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BAYESIAN INFERENCE FOR FIELLER-CREASY PROBLEM USING UNBALANCED DATA†

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ABSTRACT

In this paper, we consider Bayesian approach to the Fieller-Creasy problem using noninformative priors. Specifically we extend the results of Yin and Ghosh (2000) to the unbalanced case. We develop some noninformative priors such as the first and second order matching priors and reference priors. Also we prove the posterior propriety under the derived noninformative priors. We compare these priors in light of how accurately the coverage probabilities of Bayesian credible intervals match the corresponding frequentist coverage probabilities.

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1. INTRODUCTION

The Fieller-Creasy problem involves statistical inference about the ratio of two independent normal means. It is a challenging problem from either a frequentist or a likelihood perspective. As an alternative, we consider Bayesian analysis with noninformative priors for this problem.

Bayesian analysis for the original Fieller-Creasy problem based on noninformative priors began with Kappenman *et al.* (1970), and was addressed subsequently in Bernardo (1977), Stephens and Smith (1992), Liseo (1993), Philipe and Robert (1998), Reid (1996) and Berger *et al.* (1999). All these papers considered either Jeffreys' prior or reference priors. A Bayesian analysis based on proper priors for this problem was given in Carlin and Louis (2000).

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Recently, Yin and Ghosh (2000) developed the noninformative priors for Bayesian and likelihood-based inferences in the more generalized Fieller-Creasy setting of two location-scale models. But they considered only the balanced case. In reality there might be a necessity of the noninformative priors for the objective Bayesian analysis using unbalanced data.

The present paper focuses on developing noninformative priors for the Fieller-Creasy problem in the unbalanced case. We consider Bayesian priors such that the resulting credible intervals for the ratio of two normal means have coverage probabilities equivalent to their frequentist counterparts. Although this matching can be justified only asymptotically, our simulation results indicate that this is indeed achieved for small or moderate sample sizes as well.

This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Ghosh and Mukerjee (1992), Mukerjee and Dey (1993), Datta and Ghosh (1995a), Datta and Ghosh (1995b, 1996), Datta (1996), Mukerjee and Ghosh (1997) and Kim *et al.* (2005, 2006).

On the other hand, Ghosh and Mukerjee (1992) and Berger and Bernardo (1989, 1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

The outline of the remaining sections is as follows. In Section 2, we derive first order and second order probability matching priors for the ratio of two normal means. Also we derive reference priors for different groups of ordering for the parameters. It turns out that among the reference priors, only two group reference prior satisfies a second order probability matching criterion. In Section 3, we provide the propriety of the posterior distribution for a general class of prior distributions which include all reference priors. In Section 4, simulated frequentist coverage probabilities under the proposed priors are investigated.

2. NONINFORMATIVE PRIORS

Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ be two independent random samples from $N(\mu, \sigma^2)$ and $N(\theta\mu, \sigma^2)$, respectively. Here the parameter of interest is θ, the ratio of means.

In order to find probability matching priors, it is convenient to introduce an orthogonal parametrization (Cox and Reid, 1987; Tibshirani, 1989). To this end, let

$$\theta_1 = \theta, \quad \theta_2 = \mu(n + m\theta^2)^{1/2} \quad \text{and} \quad \theta_3 = \sigma^2.$$

With this parametrization, the likelihood function is given by

$$L(\theta_1, \theta_2, \theta_3) \propto \theta_3^{-N/2} \times \exp\left\{-\frac{1}{2\theta_3}\left[\sum_{i=1}^{n}\{x_i - \theta_2(n + m\theta_1^2)^{-1/2}\}^2 + \sum_{j=1}^{m}\{y_j - \theta_1\theta_2(n + m\theta_1^2)^{-1/2}\}^2\right]\right\}, \tag{2.1}$$

where $N = n + m$. From the above likelihood function (2.1), the Fisher information matrix is given by

$$\mathbf{I} = \begin{pmatrix} \theta_3^{-1}\theta_2^2 nm(n + m\theta_1^2)^{-2} & 0 & 0 \\ 0 & \theta_3^{-1} & 0 \\ 0 & 0 & \dfrac{N}{2\theta_3^2} \end{pmatrix}.$$

Following Tibshirani (1989), the class of a first order probability matching prior is given by

$$\pi_M^{(1)}(\theta_1, \theta_2, \theta_3) \propto |\theta_2|\theta_3^{-1/2}(n + m\theta_1^2)^{-1}g(\theta_2, \theta_3), \tag{2.2}$$

where $g(\cdot, \cdot)$ is an arbitrary positive and differentiable function in its arguments.

