4}(X_{t-1} + X_{t-2} + X_{t-3} + X_{t-4}), \\ X_{2,t} &= -\frac{1}{4}(X_{t-1} - X_{t-2} + X_{t-3} - X_{t-4}), \\ X_{R,t} &= \frac{1}{4}(X_{t-2} - X_{t-4}), \\ X_{I,t} &= -\frac{1}{4}(X_{t-1} - X_{t-3}). \end{aligned}$$

Note that $X_{1,t}$ and $X_{2,t}$ are integrated at zero and 1/2 frequencies, and $X_{R,t}$ and $X_{I,t}$ are the real and imaginary parts of the integrated process at 1/4 frequency, respectively.

If we permit parameters to change their values at the change points $T_1, \ldots,$ T_{m-1}, where $0 < T_1 < \cdots < T_{m-1} < T$, the seasonally cointegrated VAR model with structural changes can be written as

$$\triangle_4 X_t = \sum_{i=1}^{2} \alpha_i(t)\beta_i(t)' X_{i,t}{}^* + 2\{\alpha_R(t)\beta_R(t)' + \alpha_I(t)\beta_I(t)'\}X^*_{R,t} + 2\{\alpha_R(t)\beta_I(t)'$$
$$-\alpha_I(t)\beta_R(t)'\}X^*_{I,t} + \sum_{j=1}^{k-4} \Gamma_j \triangle_4 X_{t-j} + \phi D_t + \epsilon_t, \quad t = 1, \ldots, T. \qquad (2.2)$$

The dimensions of α_i and β_i are $n \times r_i$ and $n_1 \times r_i$, respectively, where $n_1 = n+1$ is the dimension of $X^*_{i,t}$ with $i = 1, 2, R$ and I. Variable $X^*_{i,t}$ comprises $X_{i,t}$ and restricted deterministic variables, $\{\epsilon_t\}$ is a sequence of independent Gaussian variables with mean zero and variance $\Omega(t)$, and

$$\alpha_1(t)\beta_1(t)' = \alpha_{11}\beta_{11}1_{1t} + \cdots + \alpha_{1m}\beta_{1m}1_{mt},$$
$$\alpha_2(t)\beta_2(t)' = \alpha_{21}\beta_{21}1_{1t} + \cdots + \alpha_{1m}\beta_{2m}1_{mt},$$
$$\alpha_R(t)\beta_R(t)' = \alpha_{R1}\beta_{R1}1_{1t} + \cdots + \alpha_{Rm}\beta_{Rm}1_{mt},$$
$$\alpha_I(t)\beta_I(t)' = \alpha_{I1}\beta_{I1}1_{1t} + \cdots + \alpha_{Im}\beta_{Im}1_{mt},$$
$$\alpha_R(t)\beta_I(t)' = \alpha_{R1}\beta_{I1}1_{1t} + \cdots + \alpha_{Rm}\beta_{Im}1_{mt},$$
$$\alpha_I(t)\beta_R(t)' = \alpha_{I1}\beta_{R1}1_{1t} + \cdots + \alpha_{Im}\beta_{Rm}1_{mt},$$
$$\Gamma_i = \Gamma_{1,i}1_{1,t} + \cdots + \Gamma_{m,i}1_{m,t}, \ (i = 1, \ldots, k-4),$$
$$\Phi(t) = \Phi_1 1_{1,t} + \cdots + \Phi_m 1_{mt},$$
$$\Omega(t) = \Omega_1 1_{1,t} + \cdots + \Omega_m 1_{mt},$$

where

$$1_{j,t} = 1(T_{j-1} + 1 \le t \le T_j), \ j = 1, \ldots, m, \ (T_0 = 0, \ T_m = T)$$

is an indicator function as defined in Hansen (2003). It is noted that the rank of cointgrated vector β^*_{ij} ($i = 1, 2, R, I$ and $j = 1, \ldots, m$) may vary across the subsamples. Therefore, the rank of $\alpha_i(t)\beta_i(t)$ may change across the subsamples to permit a change in the cointegration rank r_i while the dimensions of other parameters are constant.

Let

$$
\begin{aligned}
&A_1 = (\alpha_{11}, \ldots, \alpha_{m1}), \quad A_2 = (\alpha_{12}, \ldots, \alpha_{m2}),\\
&A_R = (\alpha_{1R}, \ldots, \alpha_{mR}), \quad A_I = (\alpha_{1I}, \ldots, \alpha_{mI}),\\
&B_1 = \begin{pmatrix} \beta_{11} & 0 & \cdots & 0 \\ 0 & \beta_{21} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{m1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \beta_{12} & 0 & \cdots & 0 \\ 0 & \beta_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{m2} \end{pmatrix},\\
&B_R = \begin{pmatrix} \beta_{1R} & 0 & \cdots & 0 \\ 0 & \beta_{2R} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{mR} \end{pmatrix}, \quad B_I = \begin{pmatrix} \beta_{1I} & 0 & \cdots & 0 \\ 0 & \beta_{2I} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{mI} \end{pmatrix},\\
&C = (\psi_1, \ldots, \psi_m), \quad \psi_j = (\Gamma_{j,1}, \ldots, \Gamma_{j,k-4}, \Phi_j), \quad j = 1, \ldots, m
\end{aligned}
$$

and

$$
\begin{aligned}
Z_{0t} &= \triangle_4 X_t : n \times 1,\\
Z_{1t} &= (1_{1,t}X_{1,t}^{*\,\prime}, \ldots, 1_{m,t}X_{m,t}^{*\,\prime}) : (mn_1) \times 1,\\
Z_{2t} &= (1_{1,t}X_{2,t}^{*\,\prime}, \ldots, 1_{m,t}X_{m,t}^{*\,\prime}) : (mn_1) \times 1,\\
Z_{Rt} &= (1_{1,t}X_{R,t}^{*\,\prime}, \ldots, 1_{m,t}X_{m,t}^{*\,\prime}) : (mn_1) \times 1,\\
Z_{It} &= (1_{1,t}X_{I,t}^{*\,\prime}, \ldots, 1_{m,t}X_{m,t}^{*\,\prime}) : (mn_1) \times 1,\\
\tilde{Z}_{nt} &= (\triangle_4 X_{t-1}', \ldots, \triangle_4 X_{t-k+4}', D_t')' : n_2 \times 1,\\
Z_{ct} &= (1_{1t}\widetilde{Z}_{nt}', \ldots, 1_{mt}\widetilde{Z}_{nt}')' : (mn_2) \times 1.
\end{aligned}
$$

Then, (2.2) can be written as

$$
\begin{aligned}
Z_{0t} = &\sum_{i=1}^{2} A_i B_i' Z_{it} + 2(A_R B_R' + A_I B_I') Z_{Rt} + 2(A_R B_I' - A_I B_R') Z_{It}\\
&+ C Z_{ct} + \epsilon_t, \quad t = 1, \ldots, T
\end{aligned} \tag{2.3}
$$

and it is the same form as (2.1).

For each of the m equations in (2.2),

$$\begin{aligned}
\triangle_4 X_t &= \sum_{i=1}^{2} \alpha_{i1}\beta_{i1}' X_{i,t}{}^* + 2(\alpha_{R1}\beta_{R1}' + \alpha_{I1}\beta_{I1}')X_{R,t}^* + 2(\alpha_{R1}\beta_{I1}' - \alpha_{I1}\beta_{R1}')X_{I,t}^* \\
&\quad + \sum_{j=1}^{k-4} \Gamma_{1,j}\triangle_4 X_{t-j} + \phi_1 D_t + \epsilon_t, \quad t = 1, \ldots, T_1, \\
\triangle_4 X_t &= \sum_{i=1}^{2} \alpha_{i2}\beta_{i2}' X_{i,t}{}^* + 2(\alpha_{R2}\beta_{R2}' + \alpha_{I2}\beta_{I2}')X_{R,t}^* + 2(\alpha_{R2}\beta_{I2}' - \alpha_{I2}\beta_{R2}')X_{I,t}^* \\
&\quad + \sum_{j=1}^{k-4} \Gamma_{2,j}\triangle_4 X_{t-j} + \phi_2 D_t + \epsilon_t, \quad t = T_1 + 1, \ldots, T_2, \\
&\ \ \vdots \\
\triangle_4 X_t &= \sum_{i=1}^{2} \alpha_{im}\beta_{im}' X_{i,t}{}^* + 2(\alpha_{Rm}\beta_{Rm}' + \alpha_{Im}\beta_{Im}')X_{R,t}^* + 2(\alpha_{Rm}\beta_{Im}' \\
&\quad - \alpha_{Im}\beta_{Rm}')X_{I,t}^* + \sum_{j=1}^{k-4} \Gamma_{m,j}\triangle_4 X_{t-j} + \phi_m D_t + \epsilon_t, \quad t = T_{m-1} + 1, \ldots, T.
\end{aligned}$$

