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# NEW LM TESTS FOR UNIT ROOTS IN SEASONAL AR PROCESSES†

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ABSTRACT

On the basis of marginal likelihood of the residual vector which is free of nuisance mean parameters, we propose new Lagrange Multiplier seasonal unit root tests in seasonal autoregressive process. The limiting null distribution of the tests is the standardized $\chi^2$-distribution. A Monte-Carlo simulation shows the new tests are more powerful than the tests based on the ordinary least squares (OLS) estimator, especially for large number of seasons and short time spans.

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*Keywords.* $\chi^2$-distribution, marginal likelihood, nuisance mean parameters, seasonal unit root tests.

## 1. INTRODUCTION

We propose a new test of the random walk hypothesis for seasonal time series. There have been several researches in the literature on this subject such as Dickey *et al.* (1984) and Hylleberg *et al.* (1990). However, most of these tests are based on the OLS estimator and have complicated non-standard null distributions depending on the type of mean adjustment and the period of seasonality.

In this paper we propose new Lagrange Multiplier (LM) tests based on a marginal likelihood of the residual vectors which is free from nuisance mean parameters. Neyman and Scott (1948) pointed out that when the number of nuisance parametners grows in proportion to individuals, maximum-likelihood estimates need not be consistent. Even if they are consistent, they need not be

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efficient. The use of marginal likelihoods based on error contrasts, for the estimation of variance components, has been recommended by Patterson and Thompson (1971). The method is also known as restricted maximum likelihood (REML). Kitanidis (1983, 1987) points out that the marginal likelihood is superior in this context to full, or profile, likelihood. See Chapter 7 of McCullagh and Nelder (1989) for more details of marginal likelihood.

From these reasons we develop LM tests for seasonal unit roots on the basis of marginal likelihood. This will be very useful in handling large number of nuisance mean parameters whose number grows in proportion to number of seasons. Specifically, the limiting null distribution of the proposed tests is not influenced by the mean parameters, and follows a standardized $\chi^2$-distribution.

The remainder of this paper is organized as follows. Section 2 proposes the test statistics and investigates asymptotic null distributions of the tests. Section 3 provides critical values for the proposed tests. Section 4 shows results of Monte-Carlo simulation. Concluding remarks are given in Section 5. All proofs are given in Appendix.

## 2. Test Statistics

### 2.1. The marginal likelihood

We consider a stochastic process $y_t$ which is generated by the seasonal AR(1) process

$$\begin{cases} y_t = \sum_{i=1}^{d} \mu_i \delta_{it} + u_t, & \\ u_t = \rho u_{t-d} + \epsilon_t, & t = 1, \ldots, dT, \end{cases} \tag{2.1}$$

where $\delta_{it} = 1$ if $t = i \pmod{d}$ and $\delta_{it} = 0$ if otherwise. Let $d$ denote seasonal lag that $d = 4$ for quarterly data and $d = 12$ for monthly data, $y_{-d+1}, y_{-d+2}, \ldots, y_0$ are initial values, and $\epsilon_t$ are independent and identically distributed with mean 0 and variance $\sum_{i=1}^{d} \sigma_i^2 \delta_{it}$. We are interested in testing for the seasonal random walk null hypothesis $H_0 : \rho = 1$ against the stationary alternative $H_1 : |\rho| < 1$. The conventional LM statistic is developed with full likelihood with parameters $\rho$, $\sigma_1^2, \ldots, \sigma_d^2$ and $d$ nuisance parameters, $\mu_1, \ldots, \mu_d$. However we will use an alternative likelihood referred to the marginal likelihood which contains only $\rho$ and $\sigma_i^2$s. The marginal likelihood is calculated from the full likelihood, $\mathrm{L}(\rho, \sigma_1^2, \ldots, \sigma_d^2 | \mu)$, which is presented as follows:

$$\begin{aligned} &\mathrm{L}(\rho, \sigma_1^2, \ldots, \sigma_d^2 |\ \mu_1, \ldots,\ \mu_d) \\ &= \prod_{i=1}^{d} \frac{|V_i^{-1}|^{1/2}}{(\sqrt{2\pi})^T} \exp\left[ -\frac{1}{2}(Y_i - \mu_i X)' V_i^{-1} (Y_i - \mu_i X) \right], \end{aligned} \tag{2.2}$$