Since the class of the first order probability matching prior is quite large, one needs to narrow down this class. Specially, Murkerjee and Ghosh (1997) developed a second order probability matching prior. Among the first order matching prior, the second order matching prior satisfies the following differential equation.

$$\frac{1}{6}g(\theta_2, \theta_3)\frac{\partial}{\partial\theta_1}\left\{I_{11}^{-3/2}L_{111}\right\} + \sum_{v=2}^{3}\sum_{s=2}^{3}\left\{I_{11}^{-1/2}L_{11s}I^{sv}g(\theta_2, \theta_3)\right\} = 0, \tag{2.3}$$

where I_{ij} is the $(i, j)^{th}$ element of Fisher information matrix, I^{sv} is the $(i, j)^{th}$ element of inverse of Fisher information matrix,

$$L_{111} = E\left[\left(\frac{\partial \log L(\theta_1, \theta_2, \theta_3)}{\partial\theta_1}\right)^3\right] \text{ and } L_{ijk} = E\left[\frac{\partial^3 \log L(\theta_1, \theta_2, \theta_3)}{\partial\theta_i\partial\theta_j\partial\theta_k}\right].$$

After some algebraic calculations, one can get

$$I^{22} = \theta_3, \quad I^{23} = I^{32} = 0, \quad I^{33} = \frac{2\theta_3^2}{n + m},$$

$L_{111} = 0, L_{112} = -nm\theta_2\theta_3^{-1}(n + m\theta_1^2)^{-2}$ and $L_{113} = nm\theta_2^2\theta_3^{-2}(n + m\theta_1^2)^{-2}$.

Then the differential equation (2.3) reduces to

$$-\theta_3^{1/2}\frac{\partial}{\partial\theta_2}g(\theta_2, \theta_3) + \frac{2\theta_2}{n+m}\frac{\partial}{\partial\theta_3}\theta_3^{1/2}g(\theta_2, \theta_3) = 0.$$

A solution of the above equation is

$$g(\theta_2, \theta_3) = \theta_3^{-1/2}h\left(\frac{\theta_2^2}{n+m} + \theta_3\right),$$

where $h(\cdot)$ is an arbitrary positive differentiable function in its arguments. So, if one takes $h(\cdot) = 1$, then the second order probability matching prior is given by

$$\pi_M^{(2)}(\theta_1, \theta_2, \theta_3) = |\theta_2|\theta_3^{-1}(n + m\theta_1^2)^{-1}. \tag{2.4}$$

REMARK 2.1. The second order matching prior given in (2.4) is not an alternative coverage probability matching prior introduced by Mukerjee and Reid (1999). The alternative coverage probability matching priors is the prior such that the probability for a confidence set to include an alternative value of the interesting parameter matches true coverage asymptotically. Mukerjee and Reid (1999) gave the simple differential equations that a second order probability matching prior matches alternative coverage probabilities up to the second order. But in our case we can easily show that some conditions are not satisfied.

REMARK 2.2. Datta (1996) showed that if $I_{11}^{-3/2}L_{111}$ does not depend on θ_1, then the second order matching prior is highest posterior distribution (HPD) matching prior within the first order matching priors. But the second order matching prior given in (2.4) is not a HPD matching prior.

Following Datta and Ghosh (1995b), the reference prior introduced by Berger and Bernardo (1989) can be obtained easily from the information matrix, if parameters orthogonality is satisfied. From the information matrix, the reference priors by the order of inferential importance are given as follows:

THE ORDER OF IMPORTANCE	REFERENCE PRIOR		
$(\{\theta_1\}, \{\theta_2\}, \{\theta_3\})$	$\pi_R^1(\theta_1, \theta_2, \theta_3)$	$\propto$	$(n + m\theta_1^2)^{-1}\theta_3^{-1}$
$(\{\theta_1, \theta_2\}, \{\theta_3\})$	$\pi_R^2(\theta_1, \theta_2, \theta_3)$	$\propto$	$(n + m\theta_1^2)^{-1}\theta_3^{-1}\|\theta_2\|$
$(\{\theta_1, \theta_2, \theta_3\})$	$\pi_R^3(\theta_1, \theta_2, \theta_3)$	$\propto$	$(n + m\theta_1^2)^{-1}\theta_3^{-2}\|\theta_2\|$
$(\{\theta_1\}, \{\theta_2, \theta_3\}), (\{\theta_1, \theta_3\}, \{\theta_2\}), (\{\theta_2, \theta_3\}, \{\theta_1\})$	$\pi_R^4(\theta_1, \theta_2, \theta_3)$	$\propto$	$(n + m\theta_1^2)^{-1}\theta_3^{-3/2}$

Note that, the prior π_R^1 is called the one-at-a-time reference prior. The two group reference prior π_R^2 is actually the second order matching prior. And π_R^3 is Jeffreys' prior.