Regressing $\triangle_4 X_{0t}$, X_{1t}^*, $X_{2,t}^*$, $X_{R,t}^*$ and X_{It}^* on $\sum_{j=1}^{k-4} \Gamma_{m,j}\triangle_4 X_{t-j}$ and D_t, we obtain the corresponding residuals r_{0t}, r_{1t}, r_{2t}, r_{Rt} and r_{It}, respectively. Since the MLEs of $\alpha_i(t)\beta_i(t)$ are equivalent to the estimates of the following equations:

$$\begin{aligned}
r_{0t} &= \sum_{i=1}^{2} \alpha_{i1}\beta_{i1}' r_{it} + 2(\alpha_{R1}\beta_{R1}' + \alpha_{I1}\beta_{I1}')r_{Rt} + 2(\alpha_{R1}\beta_{I1}' - \alpha_{I1}\beta_{R1}')r_{It} \\
&\quad + r_{\epsilon t}, \quad t = 1, \ldots, T_1, \\
r_{0t} &= \sum_{i=1}^{2} \alpha_{i2}\beta_{i2}' r_{it} + 2(\alpha_{R2}\beta_{R2}' + \alpha_{I2}\beta_{I2}')r_{Rt} + 2(\alpha_{R2}\beta_{I2}' - \alpha_{I2}\beta_{R2}')r_{It} \\
&\quad + r_{\epsilon t}, \quad t = T_1 + 1, \ldots, T_2, \\
&\ \ \vdots \\
r_{0t} &= \sum_{i=1}^{2} \alpha_{im}\beta_{im}' r_{it} + 2(\alpha_{Rm}\beta_{Rm}' + \alpha_{Im}\beta_{Im}')r_{Rt} + 2(\alpha_{Rm}\beta_{Im}' - \alpha_{Im}\beta_{Rm}')r_{It} \\
&\quad + r_{\epsilon t}, \quad t = T_{m-1} + 1, \ldots, T,
\end{aligned}$$

using the notation for $\alpha_i(t)\beta_i(t)$ as defined previously, we obtain

$$r_{0t} = \sum_{i=1}^{2} \alpha_i(t)\beta_i(t)' r_{it} + 2\{\alpha_R(t)\beta_R(t)' + \alpha_I(t)\beta_I(t)'\} r_{Rt} + 2\{\alpha_R(t)\beta_I(t)' \\ -\alpha_I(t)\beta_R(t)'\} r_{It} + r_{\epsilon t}, \qquad t = 1, \ldots, T. \tag{2.4}$$

Again, for each of the m equations in (2.4), by regressing r_{0t}, r_{Rt}, r_{It} on r_{1t}, r_{2t}, we obtain the residuals u_{0t}, u_{Rt}, u_{It} and (2.5) as follows:

$$u_{0t} = 2\{\alpha_R(t)\beta_R(t)' + \alpha_I(t)\beta_I(t)'\} u_{Rt} + 2\{\alpha_R(t)\beta_I(t)' - \alpha_I(t)\beta_R(t)'\} u_{It} \\ + u_{\epsilon t}, \quad t = 1, \ldots, T. \tag{2.5}$$

We can rewrite (2.5) as

$$U_{0t} = 2(A_R B_R' + A_I B_I') U_{Rt} + 2(A_R B_I' - A_I B_R') U_{It} + U_{\epsilon t}, \\ t = 1, \ldots, T, \tag{2.6}$$

where

$$U_{0t} = \Delta_4 u_t : n \times 1,$$
$$U_{Rt} = (1_{1,t} u_{R,t}^{*}{}', \ldots, 1_{m,t} u_{R,t}^{*}{}') : (mn_1) \times 1,$$
$$U_{It} = (1_{1,t} u_{I,t}^{*}{}', \ldots, 1_{m,t} u_{I,t}^{*}{}') : (mn_1) \times 1,$$

which is the same form as (2.1) without the structural change. For estimating the parameter at frequency 1/4, we use (2.6) to obtain the generalized reduced rank regression (GRRR) estimator in the next section.

If we concentrate on the parameters at frequency zero, we have

$$U_{0t}^* = A_1 B_1 U_{1t}^* + U_{\epsilon t}^*, \quad t = 1, \ldots, T, \tag{2.7}$$

where U_{0t}^* and U_{1t}^* can be obtained for each of the m equations in (2.4) by regressing r_{0t}, r_{1t} on r_{2t}, r_{Rt}, r_{It}. It should be noted that (2.7) has a form that is similar to (2) of Hansen (2003). By applying procedures similar to those used in Hansen (2003), we can estimate the parameters A_1 and B_1 in (2.4).

3. Estimation and the Asymptotic Result

Estimation of the parameters in (2.6) can be done by using the GRRR technique of Hansen (2003).

Since a cointegrating vector is not unique, multiplying it by a nonzero constant yields a more cointegrating vector. In order to obtain a unique parameterization, we adopt the following structures of B_R and B_I using parameter restrictions of the form $\mathrm{vec}(B_R) = H_R\varphi_R + h_R$, $\mathrm{vec}(B_I) = H_I\varphi_I + h_I$, $\mathrm{vec}(A_R) = G_R\psi_R$ and $\mathrm{vec}(A_I) = G_I\psi_I$, where $\mathrm{vec}(\cdot)$ is the vectorization operator. Matrices H_R and H_I are known as $mp_1(r_1 + \cdots + r_m) \times p_\varphi$ matrices, h_R and h_I are known as $mp_1(r_1 + \cdots + r_m)$ dimensional vectors, φ_R and φ_I are vectors that consist of p_φ parameters, G_R and G_I are known as $p(r_1 + \cdots + r_m) \times p_\psi$ matrices and ψ_R and ψ_I are vectors with p_ψ free parameters. These restrictions are equal to those of Hansen (2003). See Hansen (2003) for the advantages of these restrictions. For example, consider the bivariate process with a seasonal cointegrating rank $r = 1$ that consists of two subsamples m=2. In this setting, $B_R = \mathrm{diag}(\beta_{R1}, \beta_{R2})$, where $\beta_{R1} = (\beta_{R,1,1}, \beta_{R,1,2})$ and $\beta_{R2} = (\beta_{R,2,1}, \beta_{R,2,2})$. If $\beta_{R,1,1} = \beta_{R,2,1}$ and $\beta_{R,1,2} = \beta_{R,2,2}$, then

$$H_R = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,1\,0 \\ 0\,1\,0\,0\,0\,0\,0\,1 \end{pmatrix}', \quad \varphi_R = \begin{pmatrix} \beta_{R,1,1} \\ \beta_{R,1,2} \end{pmatrix},$$
$$h_R = \begin{pmatrix} 0\,0\,0\,0\,0\,0\,0\,0 \end{pmatrix}'.$$

If we impose the normalization $\beta_{R,1,1} = \beta_{R,2,1} = 1$, then

$$H_R = \begin{pmatrix} 0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix}', \quad \varphi_R = \begin{pmatrix} \beta_{R,1,2} \\ \beta_{R,2,2} \end{pmatrix},$$
$$h_R = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,1\,0 \end{pmatrix}'.$$

If we impose the normalization $\beta_{R,1,1} = \beta_{R,2,1} = 1$ and cointegrating vectors unchanged, then

$$H_R = \begin{pmatrix} 0\,1\,0\,0\,0\,0\,0\,1 \end{pmatrix}', \quad \varphi_R = \beta_{R,1,2},$$
$$h_R = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,1\,0 \end{pmatrix}'.$$

If the cointegrating vectors are not changed but the adjustment vectors are changed, this restriction is useful in estimating the parameters. We also estimate the adjustment vector α_R and α_I by using G_R and G_I. These restrictions enable us to estimate adjustment vector α and cointegrating vector β if only one of the vectors α and β is changed in the cointegration system.