where $Y_i = [y_i, y_{i+d}, \ldots, y_{i+(T-1)d}]'$ and $X = 1_T = [1, 1, \ldots, 1]'$ are $T \times 1$ vectors, and $Y_i$ includes elements which are observed in the $i^{th}$ season. Let $|\cdot|$ be the determinant of a matrix. Let $L_i$=$(|V_i^{-1}|^{1/2})/((\sqrt{2\pi})^T) \exp\{-(1/2)(Y_i - \mu_i X)' V_i^{-1}$ $(Y_i - \mu_i X)\}$ and $L_i$ be the full likelihood of the $i^{th}$ season. Let $Y_i - \mu_i X \sim N_T(0, V_i(\theta_i))$ where $\theta_i = (\rho, \sigma_i^2)$, $(V_i)_{k,k} = \sigma_i^2/(1-\rho^2)$ and $(V_i)_{k,s} = \sigma_i^2 \rho^{|k-s|}/(1-\rho^2)$, for $k, s = 1, \ldots, T$, where $(V_i)_{k,s}$ denotes the $(k, s)$ element of matrix $V_i(\theta_i)$. We use $V_i$ in lieu of $V_i(\theta_i)$ as a matter of convenience. Let us now formulate the marginal likelihood, $\mathrm{L}_M(\rho, \sigma^2)$. We will construct a LM test based on a $\mu$-free likelihood obtained by integrating out the nuisance parameters from the full likelihood. Therefore the marginal likelihood is defined as

$$\begin{aligned} \mathrm{L}_M(\rho, \sigma_1^2, \ldots, \sigma_d^2) &= \int \cdots \int \mathrm{L}(\rho, \sigma_1^2, \ldots, \sigma_d^2 |\ \mu_1, \ldots,\ \mu_d)\, d\mu_1 d\mu_2 \cdots d\mu_d \\ &= \int \cdots \int \prod_{i=1}^{d} \mathrm{L}_i \ d\mu_1 d\mu_2 \cdots d\mu_d \\ &= \int \mathrm{L}_1 d\mu_1 \ \cdots \ \int \mathrm{L}_d d\mu_d \,. \end{aligned} \tag{2.3}$$

The first integral can be represented as

$$\begin{aligned} \int \mathrm{L}_1 d\mu_1 &= \frac{|V_1^{-1}|^{1/2}}{(\sqrt{2\pi})^T} \int \exp\left\{ -\frac{1}{2}(Y_1 - \mu_1 X)' V_1^{-1} (Y_1 - \mu_1 X) \right\} d\mu_1 \\ &= \frac{|V_1^{-1}|^{1/2}}{(\sqrt{2\pi})^T} \\ &\quad \times \int \exp\left\{ -\frac{1}{2} Y_1' V_1^{-1} Y_1 + \mu_1 Y_1' V_1^{-1} X - \frac{\mu_1^2}{2} X' V_1^{-1} X \right\} d\mu_1. \end{aligned} \tag{2.4}$$

Let $\hat{\mu}_1 = (X'V_1^{-1}X)^{-1}(X'V_1^{-1}Y_1) = \left[2\rho\{(y_1 + y_{1+(T-1)d})/2\} + (1-\rho)T\bar{y}_1\right]/(2\rho + T(1-\rho))$, where $\bar{y}_1 = (1/T)\sum_{t=1}^{T} y_{i+(t-1)d}$ and the $\hat{\mu}_1$ is identical to the generalized least squares estimate (GLSE) of $\mu$. The formula (2.4) can be written as

$$\frac{|V_1^{-1}|^{1/2}}{(\sqrt{2\pi})^T} \exp\left\{ -\frac{1}{2} Y_1' V_1^{-1} Y_1 + \frac{\hat{\mu}_1^2}{2} (X' V_1^{-1} X) \right\}$$

$$\times \int \exp\left\{-\frac{(\mu_1-\hat{\mu}_1)^2}{2}(X'V_1^{-1}X)\right\}d\mu_1. \tag{2.5}$$

The integration part of formula (2.5) is equal to $(2\pi)^{1/2}|X'V_1^{-1}X|^{-1/2}$ because

$$(2\pi)^{-1/2}|X'V_1^{-1}X|^{1/2}\int \exp\left\{-\frac{(\mu_1-\hat{\mu}_1)^2}{2}(X'V_1^{-1}X)\right\}d\mu_1 = 1\,.$$