3. PROPRIETY OF POSTERIORS

In this section, we will show the propriety of posterior distributions under various noninformative priors given in the previous section. The noninformative priors proposed in the previous section can be represented in a general form as follows:

$$\pi_G(\theta_1, \theta_2, \theta_3) \propto (n + m\theta_1^2)^{-1}|\theta_2|^a\theta_3^{-b}, \tag{3.1}$$

where $a = 0, 1$ and $b = 1/2, 1, 3/2, 2$. Using the above prior, the joint posterior of θ_1, θ_2 and θ_3 is given by

$$\begin{aligned}\pi_G(\theta_1, \theta_2, \theta_3|\underline{x}, \underline{y}) \propto\ & (n + m\theta_1^2)^{-1}|\theta_2|^a\theta_3^{-(N/2+b)} \\ & \times \exp\left\{-\frac{1}{2\theta_3}\left[s_x + s_y + n\{\overline{x} - \theta_2(n + m\theta_1^2)^{-1/2}\}^2\right.\right. \\ & \left.\left. + m\{\overline{y} - \theta_1\theta_2(n + m\theta_1^2)^{-1/2}\}^2\right]\right\}.\end{aligned}$$

Let $\theta = \theta_1$, $\mu = \theta_2(n + m\theta_1^2)^{-1/2}$ and $\tau = \theta_3^{-1}$. Then the above joint posterior changes to

$$\begin{aligned}\pi_G(\theta, \mu, \tau|\underline{x}, \underline{y}) \propto\ & |\mu|^a(n + m\theta^2)^{\frac{a-1}{2}}\tau^{\frac{N}{2}+b-2} \\ & \times \exp\left\{-\frac{\tau}{2}\left[s_x + s_y + n(\overline{x} - \mu)^2 + m(\overline{y} - \theta\mu)^2\right]\right\}.\end{aligned} \tag{3.2}$$

Now, we will consider the propriety of the posteriors given by (3.2).

THEOREM 3.1. *If $N/2 + b - 3/2 > 0$, then the joint posterior distribution of θ, μ and τ is proper.*

PROOF. For the convenance, we consider the proof with respect to the values of a. When $a = 0$, the posterior is given by

$$\begin{aligned}\pi_G(\theta, \mu, \tau|\underline{x}, \underline{y}) \propto\ & (n + m\theta^2)^{-\frac{1}{2}}\tau^{\frac{N}{2}+b-2} \\ & \times \exp\left\{-\frac{\tau}{2}\left[s_x + s_y + n(\overline{x} - \mu)^2 + m(\overline{y} - \theta\mu)^2\right]\right\}.\end{aligned}$$

By integrating with respect to μ, one gets

$$\pi_G(\theta, \tau|\underline{x}, \underline{y}) \propto (n + m\theta^2)^{-1}\tau^{\frac{N}{2}+b-\frac{5}{2}} \exp\left\{-\frac{\tau}{2}\left(s_x + s_y + \frac{nm(\overline{y} - \theta\overline{x})^2}{n + m\theta^2}\right)\right\}.$$

Next by integrating with respect to τ, it follows that if $N/2+b-3/2>0$,

$$\begin{aligned}\pi_G(\theta|\underline{x},\underline{y}) &\propto (n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-\frac{3}{2})} \\ &\propto (n+m\theta^2)^{-1}\left[1+\frac{nm(\overline{y}-\theta\overline{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-\frac{3}{2})} \\ &\leq (n+m\theta^2)^{-1},\end{aligned}$$

since $\left[1+\{nm(\overline{y}-\theta\overline{x})^2\}/\{(n+m\theta^2)(s_x+s_y)\}\right]^{-(\frac{N}{2}+b-\frac{3}{2})}\leq 1$. Finally the integration with respect to θ results in

$$\int_{-\infty}^{\infty}\frac{1}{n+m\theta^2}d\theta=\frac{\sqrt{nm}}{\pi}.$$

Hence the posterior distribution is proper when $a=0$.