Assumptions 3.1 through 3.3 are based on Hansen (2003) for the nonseasonally cointegrated model, which is still applicable to seasonally cointegrated models. Assumption 3.1 is needed to ensure the assumed rank of A and B. Assumption 3.2 is necessary for testing some of the hypotheses, and Assumption 3.3 is needed for the construction of a useful iterative algorithm to estimate the model.

ASSUMPTION 3.1. Matrices H_R, H_I, G_R and G_I have full column ranks and H_R, H_I, G_R, G_I, h_R, and h_I are such that A_R, A_I, B_R, and B_I have a full column rank for all $(\psi_R', \psi_I', \varphi_R', \varphi_I')' \in \mathbb{R}^n$, except on a set with Lebesque measure zero, where n is the total number of columns in H_R, H_I, G_R and G_I.

ASSUMPTION 3.2. Matrices H_R, H_I, G_R and G_I and the vector h_R and h_I are such that $\psi_R, \psi_I, \varphi_R$ and φ_I are identified.

ASSUMPTION 3.3. Parameters $\psi_R, \psi_I, \varphi_R, \varphi_I$ and θ are variation free, *i.e.*, the parameter space for $(\psi_R,\varphi_R,\theta)$ is given by a product space $\Theta_\psi \times \Theta_\varphi \times \Theta_\theta$, where Θ_κ is the parameter space for $\kappa = \psi_R, \varphi_R$ or θ and the parameter space for $(\psi_I,\varphi_I,\theta)$ is given by a product space $\Theta_\psi \times \Theta_\varphi \times \Theta_\theta$, where Θ_κ is the parameter space for $\kappa = \psi_I, \varphi_I$ or θ.

For the estimation, we impose the following restrictions of the form:

$$\text{vec}(A) = \begin{pmatrix} \text{vec}(A_R) \\ \text{vec}(A_I) \end{pmatrix} = G\varphi, \tag{3.1}$$

where

$$G = \begin{pmatrix} G_R & 0 \\ 0 & G_I \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_R \\ \varphi_I \end{pmatrix}$$

and

$$\text{vec}(B) = \begin{pmatrix} \text{vec}(B_R) \\ \text{vec}(B_I) \end{pmatrix} = H\phi + h, \tag{3.2}$$

where

$$H = \begin{pmatrix} H_R & 0 \\ 0 & H_I \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}, \quad h = \begin{pmatrix} h_R \\ h_I \end{pmatrix}$$

and

$$
\begin{aligned}
&U_0 \equiv (U_{01}, \ldots, U_{0T}), \quad U_R \equiv (U_{R1}, \ldots, U_{RT}),\\
&U_I \equiv (U_{I1}, \ldots, U_{IT}), \quad U_E \equiv (U_{\epsilon 1}, \ldots, U_{\epsilon T}),\\
&\Sigma = \mathrm{diag}(I_{T_1} \otimes \Omega_1, I_{T_2-T_1} \otimes \Omega_2, \ldots, I_{T-T_{m-1}} \otimes \Omega_m),\\
&U_B = 2\begin{pmatrix} (U_R' B_R \otimes I_p)' + (U_I' B_I \otimes I_p)' \\ (U_R' B_I \otimes I_p)' + (U_I' B_R \otimes I_p)' \end{pmatrix}',\\
&U_A = 2\begin{pmatrix} ([(U_R' \otimes A_R) - (U_I' \otimes A_I)]K_{p_1,r})' \\ ([(U_R' \otimes A_I) - (U_I' \otimes A_R)]K_{p_1,r})' \end{pmatrix}'
\end{aligned}
$$

and $K_{p_1,r}$ is the communication matrix, uniquely defined by $K_{p_1,r}\mathrm{vec}(A) \equiv \mathrm{vec}(A')$ for any $p_1 \times r$ matrix. Then, (2.6) can be written as

$$
U_0 = 2(A_R B_R' + A_I B_I')U_R + 2(A_R B_I' - A_I B_R')U_I + U_E. \tag{3.3}
$$

THEOREM 3.1. *Let parameters A and B be restricted by (3.1) and (3.2) and assumptions 3.1 and 3.3 hold. Then, the maximum likelihood estimates of A, B and $\Omega(t)$ satisfy*

$$
\begin{aligned}
\mathrm{vec}(\hat{A}) &= G\hat{\psi}\\
&= G[G'U_B\Sigma^{-1}U_B G]^{-1}G'U_B'\Sigma \mathrm{vec}(U_0), && (3.4)\\
\mathrm{vec}(\hat{B}) &= H\hat{\phi} + h\\
&= H[H'U_A\Sigma^{-1}U_A H]^{-1}H'U_A'\Sigma \mathrm{vec}(U_0) + h, && (3.5)\\
\hat{\Omega}_j &= (T_j - T_{j-1})^{-1} \sum_{t=T_{j-1}+1}^{T_j} \hat{U}_{\epsilon t}\hat{U}_{\epsilon t}', \quad j = 1, \ldots, m, && (3.6)
\end{aligned}
$$

where $\hat{U}_{\epsilon t} = U_{0t} - 2(\hat{A}_R\hat{B}_R' + \hat{A}_I\hat{B}_I')U_{Rt} - 2(\hat{A}_R\hat{B}_I' - \hat{A}_I\hat{B}_R')U_{It}$. The maximum value of the likelihood function is given by

$$
L_{\max}(\hat{A}_R, \hat{A}_I, \hat{B}_R, \hat{B}_I, \hat{\Omega}(t)) = (2\pi)^{Tn/2}\prod_{t=1}^{T}|\hat{\Omega}(t)|^{-1/2}\exp\left(-\frac{Tn}{2}\right).
$$

The parameter estimates can be obtained by iterating on these three equations (3.4), (3.5) and (3.6) until they converge by using some initial values of the parameters. For a given value of $\hat{B}$, selected properly in the first iteration, the estimates of $\hat{A}$ and then $\hat{\Omega}_j$ are computed. We can compute a new $\hat{B}$ and a likelihood function and repeat this procedure until the value of the likelihood function converges. As being pointed out by Hansen (2003), the local maxima may exist. Hence, different initial values of the parameters should be tested to check whether the algorithm converges to the same value of the likelihood function.

We need Assumption 3.4 to guarantee the process to have a (quarterly) seasonally cointegrated relation in each subsample. For the general assumptions used in the seasonal cointegrated case, see Johansen and Schaumburg (1999).

ASSUMPTION 3.4 (Johansen and Schaumburg, 1999). Let $A_j(z) \equiv I - \Pi_{j,1}z - \Pi_{j,2}z^2 - \cdots - \Pi_{j,k}z^k$, where z $\in \mathbb{C}$, $j = 1, \ldots, m$,
(i) the roots of $|A_j(z)| = 0$ satisfy $|z| > 1 + \delta$ or $z \in (1, -1, +i, -i)$ with $|z_i| = 1$ for some $\delta > 0$ and $A_j(z_i) = -\alpha_i \beta_i^*$, and
(ii) $|\alpha_{j,i\perp}^* \dot{A}(z_i) \beta_{j,i\perp}| = 0$, $i = 1, 2, 3, 4$ and $\alpha_{j.1}, \beta_{j.1}$, $\alpha_{j.2}, \beta_{j.2}$, $\alpha_{j.3}, \beta_{j.3}$ and $\alpha_{j.4}, \beta_{j.4}$ are the adjustment vectors and the cointegrated vectors in zero, π, $\pi/2$ and $-\pi/2$, respectively.

Hansen (2003) used the Granger's representation theorem when a structural change exists by using a closed-form expression for the $I(1)$ process in Hansen (2000), and obtain the distribution of $\hat{\beta}$ after the structural change. Although we do not propose an explicit form of the distribution of $\hat{\beta}$ after the structural change, Theorem 3.2 is still necessary to prove Theorem 3.3. Details of this proof are given in Appendix.