Thus, the first integral can be represented as

$$\frac{|V_1^{-1}|^{1/2}}{(\sqrt{2\pi})^{T-1}|X'V_1^{-1}X|^{1/2}}\exp\left\{-\frac{1}{2}(Y_i-\hat{\mu}_1X)'V_1^{-1}(Y_i-\hat{\mu}_1X)\right\} \tag{2.6}$$

and the nuisance parameter $\mu_1$ is eliminated from the full likelihood, $\mathrm{L}_1(\rho,\sigma_1^2)$. By replicating the integration for $\mu_i$, $i=2,\ldots,d$, we can finally have the marginal likelihood as the following:

$$\begin{aligned}\mathrm{L}_\mathrm{M}(\rho,\sigma_1^2,\ldots,\sigma_d^2) = \prod_{i=1}^{d}&\frac{|V_i^{-1}|^{1/2}}{(\sqrt{2\pi})^{T-1}|X'V_i^{-1}X|^{1/2}}\\ &\times\exp\left\{-\frac{1}{2}(Y_i-\hat{\mu}_iX)'V_i^{-1}(Y_i-\hat{\mu}_iX)\right\}.\end{aligned} \tag{2.7}$$

We note that both $\hat{\mu}_i$ and $|V_i^{-1}|/|X'V_i^{-1}X|$, $i=1,\ldots,d$ are well-defined at $\rho=1$ in (2.3) and (2.7). Now the proposed likelihood has been obtained by integrating out the nuisance parameters, $\mu_1,\ldots,\mu_d$, from the full likelihood.

### *2.2. The test statistic*

On the basis of marginal likelihood, we can construct a score test for the null hypothesis of seasonally nonstationary time series, $H_0:\rho=1$. First, we are going to establish a test statistic for each season. Next, we construct the proposed test statistic by pooling seasonal test statistics. In order to build our test statistic, let us now derive the score vector and the Fisher information matrix. Let $S^{(i)}(\theta_i)$ be the score vector for the $i^{th}$ season with $S^{(i)}(\theta_i)=[S_1^{(i)}(\theta_i),S_2^{(i)}(\theta_i)]=[\partial l_M^{(i)}/\partial\rho,\partial l_M^{(i)}/\partial\sigma_i^2]'$ and $l_M^{(i)}=\ln L_M^{(i)}(\rho,\sigma_i^2)$ be the $i^{th}$ marginal log likelihood. Let $J^{(i)}$ be the inverse of the Fisher information matrix for the $i^{th}$ individual with $J_{mn}^{(i)}$ representing the $(m,n)^{th}$ components of $J^{(i)}$. Therefore we obtain the followings from the direct computation given in Appendix,

$$\tilde{S}_1^{(i)} = S_1^{(i)}(\tilde{\theta}_i) = \frac{T-1}{4}+\frac{1}{\tilde{\sigma}_i^2}\left\{\frac{\left(y_{i+(T-1)d}-y_i\right)^2}{4}-\frac{1}{2}\sum_{t=2}^{T}(\Delta_d y_{i+(t-1)d})^2\right\} \tag{2.8}$$

and

$$J_{11}^{(i)} = \frac{8}{(T-1)(T-2)}, \tag{2.9}$$

where $\tilde{\theta}_i = (1, \tilde{\sigma}^2)'$ and $\tilde{\sigma}^2 = \{1/(T-1)\} \sum_{t=2}^{T} (\Delta_d y_{i+(t-1)d})^2$, $\Delta_d = (1 - B^d)$ and $B$ is the usual back shift operator. In this context, we are able to construct the proposed LM test statistic, $t_M$, from the score vector and the inverse of Fisher information matrix based on the marginal likelihood, $L_M(\rho, \sigma^2)$. The test statistic is given by

$$t_M = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} t_M^{(i)} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \tilde{S}_1^{(i)} \sqrt{J_{11}^{(i)}} \,. \tag{2.10}$$

In the following theorem, we derive the null distribution of the proposed score test statistic.