When $a=1$, the joint posterior is given by

$$\begin{aligned}\pi_G(\theta,\mu,\tau|\underline{x},\underline{y}) &\propto |\mu|\tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}\left[s_x+s_y+n(\overline{x}-\mu)^2+m(\overline{y}-\theta\mu)^2\right]\right\} \\ &\propto |\mu|\exp\left\{-\frac{\tau(n+m\theta^2)}{2}\left(\mu-\frac{n\overline{x}+\theta m\overline{y}}{n+m\theta^2}\right)^2\right\} \\ &\quad\times\tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}\left(s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right)\right\}.\end{aligned}$$

It is well known that

$$\int_{-\infty}^{\infty}|x|e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\sigma^2\left\{2e^{-\frac{\mu^2}{2\sigma^2}}+\sqrt{\frac{2\pi\mu^2}{\sigma^2}}Erf\left(\sqrt{\frac{\mu^2}{2\sigma^2}}\right)\right\},$$

where $Erf(a)=\int_0^a(1/\sqrt{2\pi})e^{-x^2/2}dx$, with $a>0$. Using this result, the integration with respect to μ results in

$$\begin{aligned}&\int_{-\infty}^{\infty}|\mu|\exp\left\{-\frac{\tau(n+m\theta^2)}{2}\left(\mu-\frac{n\overline{x}+\theta m\overline{y}}{n+m\theta^2}\right)^2\right\}d\mu \\ &=\left\{\tau(n+m\theta^2)\right\}^{-1}\left[2\exp\left\{-\frac{\tau(n\overline{x}+\theta m\overline{y})^2}{2(n+m\theta^2)}\right\}\right. \\ &\quad\left.+\sqrt{2\pi}\sqrt{\frac{\tau(n\overline{x}+\theta m\overline{y})^2}{2(n+m\theta^2)}}Erf\left(\sqrt{\frac{\tau(n\overline{x}+\theta m\overline{y})^2}{2(n+m\theta^2)}}\right)\right].\end{aligned}$$

Since $\tau > 0$, $\exp\left\{-\tau(n\overline{x}+\theta m\overline{y})^2/2(n+m\theta^2)\right\} \leq 1$ and $Erf(\cdot) \leq 1$, the joint posterior distribution of θ and τ is bounded by

$$\begin{aligned}
&\pi_G(\theta,\tau|\underline{x},\underline{y}) \\
&\leq \frac{\tau^{\frac{N}{2+b-3}}}{n+m\theta^2}\left\{2+\sqrt{2\pi}\sqrt{\frac{\tau(n\overline{x}+m\theta\overline{y})^2}{(n+m\theta^2)}}\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\tau\left(s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right)\right\} \\
&= 2\frac{\tau^{\frac{N}{2+b-3}}}{n+m\theta^2}\exp\left\{-\frac{1}{2}\tau\left(s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right)\right\} \\
&\quad +\frac{\sqrt{2\pi}\tau^{\frac{N}{2+b-\frac{5}{2}}}}{n+m\theta^2}\sqrt{\frac{(n\overline{x}+m\theta\overline{y})^2}{(n+m\theta^2)}}\exp\left\{-\frac{\tau}{2}\left(s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right)\right\}.
\end{aligned}$$

Integration with respect to τ in the right side of the last equality is proportional to

$$\begin{aligned}
&(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-2)} \\
&+(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-\frac{3}{2})}\sqrt{\frac{(n\overline{x}+m\theta\overline{y})^2}{n+m\theta^2}}.
\end{aligned}$$

Now, the first term of the above quantity is proportional to

$$\begin{aligned}
&(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-2)} \\
&\quad \propto (n+m\theta^2)^{-1}\left[1+\frac{nm(\overline{y}-\theta\overline{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-2)} \\
&\quad \leq (n+m\theta^2)^{-1},
\end{aligned}$$

which the integration with respect to θ results in a finite value. And the second term is proportional to

$$\begin{aligned}
&\left[1+\frac{nm(\overline{y}-\theta\overline{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-\frac{3}{2})}\frac{|n\overline{x}+m\theta\overline{y}|}{(n+m\theta^2)^{\frac{3}{2}}} \\
&\quad \leq \frac{|n\overline{x}+m\theta\overline{y}|}{(n+m\theta^2)^{\frac{3}{2}}} \\
&\quad \leq \frac{|n\overline{x}|}{(n+m\theta^2)^{\frac{3}{2}}}+\frac{|m\theta\overline{y}|}{(n+m\theta^2)^{\frac{3}{2}}}.
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \frac{1}{(n+m\theta^2)^{\frac{3}{2}}} d\theta = \frac{2}{n\sqrt{m}} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{|\theta|}{(n+m\theta^2)^{\frac{3}{2}}} d\theta = \frac{2}{m\sqrt{n}},$$