THEOREM 3.2 (Consistency). *The maximum likelihood estimators given by (3.4), (3.5), and (3.6) in Theorem 3.1 are consistent for the true parameters.*

THEOREM 3.3 (Asymptotic distribution of LR tests). *Let M_0 and M_1 be two models defined by restrictions (3.1) and (3.2) both satisfying assumptions 3.1 – 3.4, and both having the same cointegration rank in each subsample. If M_1 is a submodel of M_0 with q fewer parameters, then the asymptotic distribution of the likelihood ratio test of M_1, which is a test against M_0, is χ^2 with q degrees of freedom.*

For the computation of the degrees of freedom in Theorem 3.3, we use the following lemma by Johansen (1996) and Hansen (2003).

LEMMA 3.1. *The function $f(x, y) = xy'$, where x is $n \times r$ $(r \leq n)$ and y is $n_1 \times r$ $(r \leq n_1)$, is differentiable at all points with a differential given by*

$$Df(x, y) = x(dy)' + (dx)y',$$

where dy is $n \times r$ and dx is $n_1 \times r$. If x and y have full rank r, then the tangent space at (x, y) given by $\{x(dy)' + (dx)y' : dx \in \mathbb{R}^{n_1 \times r}, dy \in \mathbb{R}^{n \times r}\}$ has dimension $(n + n_1 - r)r$.

From Lemma 3.1, we can observe that the number of free parameters $\alpha(t)\beta(t)$ is $\sum_{i=1}^{m}(n + n_1 - r_i)r_i$. For example, let m=2, $X_t' = (X_{1t}, X_{2t})$ be a bivariate process and $r_1 = r_2 = 2$ with hypothesis

$$H_0 : \alpha_{11} = \alpha_{12} \text{ and } \beta_{11} = \beta_{12} \quad vs. \quad H_1 : \text{ not } H_0$$

then the degrees of freedom is $(n + n_1 - r)r = 3$.

See Hansen (2003) for more details for computing the degrees of freedom under different situations.

4. SIMULATION STUDY

In order to examine the properties of the estimators, we perform a simulation study. Let m=2 and $X_t' = (X_{1t}, X_{2t})$ be a bivariate process. We consider the following model for the simulation study.

$$\begin{aligned}
\triangle_4 X_t &= \sum_{i=1}^{2} \alpha_i(t)\beta_i(t)' X_{i,t} + 2\{\alpha_R(t)\beta_R(t)' + \alpha_I(t)\beta_I(t)'\}X_{R,t} + 2\{\alpha_R(t)\beta_I(t)' \\
&\quad - \alpha_I(t)\beta_R(t)'\}X_{I,t} + \epsilon_t \\
&= \Big[\sum_{i=1}^{2} \alpha_{i1}\beta_{i1}' X_{i,t} + 2(\alpha_{R1}\beta_{R1}' + \alpha_{I1}\beta_{I1}')X_{R,t} \\
&\qquad + 2(\alpha_{R1}\beta_{I1}' - \alpha_{I1}\beta_{R1}')X_{I,t} + \epsilon_t\Big]\big|_{t \le T_1} \\
&\quad + \Big[\sum_{i=1}^{2} \alpha_{i2}\beta_{i2}' X_{i,t} + 2(\alpha_{R2}\beta_{R2}' + \alpha_{I2}\beta_{I2}')X_{R,t} \\
&\qquad + 2(\alpha_{R2}\beta_{I2}' - \alpha_{I2}\beta_{R2}')X_{I,t} + \epsilon_t\Big]\big|_{t > T_1}.
\end{aligned}$$

Parameter values for the simulations are given in Table 4.1. Model 1 allows all the parameters to change except for the nonseasonally and seasonally cointegrated vectors, and Model 2 allows all the parameters to change except for the nonseasonal and seasonal adjustment vectors.

Table 4.2 and Table 4.3 are based on 1,000 replications. Since the property of the estimators at frequency 1/2 is the same as that at zero frequency, we did not deal with the estimators at frequency 1/2. We use the following restrictions

TABLE 4.1 *Parameter values for Model 1 and Model 2*

Model 1				Model 2			
Before Change ($T=1,\dots,T_1$)		After Change ($T=T_1+1,\dots,T$)		Before Change ($T=1,\dots,T_1$)		After Change ($T=T_1+1,\dots,T$)	
α_{11}	$[-1,-0.5]'$	α_{12}	$[-1,-0.7]'$	α_{11}	$[\mathbf{-1},\mathbf{-0.5}]'$	α_{12}	$[\mathbf{-1},\mathbf{-0.5}]'$
β_{11}	$[\mathbf{1},\mathbf{-1}]'$	β_{12}	$[\mathbf{1},\mathbf{-1}]'$	β_{11}	$[1,-1]'$	β_{12}	$[1,-0.8]'$
α_{21}	$[1,-0.4]'$	α_{22}	$[1,-0.5]'$	α_{21}	$[\mathbf{1},\mathbf{-0.5}]'$	α_{22}	$[\mathbf{1},\mathbf{-0.5}]'$
β_{21}	$[\mathbf{1},\mathbf{-0.7}]'$	β_{22}	$[\mathbf{1},\mathbf{-0.7}]'$	β_{21}	$[1,-0.7]'$	β_{22}	$[1,-0.8]'$
α_{R1}	$[1,0.2]'$	α_{R2}	$[1,0.5]'$	α_{R1}	$[\mathbf{1},\mathbf{0.4}]'$	α_{R2}	$[\mathbf{1},\mathbf{0.4}]'$
β_{R1}	$[\mathbf{1},\mathbf{-1}]'$	β_{R2}	$[\mathbf{1},\mathbf{-1}]'$	β_{R1}	$[1,-1]'$	β_{R2}	$[1,-0.9]'$
α_{I1}	$[-1,-1]'$	α_{I2}	$[-0.4,-0.9]'$	α_{I1}	$[\mathbf{-1},\mathbf{-1}]'$	α_{I2}	$[\mathbf{-1},\mathbf{-1}]'$
β_{I1}	$[\mathbf{1},\mathbf{-0.8}]'$	β_{I2}	$[\mathbf{1},\mathbf{-0.8}]'$	β_{I1}	$[1,-0.8]'$	β_{I2}	$[1,-0.7]'$
Ω_1	$\begin{pmatrix}1 & 0.3\\0.3 & 1\end{pmatrix}$	Ω_2	$\begin{pmatrix}1 & 0.5\\0.5 & 1.5\end{pmatrix}$	Ω_1	$\begin{pmatrix}1 & 0.3\\0.3 & 1\end{pmatrix}$	Ω_2	$\begin{pmatrix}1 & 0.5\\0.5 & 1.5\end{pmatrix}$

TABLE 4.2 *Simulation results of the GRRR estimation for Model 1 based on 1,000 replications for each sample size T*

	$T_1=T_2=50$		$T_1=T_2=100$		$T_1=T_2=200$		$T_1=T_2=400$	
True value	Mean	STD	Mean	STD	Mean	STD	Mean	STD
$\alpha_{11,1}=-1$	−1.0866	0.3087	−1.0590	0.1831	−1.0259	0.1255	−1.0090	0.0801
$\alpha_{11,2}=-0.5$	−0.4726	0.2276	−0.4849	0.1679	−0.4903	0.1180	−0.4980	0.0769
$\alpha_{12,1}=-1$	−1.0004	0.2661	−1.0013	0.1488	−0.9989	0.0948	−1.0027	0.0659
$\alpha_{12,2}=-0.7$	−0.6245	0.2446	−0.6509	0.1537	−0.6788	0.0947	−0.6881	0.0602
$\beta_{1,1}=1$	1.0000	0	1.0000	0	1.0000	0	1.0000	0
$\beta_{1,2}=-1$	−0.9933	0.1229	−0.9999	0.0013	−1.0001	0.0009	−1.0000	0.0006
$\alpha_{R1,1}=1$	0.9524	0.1208	0.9791	0.0826	0.9887	0.0560	0.9947	0.0390
$\alpha_{R1,2}=0.2$	0.2183	0.1219	0.2066	0.0808	0.2066	0.0542	0.2040	0.0396
$\alpha_{I1,1}=-1$	−0.9491	0.1528	−0.9832	0.1074	−0.9900	0.0737	−0.9947	0.0501
$\alpha_{I1,2}=-1$	−1.0096	0.1477	−1.0050	0.1103	−1.0031	0.0715	−1.0024	0.0509
$\alpha_{R2,1}=1$	1.0033	0.0809	1.0037	0.0691	1.0016	0.0532	1.0020	0.0416
$\alpha_{R2,2}=0.5$	0.5098	0.0996	0.5042	0.0831	0.5056	0.0637	0.5021	0.0487
$\alpha_{I2,1}=-0.4$	−0.4068	0.0730	−0.4035	0.0590	−0.4001	0.0497	−0.3976	0.0399
$\alpha_{I2,2}=-0.9$	−0.8960	0.0852	−0.9005	0.0760	−0.8971	0.0625	−0.8988	0.0493
$\beta_{R,1}=1$	1.0000	0	1.0000	0	1.0000	0	1.0000	0
$\beta_{R,2}=-1$	−0.9997	0.0061	−1.0002	0.0045	−1.0001	0.0028	−0.9999	0.0017
$\beta_{I,1}=1$	1.0000	0	1.0000	0	1.0000	0	1.0000	0
$\beta_{I,2}=-0.8$	−0.8002	0.0062	−0.8001	0.0042	−0.8000	0.0029	−0.7999	0.0016