THEOREM 2.1. *Consider model (2.1). Let $\{\epsilon_t\}$ be a sequence of independent and identically distributed random variables with mean 0 and variance $\sum_{i=1}^{d} \sigma_i^2 \delta_{it}$. Then under the null hypothesis, $H_0 : \rho = 1$, we have $t_M \xrightarrow{d} (1/\sqrt{2d})\{\sum_{i=1}^{d} W_i^2(1) - d\}$ as $T \to \infty$.*

PROOF. A proof of Theorem 2.1 is in Appendix. □

Here $\xrightarrow{d}$ stands for the convergence in distribution, and $W_i(\cdot)$ denotes the standard Brownian motion. We note that the limiting null distribution is the standardized $\chi^2$-distribution and is approximately $N(0,1)$ for large $d$.

## 3. CRITICAL VALUES

For fixed $d$ and $T$, sample critical values of the proposed test statistics are computed via stochastic simulation with 10,000 replications. The critical values are tabulated in Table 3.1 for $d$ =2, 4, 12 and $T$ =5, 10, 15, 20, 25, 30, 40, 50, 60, 70, 100 and $\infty$. For the case of $\infty$, we did not simulate but calculate from the $\chi^2$-distribution table. We generate data by the model

$$y_t = \sum_{i=1}^{d} \mu_i \delta_{it} + \rho \left( y_{t-d} - \sum_{i=1}^{d} \mu_i \delta_{it} \right) + e_t \tag{3.1}$$

with $y_{-d+1} = \mu_1, y_{-d+2} = \mu_2, \ldots, y_0 = \mu_d$ and $\mu_i \sim N(0,1)$ for $i = 1, \ldots, d$. Furthermore we set $e_t \sim N(0, \sum_{i=1}^{d} \sigma_i^2 \delta_{it})$ and $\sigma_i^2 \sim U(0.5, 1.5)$ for $i = 1, \ldots, d$.

TABLE 3.1 *Critical values of the test statistic (left 5%)*

| $d\backslash T$ | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 50 | 60 | 70 | 100 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | −1.23 | −1.23 | −1.23 | −1.22 | −1.22 | −1.22 | −1.22 | −1.21 | −1.21 | −1.21 | −1.20 | −1.20 |
| 4 | −1.23 | −1.23 | −1.23 | −1.23 | −1.23 | −1.22 | −1.22 | −1.22 | −1.22 | −1.22 | −1.22 | −1.22 |
| 12 | −1.30 | −1.33 | −1.34 | −1.35 | −1.36 | −1.36 | −1.36 | −1.37 | −1.37 | −1.38 | −1.38 | −1.38 |

NOTE : *Tests for* $H_0 : \rho = 1$ *in model* $y_t = y_{t-d} + e_t$*;* $e_t \sim N(0,1)$*; initial values are set such as* $y_{-d-50} = 0, y_{-d-50} = 0, \ldots, y_{-50} = 0$*; Size of tests = 5%; number of replications = 50,000.*

Here $U(a,b)$ stands for a uniform distribution on interval $(a,b)$. Once $\mu_i$ and $\sigma_i^2$ are generated for each season independently of disturbances, $e_t$, they are fixed throughout replications.

## 4. SIMULATION

In this section, we present the results of Monte-Carlo experiments to investigate finite sample performances of the proposed seasonal unit root test. We compare the proposed test with the test of Dickey *et al.* (1984, DHF test hereafter). The DHF test is now most extensively used in the empirical seasonal time series literature. The experiment examines power performances of the two seasonal unit root tests for the model (2.1) under the general setup of the null hypothesis $H_0 : \rho = 1$ and the alternative that $H_1 : |\rho| < 1$. Table 4.1 and Table 4.2 show rejection probabilities of the two test statistics with 5% level. We set as alternative $\rho = 0.99,\ 0.95,\ 0.90,\ 0.80,\ 0.70,\ i = 1, \ldots, d$ for Table 4.1 and Table 4.2. We consider normal disturbances in Table 4.1, and ARCH errors with $\theta = 0.5$ for possible serial correlations in Table 4.2.

The simulation results show clearly that the proposed test is more powerful than the DHF test especially for the seasonal time series models with large number of seasons $d$ and small time span $T \leq 20$.

## 5. CONCLUDING REMARKS

We propose new score type seasonal unit root tests for seasonal time series model. The proposed tests are based on the marginal likelihood which is free of nuisance mean parameters. The Monte-Carlo simulation results show that the proposed tests are more powerful than the tests based on the ordinary least squares estimates especially for short time series with large number of seasons $d$.