this completes the proof. □

Under the prior π_G, the marginal posterior density function of θ is given by

$$\pi_G(\theta|\underline{x}, \underline{y}) \propto \int_{-\infty}^{\infty} \frac{|\mu|^a (n+m\theta^2)^{\frac{a-1}{2}}}{[s_x + s_y + n(\bar{x}-\mu)^2 + m(\bar{y}-\theta\mu)^2]^{\frac{N}{2}+b-1}} d\mu.$$

The normalizing constant for the marginal posterior density of θ requires two-dimensional integration.

4. Simulation Results

In this section, we perform some simulations to compare the frequentist coverage probabilities with respect to the priors given in the previous section. We calculate the frequentist coverage probabilities by investigating the credible intervals of the marginal posterior density of θ under the proposed priors π_G for several values of θ, n and m. We show numerically how the frequentist coverage of a $(1-\alpha)^{th}$ posterior quantile is close to $1-\alpha$.

To find the estimated coverage probabilities, we use Markov Chain Monte Carlo (MCMC) numerical integration. We describe the details for MCMC. In the joint posterior distributions given in (3.2), let $\omega = \theta\mu$. Then the joint posterior distribution of ω, μ and τ is given by

$$\begin{aligned}\pi_G(\omega, \mu, \tau|\underline{x}, \underline{y}) \propto\ & (n\mu^2 + m\omega^2)^{\frac{a-1}{2}} \tau^{\frac{N}{2}+b-2} \\ & \times \exp\left\{-\frac{\tau}{2}\left[s_x + s_y + n(\bar{x}-\mu)^2 + m(\bar{y}-\omega)^2\right]\right\}.\end{aligned}$$

This leads to the full conditionals

$$\begin{aligned}\omega|\mu, \tau, \underline{x}, \underline{y} &\propto (n\mu^2 + m\omega^2)^{\frac{a-1}{2}} \exp\left\{-\frac{\tau}{2}\left[m(\bar{y}-\omega)^2\right]\right\}, \\ \mu|\omega, \tau, \underline{x}, \underline{y} &\propto (n\mu^2 + m\omega^2)^{\frac{a-1}{2}} \exp\left\{-\frac{\tau}{2}\left[n(\bar{x}-\mu)^2\right]\right\}, \\ \tau|\omega, \mu, \underline{x}, \underline{y} &\sim \Gamma\left(\tau\middle|\frac{N}{2}+b-2, \frac{1}{2}\left[s_x + s_y + n(\bar{x}-\mu)^2 + m(\bar{y}-\omega)^2\right]\right),\end{aligned}$$

where a probability density function $\Gamma(x|b,c)$ is given by

$$\frac{c^b}{\Gamma(b)} x^{b-1} \exp(-bx).$$

TABLE 4.1 *The estimated coverage probabilities for $\theta = 0.1$*

			π_R^1		π_R^2		π_R^3		π_R^4	
	n	m	0.05	0.95	0.05	0.95	0.05	0.95	0.05	0.95
$\mu = 0.1$	5	10	0.0012	0.9953	0.0029	0.9913	0.0044	0.9883	0.0014	0.9946
	10	15	0.0013	0.9964	0.0029	0.9917	0.0024	0.9901	0.0023	0.9955
	15	20	0.0010	0.9967	0.0026	0.9927	0.0038	0.9922	0.0011	0.9972
	20	25	0.0015	0.9960	0.0030	0.9920	0.0032	0.9915	0.0019	0.9964
$\mu = 1$	5	10	0.0176	0.9732	0.0258	0.9635	0.0366	0.9539	0.0203	0.9736
	10	15	0.0273	0.9653	0.0357	0.9558	0.0460	0.9441	0.0358	0.9622
	15	20	0.0349	0.9566	0.0412	0.9488	0.0525	0.9482	0.0401	0.9580
	20	25	0.0421	0.9569	0.0479	0.9520	0.0514	0.9507	0.0457	0.9523
$\mu = 10$	5	10	0.0505	0.9491	0.0506	0.9491	0.0641	0.9375	0.0577	0.9470
	10	15	0.0451	0.9518	0.0450	0.9517	0.0578	0.9400	0.0558	0.9463
	15	20	0.0447	0.9470	0.0446	0.9466	0.0571	0.9464	0.0540	0.9493
	20	25	0.0498	0.9517	0.0496	0.9516	0.0520	0.9502	0.0519	0.9464
$\mu = 100$	5	10	0.0505	0.9490	0.0497	0.9514	0.0638	0.9361	0.0586	0.9453
	10	15	0.0449	0.9517	0.0480	0.9510	0.0568	0.9425	0.0536	0.9434
	15	20	0.0446	0.9466	0.0477	0.9493	0.0570	0.9428	0.0559	0.9476
	20	25	0.0497	0.9516	0.0497	0.9516	0.0520	0.9502	0.0519	0.9464