TABLE 4.3 *Simulation results of the GRRR estimation for Model 2 based on 1000 replications for each sample size T*

	$T_1 = T_2 = 50$		$T_1 = T_2 = 100$		$T_1 = T_2 = 200$		$T_1 = T_2 = 400$	
True value	Mean	STD	Mean	STD	Mean	STD	Mean	STD
$\alpha_{1,1} = -1$	−0.9965	0.0487	−0.9983	0.0411	−0.9996	0.0342	−1.0013	0.0318
$\alpha_{1,2} = -0.5$	−0.5047	0.04	−0.5011	0.032	−0.5001	0.0283	−0.5013	0.0256
$\beta_{11,1} = 1$	1.0000	0	1.0000	0	1.0000	0	1.0000	0
$\beta_{11,2} = -1$	−1	0.0014	−1	0.001	−1	0.0007	−1	0.0005
$\beta_{12,1} = 1$	1.0000	0	1.0000	0	1.0000	0	1.0000	0
$\beta_{12,2} = -0.8$	−0.8	0.0021	−0.8	0.0015	−0.8	0.001	−0.8	0.0007
$\alpha_{R,1} = 1$	0.9979	0.0575	0.9978	0.0393	0.9983	0.0275	0.9999	0.0205
$\alpha_{R,2} = 0.4$	0.4085	0.0511	0.4053	0.0357	0.4015	0.0254	0.4013	0.0183
$\alpha_{I,1} = -1$	−0.9774	0.0938	−0.9869	0.0636	−0.9942	0.0446	−0.9964	0.0325
$\alpha_{I,2} = -1$	−1.0173	0.0862	−1.007	0.0617	−1.0035	0.0429	−1.0011	0.0288
$\beta_{R1,1} = 1$	1	0	1	0	1	0	1	0
$\beta_{R1,2} = -1$	−1.001	0.0103	−1.0001	0.0062	−1.0001	0.0041	−1.0001	0.0023
$\beta_{I1,1} = 1$	1	0	1	0	1	0	1	0
$\beta_{I1,2} = -0.8$	−0.8003	0.0112	−0.7999	0.0073	−0.8	0.0041	−0.8	0.0024
$\beta_{R2,1} = 1$	1	0	1	0	1	0	1	0
$\beta_{R2,2} = -0.9$	−0.9011	0.0114	−0.9006	0.0068	−0.9003	0.004	−0.9002	0.0023
$\beta_{I2,1} = 1$	1	0	1	0	1	0	1	0
$\beta_{I2,2} = -0.7$	−0.7003	0.0104	−0.7006	0.007	−0.7004	0.0038	−0.7003	0.0021

when the cointegrating vectors are not changed:

$$H_j = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}', \quad \varphi_j = \beta_{j,1,2},$$
$$h_j = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}',$$

where $j = 1, R, I$. When the cointegrating vectors are changed, we use

$$H_j = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}', \quad \varphi_j = \begin{pmatrix} \beta_{j,1,2} \\ \beta_{j,2,2} \end{pmatrix},$$
$$h_j = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}'.$$

It is observed that the estimators approach to the true values and the standard deviations decrease as the sample size is increased. We can observe that the standard deviations of the nonseasonally and seasonally cointegrated vectors decrese much faster than those of the nonseasonal and seasonal adjustment vectors

TABLE 4.4 *Means and quantiles of the test statistic of Theorem 3.3, under the null hypothesis at zero frequency and at 1/4 frequency*

Frequency 0		Mean and Quantile						
		Mean	0.01	0.05	0.25	0.75	0.95	0.99
True		1	0.000	0.004	0.102	1.323	3.841	6.635
Sample Size	$T_1=T_2=50$	1.2732	0.0002	0.0047	0.1291	1.7055	4.7482	8.6111
	$T_1=T_2=100$	1.1081	0.0002	0.0048	0.1119	1.4526	4.2503	7.1033
	$T_1=T_2=200$	1.0441	0.0001	0.0037	0.1068	1.3863	3.9771	6.9925
	$T_1=T_2=400$	1.0064	0.0001	0.0036	0.1035	1.3213	3.8560	6.5048
Frequency 1/4		Mean and Quantile						
		Mean	0.01	0.05	0.25	0.75	0.95	0.99
True		2	0.020	0.103	0.575	2.773	5.991	9.210
Sample Size	$T_1=T_2=50$	2.4951	0.0249	0.1239	0.7186	3.4353	7.5114	11.9242
	$T_1=T_2=100$	2.2342	0.0196	0.1148	0.6575	3.1149	6.6917	10.1802
	$T_1=T_2=200$	2.1032	0.0216	0.1132	0.6241	2.9003	6.2436	9.6961
	$T_1=T_2=400$	2.0546	0.0191	0.1061	0.6077	2.8504	6.1009	9.3420

as we increase the sample size. This reflects the fact that the distribution of the (seasonal) cointegrating vector has a T-consistency and the adjustment vector has a $\sqrt{T}$-consistency.

In order to observe the effect of the sample size on the asymptotic result of Theorem 3.3 we use Model 1 which test the m structural changes against the $m+k$ structural changes and the linear parameter restrictions in the presence of the structural changes.

We can rewrite Model 1 using (2.7) as follows:

$$U_{0t}^* = \left[\begin{pmatrix} -1 \\ -0.5 \end{pmatrix} \Big|_{t\le T_1} + \begin{pmatrix} -1 \\ -0.7 \end{pmatrix} \Big|_{t>T_1} \right] \begin{pmatrix} 1 & -1 \end{pmatrix} U_{1t}^* + U_{\epsilon t}^*, \quad t = 1,\ldots,T,$$

where

$$U_{\epsilon t}^*\big|_{t\le T_1} \sim N\begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix} \quad \text{and} \quad U_{\epsilon t}^*\big|_{t>T_1} \sim N\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}.$$

Hence, under the assumption of $\alpha_{11} \neq \alpha_{12}$, our hypothesis becomes

$$H_0 : \beta_{11} = \beta_{12} \quad vs. \quad H_1 : \beta_{11} \neq \beta_{12}$$

and the test statistic is

$$-2\log L_{\max,1}/L_{\max,0} \xrightarrow{d} \chi^2_{(1)}.$$

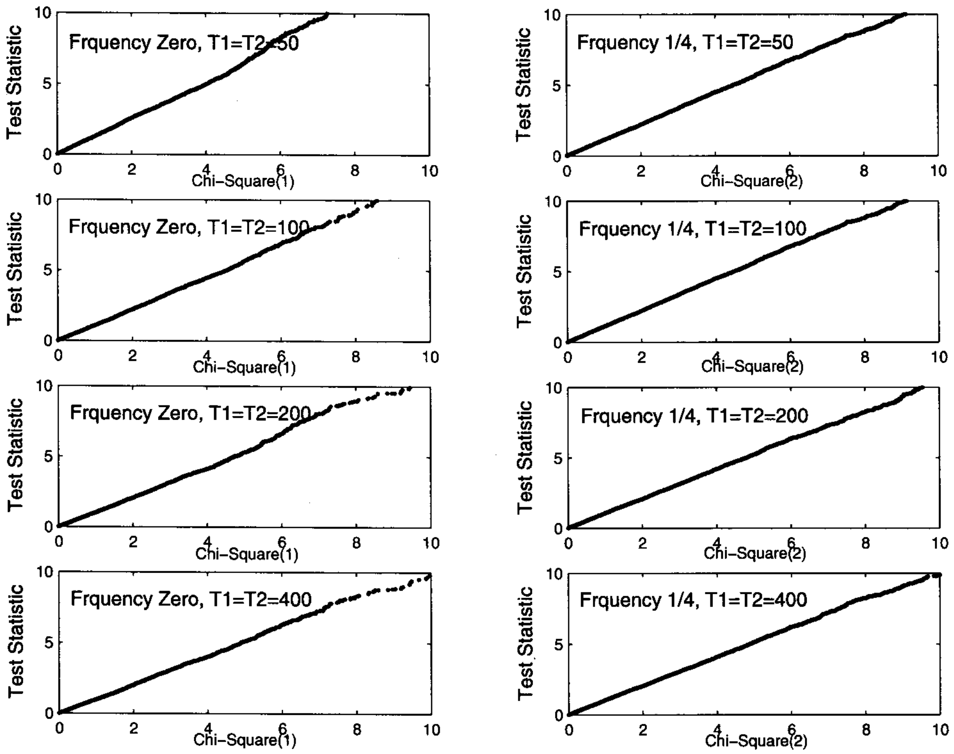

FIGURE 4.1 *Q-Q plot of the test statistic of Theorem 3.3 under the null hypothesis at zero frequency and at* 1/4 *frequency.*

Using (2.6), Model 1 can be rewritten as follows:

$$\begin{aligned} U_{0t} &= 2\Bigg[\left(\begin{pmatrix}1\\0.2\end{pmatrix}\Big|_{t\le T_1} + \begin{pmatrix}1\\0.5\end{pmatrix}\Big|_{t>T_1}\right)\begin{pmatrix}1 & -1\end{pmatrix} \\ &\qquad + \left(\begin{pmatrix}-1\\-1\end{pmatrix}\Big|_{t\le T_1} + \begin{pmatrix}-0.4\\-0.9\end{pmatrix}\Big|_{t>T_1}\right)\begin{pmatrix}1 & -0.8\end{pmatrix}\Bigg]U_{Rt} \\ &\quad +2\Bigg[\left(\begin{pmatrix}1\\0.2\end{pmatrix}\Big|_{t\le T_1} + \begin{pmatrix}1\\0.5\end{pmatrix}\Big|_{t>T_1}\right)\begin{pmatrix}1 & -0.8\end{pmatrix} \\ &\qquad - \left(\begin{pmatrix}-1\\-1\end{pmatrix}\Big|_{t\le T_1} + \begin{pmatrix}-0.4\\-0.9\end{pmatrix}\Big|_{t>T_1}\right)\begin{pmatrix}1 & -1\end{pmatrix}\Bigg]U_{It} + U_{\epsilon t} \end{aligned}$$

with $t = 1, \ldots, T$, where

$$U^*_{\epsilon t}\Big|_{t\le T_1} \sim N\begin{pmatrix}1 & 0.3\\0.3 & 1\end{pmatrix} \quad\text{and}\quad U^*_{\epsilon t}\Big|_{t>T_1} \sim N\begin{pmatrix}1 & 0.5\\0.5 & 1.5\end{pmatrix}.$$

Under the assumptions that $\alpha_{R1} \neq \alpha_{R2}$ and $\alpha_{I1} \neq \alpha_{I2}$, the hypothesis becomes

$$H_0 : \beta_{R1} = \beta_{R1} \text{ and } \beta_{I1} = \beta_{I2} \quad vs. \quad H_1 : \beta_{R1} \neq \beta_{R1} \text{ and } \beta_{I1} \neq \beta_{I2}$$

and the test statistic is

$$-2 \log L_{\max,1}/L_{\max,0} \xrightarrow{d} \chi^2_{(2)}. \tag{4.1}$$

From Table 4.4 and Figure 4.1, we can observe that the finite sample critical values are always larger than their asymptotic ones, and that the difference in the values between the asymptotic and finite sample distributions are significant when the sample size T is small.

5. Concluding Remarks

In this paper, we extend the method of Hansen (2003) to the seasonal model and propose an estimation procedure for the seasonal cointegrated vector autoregressive model permitting structural changes when change points are known. This estimation procedure enables us to impose restrictions on the seasonal cointegrating vectors and the adjustment vectors. The restrictions also enable us to estimate the adjustment vector α and the cointegrating vector β in the cointegration system. We also show that the asymptotic distributions of the likelihood ratio test for m structural changes against $m+k$ structural changes and that of the linear parameter restrictions in the presence of structural changes are χ^2 in both cases.

Acknowledgements

The authors thank two anonymous referees for their helpful comments that lead to improve this paper.

Appendix

LEMMA A.1. *From the definitions below:*

$$U_B = 2 \begin{pmatrix} (U_R' B_R \otimes I_p)' + (U_I' B_I \otimes I_p)' \\ (U_R' B_I \otimes I_p)' + (U_I' B_R \otimes I_p)' \end{pmatrix}',$$

$$U_A = 2 \begin{pmatrix} ([(U_R' \otimes A_R) - (U_I' \otimes A_I)]K_{p_1,r})' \\ ([(U_R' \otimes A_I) - (U_I' \otimes A_R)]K_{p_1,r})' \end{pmatrix}',$$

we have the identities (A.1), (A.2), (A.3) and (A.4) as follow.

$$\text{(i)}\ U_A'\Sigma^{-1}U_A = \begin{pmatrix} U_{AA(11)} & U_{AA(12)} \\ U_{AA(21)} & U_{AA(22)} \end{pmatrix}, \tag{A.1}$$

where

$$\begin{aligned}
U_{AA(11)} &= 4\sum_{t=1}^{T}[(A_R'\Omega(t)^{-1}A_R' \otimes U_R U_R') - (A_R'\Omega(t)^{-1}A_I' \otimes U_R U_I') \\
&\quad - (A_I'\Omega(t)^{-1}A_R' \otimes U_I U_R') + (A_I'\Omega(t)^{-1}A_I' \otimes U_I U_I')], \\
U_{AA(12)} &= 4\sum_{t=1}^{T}[(A_R'\Omega(t)^{-1}A_I' \otimes U_R U_R') + (A_R'\Omega(t)^{-1}A_R' \otimes U_R U_I') \\
&\quad - (A_I'\Omega(t)^{-1}A_R' \otimes U_I U_R') + (A_I'\Omega(t)^{-1}A_R' \otimes U_I U_I')], \\
U_{AA(21)} &= 4\sum_{t=1}^{T}[(A_I'\Omega(t)^{-1}A_R' \otimes U_R U_R') - (A_I'\Omega(t)^{-1}A_I' \otimes U_R U_I') \\
&\quad - (A_R'\Omega(t)^{-1}A_R' \otimes U_I U_R') + (A_R'\Omega(t)^{-1}A_I' \otimes U_I U_I')], \\
U_{AA(22)} &= 4\sum_{t=1}^{T}[(A_I'\Omega(t)^{-1}A_I' \otimes U_R U_R') + (A_I'\Omega(t)^{-1}A_R' \otimes U_R U_I') \\
&\quad + (A_R'\Omega(t)^{-1}A_I' \otimes U_I U_R') + (A_R'\Omega(t)^{-1}A_R' \otimes U_I U_I')].
\end{aligned}$$

$$\text{(ii)}\ U_B'\Sigma^{-1}U_B = \begin{pmatrix} U_{BB(11)} & U_{BB(12)} \\ U_{BB(21)} & U_{BB(22)} \end{pmatrix}, \tag{A.2}$$

where

$$\begin{aligned}
U_{BB(11)} &= 4\sum_{t=1}^{T}[(B_R'U_{Rt}U_{Rt}'B_R \otimes \Omega(t)^{-1}) + (B_R'U_{Rt}U_{It}'B_I \otimes \Omega(t)^{-1}) \\
&\quad + (B_I'U_{It}U_{Rt}'B_R \otimes \Omega(t)^{-1}) + (B_I'U_{It}U_{It}'B_I \otimes \Omega(t)^{-1})], \\
U_{BB(12)} &= 4\sum_{t=1}^{T}[(B_R'U_{Rt}U_{Rt}'B_I \otimes \Omega(t)^{-1}) - (B_R'U_{Rt}U_{It}'B_R \otimes \Omega(t)^{-1}) \\
&\quad + (B_I'U_{It}U_{Rt}'B_I \otimes \Omega(t)^{-1}) - (B_I'U_{It}U_{It}'B_R \otimes \Omega(t)^{-1})], \\
U_{BB(21)} &= 4\sum_{t=1}^{T}[(B_I'U_{Rt}U_{Rt}'B_R \otimes \Omega(t)^{-1}) + (B_I'U_{Rt}U_{It}'B_I \otimes \Omega(t)^{-1}) \\
&\quad - (B_R'U_{It}U_{Rt}'B_R \otimes \Omega(t)^{-1}) + (B_R'U_{It}U_{It}'B_I \otimes \Omega(t)^{-1})],
\end{aligned}$$

$$U_{BB(22)} = 4\sum_{t=1}^{T}[(B_I'U_{Rt}U_{Rt}'B_I \otimes \Omega(t)^{-1}) + (B_I'U_{Rt}U_{It}'B_R \otimes \Omega(t)^{-1})$$
$$- (B_R'U_{It}U_{Rt}'B_I \otimes \Omega(t)^{-1}) + (B_R'U_{It}U_{It}'B_R \otimes \Omega(t)^{-1})].$$