TABLE 4.2 *The estimated coverage probabilities for $\theta = 100$*

			π_R^1		π_R^2		π_R^3		π_R^4	
	n	m	0.05	0.95	0.05	0.95	0.05	0.95	0.05	0.95
$\mu = 0.1$	5	10	0.0000	0.6812	0.0000	0.6815	0.0000	0.6783	0.0000	0.6852
	10	15	0.0000	0.8035	0.0000	0.8030	0.0000	0.8154	0.0000	0.8089
	15	20	0.0000	0.8522	0.0000	0.8522	0.0000	0.8466	0.0000	0.8508
	20	25	0.0000	0.8753	0.0000	0.8750	0.0000	0.8707	0.0000	0.8780
$\mu = 1$	5	10	0.0006	0.9471	0.0005	0.9507	0.0005	0.9344	0.0007	0.9414
	10	15	0.0040	0.9490	0.0045	0.9479	0.0070	0.9457	0.0047	0.9443
	15	20	0.0205	0.9465	0.0182	0.9508	0.0255	0.9440	0.0233	0.9450
	20	25	0.0408	0.9492	0.0408	0.9492	0.0400	0.9407	0.0418	0.9467
$\mu = 10$	5	10	0.0514	0.9478	0.0469	0.9489	0.0641	0.9350	0.0592	0.9424
	10	15	0.0507	0.9492	0.0513	0.9532	0.0526	0.9457	0.0510	0.9443
	15	20	0.0483	0.9465	0.0511	0.9504	0.0538	0.9440	0.0558	0.9450
	20	25	0.0497	0.9492	0.0497	0.9492	0.0494	0.9407	0.0507	0.9467
$\mu = 100$	5	10	0.0530	0.9523	0.0508	0.9494	0.0615	0.9416	0.0585	0.9427
	10	15	0.0486	0.9456	0.0509	0.9502	0.0569	0.9363	0.0540	0.9470
	15	20	0.0482	0.9483	0.0508	0.9485	0.0526	0.9439	0.0540	0.9498
	20	25	0.0497	0.9492	0.0497	0.9492	0.0494	0.9407	0.0507	0.9468

Sice the conditionals of ω and μ given the rest are nonstandard distributions, the Metropolis-Hasting algorithm is used to generated samples from these conditionals following Chib and Greenberg (1995). Discarding the first 5,000 samples, we compute the 0.05^{th} and 0.95^{th} percent posterior quantiles from a sample of size 10,000 and also repeate the iterations 10,000 times to estimate the coverage probabilities.

In this simulation, we fix $\sigma = 1$, and we take $\mu = 0.1, 1, 10, 100$ and $\theta = 0.1, 100$. The sample sizes are $(n, m) = (5, 10), (10, 15), (15, 20)$ and $(20, 25)$. The results are summarized in Table 4.1 and Table 4.2. In these tables, we use the following notations for priors:

π_R^1 : one at a time reference prior,

π_R^2 : two group reference prior, Second order probability matching prior,

π_R^3 : one group reference prior, Jeffreys' prior and

π_R^4 : two group reference prior.

It is clear from the tables that the second order matching prior performs better than any other priors in matching the target coverage probabilities. And the reference prior π_R^1 is comparable to the second order matching prior π_R^2.

It appears also from our results that when $\mu = 0.1$, the values of the frequentist coverage probabilities are far from target probabilities. The poor performance of all the priors for certain regions of the parameter value is not very surprising. Gleser and Hwang (1987, Theorem 1) show that based on any sample of arbitrary but fixed size, there is a positive probability that confidence interval is infinite set. In our case, this poor performance happens when $|\mu| \approx 0$.

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