(iii) $U_A'\Sigma^{-1}vec(U_0)$ *(A.3)*

$$= 2\begin{pmatrix} \sum_{t=1}^{T}[vec(U_{Rt}U0t'\Omega(t)^{-1}A_R) - vec(U_{It}U_{0t}'\Omega(t)^{-1}A_I] \\ \sum_{t=1}^{T}[vec(U_{Rt}U0t'\Omega(t)^{-1}A_I) + vec(U_{It}U_{0t}'\Omega(t)^{-1}A_R)] \end{pmatrix}.$$

(iv) $U_B'\Sigma^{-1}vec(U_0)$ *(A.4)*

$$= 2\begin{pmatrix} \sum_{t=1}^{T}[vec(\Omega(t)^{-1}U_{0t}U_{Rt}'B_R) + vec(\Omega(t)^{-1}U_{0t}U_{It}'B_I] \\ \sum_{t=1}^{T}[vec(\Omega(t)^{-1}U_{0t}U_{Rt}'B_I) + vec(\Omega(t)^{-1}U_{0t}U_{It}'B_R] \end{pmatrix}.$$

PROOF.

$$U_A'\Sigma^{-1}U_A$$
$$= 2\begin{pmatrix} K_{p_1,r}'[(U_R \otimes A_R') - (U_I \otimes A_I')] \\ K_{p_1,r}'[(U_R \otimes A_I') + (U_I \otimes A_R')] \end{pmatrix} \times \Sigma^{-1}$$
$$\times 2\Big(K_{p_1,r}[(U_R \otimes A_R') - (U_I \otimes A_I')] \ K_{p_1,r}[(U_R \otimes A_I') + (U_I \otimes A_R')] \Big)$$
$$= 4\begin{pmatrix} K_{p_1,r}'U_{AR}'\Sigma^{-1}U_{AR}K_{p_1,r} & K_{p_1,r}'U_{AR}'\Sigma^{-1}U_{AI}K_{p_1,r} \\ K_{p_1,r}'U_{AI}'\Sigma^{-1}U_{AR}K_{p_1,r} & K_{p_1,r}'U_{AI}'\Sigma^{-1}U_{AI}K_{p_1,r} \end{pmatrix}$$
$$= \begin{pmatrix} U_{AA(11)} & U_{AA(12)} \\ U_{AA(21)} & U_{AA(22)} \end{pmatrix},$$

where

$$U_{AR} = (U_R' \otimes A_R) - (U_I' \otimes A_R), \qquad U_{AI} = (U_R' \otimes A_I) - (U_I' \otimes A_R),$$
$$U_{AA(11)} = K_{p_1,r}'U_{AR}'\Sigma^{-1}U_{AR}K_{p_1,r}, \quad U_{AA(12)} = K_{p_1,r}'U_{AR}'\Sigma^{-1}U_{AI}K_{p_1,r},$$
$$U_{AA(21)} = K_{p_1,r}'U_{AI}'\Sigma^{-1}U_{AR}K_{p_1,r}, \quad U_{AA(22)} = K_{p_1,r}'U_{AI}'\Sigma^{-1}U_{AI}K_{p_1,r}$$

and

$$K_{p_1,r}'U_{AI}'\Sigma^{-1}U_{AR}K_{p_1,r}$$
$$= 4 \times K_{p_1,r}'[U_R \otimes A_R'\Sigma^{-1}U_R \otimes A_R - U_R \otimes A_R'\Sigma^{-1}U_I \otimes A_R$$
$$- U_I \otimes A_I'\Sigma^{-1}U_I \otimes A_I - U_I \otimes A_I'\Sigma^{-1}U_I \otimes A_I]K_{p_1,r}$$
$$= 4 \times \sum_{t=1}^{T}[(A_R'\Omega(t)^{-1}A_R \otimes U_RU_R') - (A_R'\Omega(t)^{-1}A_I \otimes U_RU_I')$$
$$- (A_I'\Omega(t)^{-1}A_R \otimes U_IU_R') + (A_I'\Omega(t)^{-1}A_I \otimes U_IU_I')],$$

$K'_{p_1,r}U'_{AR}\Sigma^{-1}U_{AI}K_{p_1,r}$, $K'_{p_1,r}U'_{AI}\Sigma^{-1}U_{AR}K_{p_1,r}$ and $K'_{p_1,r}U'_{AI}\Sigma^{-1}U_{AI}K_{p_1,r}$ are calculated similarly. Hence, (A.1) is proved. (A.2), (A.3) and (A.4) can be proved similarly. □

Proof of Theorem 3.1

Applying the vector operation to (3.3) we obtain

$$\begin{aligned}
\mathrm{vec}(U_0) &= 2(U'_R B_R \otimes I_P)\mathrm{vec}(A_R) + 2(U'_R B_I \otimes I_P)\mathrm{vec}(A_I) \\
&\quad +2(U'_I B_I \otimes I_P)\mathrm{vec}(A_R) + 2(U'_I B_R \otimes I_P)\mathrm{vec}(A_I) + u \\
&= 2((U'_R B_R \otimes I_P) + (U'_R B_I \otimes I_P))\mathrm{vec}(A_R) \\
&\quad +2((U'_R B_I \otimes I_P) + (U'_I B_R \otimes I_P))\mathrm{vec}(A_I) + u \\
&= U_B \times \begin{pmatrix} \mathrm{vec}(A_R) \\ \mathrm{vec}(A_I) \end{pmatrix} + u \\
&= U_B \times G\psi + u,
\end{aligned}$$

where

$$G = \begin{pmatrix} G_R & 0 \\ 0 & G_I \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_R \\ \psi_I \end{pmatrix},$$

$\mathrm{vec}(A_R) = G_R\psi_R$, $\mathrm{vec}(A_I) = G_I\psi_I$ and $u = \mathrm{vec}(U_E)$.

For fixed values of B_R, B_I, and Σ, this becomes a restricted GLS problem; so therefore,

$$\mathrm{vec}(\hat{A}) = G[G'U_B\Sigma^{-1}U_BG]^{-1}G'U'_B\Sigma\mathrm{vec}(U_0).$$

Similarly, for fixed A_R, A_I, and Σ, we have

$$\begin{aligned}
\mathrm{vec}(U_0) &= 2(U'_R \otimes A_R)\mathrm{vec}(B'_R) + 2(U'_R \otimes A_I)\mathrm{vec}(B'_I) \\
&\quad +2(U'_I B_I \otimes I_P)\mathrm{vec}(B'_I) - 2(U'_I \otimes A_R)\mathrm{vec}(B'_R) + u \\
&= 2((U'_R \otimes A_R) - (U'_I \otimes A_I))\mathrm{vec}(B'_I) \\
&\quad +2((U'_R \otimes A_I) + (U'_I \otimes A_R)\mathrm{vec}(B'_I)) + u \\
&= U_A \times \begin{pmatrix} \mathrm{vec}(B_R) \\ \mathrm{vec}(B_I) \end{pmatrix} + u \\
&= U_A \times (H\phi + h) + u,
\end{aligned}$$

where

$$H = \begin{pmatrix} H_R & 0 \\ 0 & H_I \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}, \quad h = \begin{pmatrix} h_R \\ h_I \end{pmatrix},$$

$$\text{vec}(B_R) = H_R\phi_R + h_R, \ \ \text{vec}(B_I) = H_I\phi_I + h_I, \text{ and } u = \text{vec}(U_E),$$

which is also restricted GLS problem; so therefore,

$$\text{vec}(\hat{B}) = H[H'U_A\Sigma^{-1}U_AH]^{-1}H'U_A'\Sigma\text{vec}(U_0) + h.$$

Proof of Theorem 3.2

Within each of subsamples, the estimators are consistent as shown by Johansen and Schaumburg (1999). The estimators from m subsamples can be combined into the estimator $\hat{A}$, $\hat{B}$ and $\hat{\Omega}(t)$ that are consistent for the population parameter A, B and $\Omega(t)$ as Theorem 8 in Hansen (2003).

Proof of Theorem 3.3

The maximum value of the log-likelihood function is given by

$$\log(L_{\max})(\hat{A}_R, \hat{A}_I, \hat{B}_R, \hat{B}_I, \hat{\Omega}_{j=1,\dots,m}) \propto -\frac{T}{2}\left(\sum_{j=1}^{m}\rho_j \log|\hat{\Omega}_j|\right),$$

where $\hat{\Omega}_j = (T_j - T_{j-1})^{-1}\sum_{t=T_{j-1}+1}^{T_j}\hat{U}_{\epsilon t}\hat{U}_{\epsilon t}'$ and $\rho_j = (T_j - T_{j-1})/T$. From Theorem 10 of Hansen (2003), we can obtain

$$-\frac{T}{2}\left(\sum_{j=1}^{m}\log|\hat{\Omega}|\right) \propto \hat{U}_E'\Sigma^{-1}\hat{U}_E$$

and we can express $\hat{U}_{\epsilon t}$ as

$$\begin{aligned}
\hat{U}_{\epsilon t} &= U_{\epsilon t} + 2(A_RB_R' - A_IB_I' - \hat{A}_R\hat{B}_R - \hat{A}_I\hat{B}_I)U_R \\
&\qquad + 2(A_RB_I' - A_IB_R' - \hat{A}_R\hat{B}_I' + \hat{A}_I\hat{B}_R)U_I \\
&= U_{\epsilon t} + 2(A_R - \hat{A}_R)B_RU_R + 2A_R(B_R - \hat{B}_R)'U_R \\
&\qquad + 2(\hat{A}_R - A_R)(B_R - \hat{B}_R)'U_R \\
&\quad + 2(A_I - \hat{A}_I)B_IU_R + 2A_I(B_I - \hat{B}_I)'U_R \\
&\qquad + 2(\hat{A}_I - A_I)(B_I - \hat{B}_I)'U_R \\
&\quad + 2(A_R - \hat{A}_R)B_IU_R + 2A_R(B_I - \hat{B}_I)'U_I \\
&\qquad + 2(\hat{A}_R - A_R)(B_I - \hat{B}_I)'U_R \\
&\quad + 2(A_I - \hat{A}_I)B_RU_R + 2A_I(B_R - \hat{B}_R)'U_I \\
&\qquad + 2(\hat{A}_I - A_I)(B_R - \hat{B}_R)'U_I.
\end{aligned}$$

Since the terms $2(\hat{A}_R - A_R)(B_R - \hat{B}_R)'U_R$, $2(\hat{A}_I - A_I)(B_I - \hat{B}_I)'U_R$, $2(\hat{A}_R - A_R)(B_I - \hat{B}_I)'U_I$ and $2(\hat{A}_I - A_I)(B_R - \hat{B}_R)'U_I$ are all $O_p(T^{-1/2})O_p(T^{-1})O_p(T) = o_p(1)$, we obtain

$$\begin{aligned}\hat{U}_{\epsilon t} &= U_{\epsilon t} + 2[(A_R - \hat{A}_R)B_R + (A_I - \hat{A}_I)B_R]U_R \\ &\quad +2[(A_R - \hat{A}_R)B_I + (A_I - \hat{A}_I)B_R]U_I \\ &\quad +2[A_R(B_R - \hat{B}_R) + A_I(B_I - \hat{B}_I)]U_R \\ &\quad +2[A_R(B_I - \hat{B}_I) + A_I(B_R - \hat{B}_R)]U_I + o_p(1)\end{aligned}$$

and after some algebraic operations we obtain

$$\begin{aligned}\hat{U}_E &= U_E + U_B \begin{pmatrix} \text{vec}(A_R - \hat{A}_R) \\ \text{vec}(A_I - \hat{A}_I) \end{pmatrix} + \begin{pmatrix} \text{vec}(B_R - \hat{B}_R) \\ \text{vec}(B_I - \hat{B}_I) \end{pmatrix} U_A + o_p(1) \\ &= U_E + U_B \text{vec}(A - \hat{A}) + \text{vec}(B - \hat{B})U_A + o_p(1).\end{aligned}$$

Since

$$\begin{aligned}\text{vec}(A - \hat{A}) &= G[G'U_B'\Sigma^{-1}U_BG]^{-1}G'U_B'\Sigma^{-1}U_E \\ \text{vec}(B - \hat{B}) &= H[H'U_A'\Sigma U_AH]^{-1}H'U_A'\Sigma^{-1}U_E + o_p(T^{-1}),\end{aligned}$$

we obtain

$$\hat{U}_E\Sigma^{-1}\hat{U}_E = \eta'(I_T - P_A - P_B)\eta + o_p(1),$$

where $\eta = \Sigma^{-1/2}U_E \sim N(0, I_T)$, $P_A = \Sigma^{-1/2}U_BG[G'U_B'\Sigma^{-1}U_BG]^{-1}G'U_B'\Sigma^{-1/2}$ and $P_B = \Sigma^{-1/2}U_AH[H'U_A'\Sigma^{-1}U_AH]^{-1}H'U_A'\Sigma^{-1/2}$.
Then, as in Theorem 10 of Hansen (2003), we can obtain

$$-2\log L_{\max,1}/L_{\max,0} = \eta'Q\eta + o_p(1) \quad \xrightarrow{d} \quad \chi_{(q)}^2$$

and the detailed proof is the same as those of Theorem 10 in Hansen (2003).

REFERENCES

BOX, G. E. P. AND TIAO, G. C. (1975). "Intervention analysis with applications to economic and environmental problems", *Journal of the American Statistical Association*, **70**, 70–79.

HANSEN, P. R. (2000). "Granger representation theorem: a closed-form expression for I(1) processes", *UCSD Department of Economics*, discussion paper.

HANSEN, P. R. (2003). "Structural changes in the cointegrated vector autoregressive model", *Journal of Econometrics*, **114**, 261–295.

HYLLEBERG, S., ENGLE, R. F., GRANGER, C. W. J. AND YOO, B. S. (1990). "Seasonal integration and cointegration", *Journal of Econometrics*, **44**, 215–238.

INOUE, A. (1999). "Tests of cointegrating rank with a trend-break", *Journal of Econometrics*, **90**, 215–237.

JOHANSEN, S. (1988). "Statistical analysis of cointegration vectors", *Journal of Economic Dynamics & Control*, **12**, 231–254.

JOHANSEN, S. (1996). *Likelihood Based Inference in Cointegrated Vector Autoregressive Models,* 2nd ed., Oxford University Press, Oxford.

JOHANSEN, S. AND SCHAUMBURG, E. (1999). "Likelihood analysis of seasonal cointegration", *Journal of Econometrics*, **88**, 301–339.

KIM, S., CHO, S. AND LEE, S. (2000). "On the cusum test for parameter changes in GARCH(1, 1) models", *Communications in Statistics-Theory and Methods*, **29**, 445–462.

PICARD, D. (1985). "Testing and estimating change-points in time series", *Advances in Applied Probability*, **17**, 841–867.

SEO, B. (1998). "Tests for structural changes in cointegrated systems", *Econometric Theory*, **14**, 222–259.

WANG, L. AND WANG, J. (2006). "Testing and estimating for change in long memory parameter", *Journal of Statistical Computation and Simulation*, **76**, 317–329.

WICHERN, D. W., MILLER, R. B. AND HSU, D. A. (1976). "Changes of variance in first-order autoregressive time series models–With an application", *Applied Statistics*, **25**, 248–256.