TABLE 4.1 *Rejection probability(%) for normal errors*

| | | $T=5$ | | $T=10$ | | $T=20$ | | $T=50$ | |
|---|---|---|---|---|---|---|---|---|---|
| $d$ | $\rho$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ |
| 4 | 1.00 | 5.18 | 5.15 | 5.49 | 5.90 | 5.80 | 7.28 | 6.40 | 8.47 |
| | 0.99 | 5.57 | 5.15 | 5.62 | 5.95 | 5.69 | 6.06 | 6.21 | 8.03 |
| | 0.95 | 5.04 | 6.12 | 5.40 | 6.89 | 7.34 | 10.34 | 14.78 | 24.49 |
| | 0.90 | 5.71 | 7.30 | 6.92 | 10.11 | 10.91 | 20.93 | 38.89 | 45.91 |
| | 0.80 | 6.01 | 10.80 | 10.71 | 21.33 | 25.45 | 48.02 | 94.21 | 83.09 |
| | 0.70 | 7.19 | 13.07 | 16.32 | 30.00 | 54.53 | 57.46 | 99.98 | 95.78 |
| 12 | 1.00 | 5.40 | 6.82 | 6.39 | 8.21 | 6.77 | 11.54 | 7.63 | 16.25 |
| | 0.99 | 4.86 | 5.69 | 5.51 | 6.24 | 6.21 | 8.36 | 7.66 | 15.55 |
| | 0.95 | 5.06 | 7.48 | 6.67 | 11.75 | 11.13 | 23.70 | 33.04 | 59.77 |
| | 0.90 | 5.83 | 10.73 | 9.97 | 22.84 | 22.69 | 48.92 | 85.57 | 98.18 |
| | 0.80 | 7.03 | 15.41 | 19.99 | 49.28 | 64.35 | 91.81 | 100.00 | 100.00 |
| | 0.70 | 9.51 | 32.01 | 38.16 | 75.81 | 95.63 | 99.73 | 100.00 | 100.00 |

NOTE : *Tests for* $H_0 : \rho = 1$ *in model* $y_t = \sum_{i=1}^{d} \mu_i \delta_{it} + \rho(y_{t-d} - \sum_{i=1}^{d} \mu_i \delta_{it}) + e_t$; $y_{-d+1} = \mu_1$, $y_{-d+2} = \mu_2, \ldots, y_0 = \mu_d$ *and* $\mu_i \sim N(0,1)$ *for* $i = 1, \ldots, d$; $\rho = 1.00, 0.99, 0.95, 0.90, 0.80, 0.70$; *Size of tests* = *5%;* $e_t \sim N(0, \sum_{i=1}^{d} \sigma_i^2 \delta_{it})$, $\sigma_i^2 \sim U(0.5, 1.5)$ *for* $i = 1, \ldots, d$; *Once* $\mu_i$ *and* $\sigma_i^2$ *are generated independently of* $e_t$, *they are fixed throughout replications; number of replications* = 10,000.

TABLE 4.2 *Rejection probability(%) for ARCH errors*

| | | $T=5$ | | $T=10$ | | $T=20$ | | $T=50$ | |
|---|---|---|---|---|---|---|---|---|---|
| $d$ | $\rho$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ | $DHF$ | $t_M$ |
| 4 | 1.00 | 5.18 | 5.15 | 5.49 | 5.90 | 5.80 | 7.28 | 6.40 | 8.47 |
| | 0.99 | 4.99 | 5.52 | 5.52 | 5.98 | 5.68 | 6.81 | 6.18 | 10.59 |
| | 0.95 | 5.02 | 6.99 | 6.09 | 10.08 | 6.76 | 18.01 | 13.48 | 49.31 |
| | 0.90 | 5.40 | 9.33 | 6.55 | 16.32 | 10.06 | 36.90 | 33.46 | 75.13 |
| | 0.80 | 4.80 | 10.92 | 8.49 | 26.73 | 21.69 | 59.58 | 88.48 | 93.07 |
| | 0.70 | 5.56 | 18.18 | 12.32 | 40.66 | 44.37 | 75.74 | 99.32 | 98.15 |
| 12 | 1.00 | 5.40 | 6.82 | 6.39 | 8.21 | 6.77 | 11.54 | 7.63 | 16.25 |
| | 0.99 | 5.09 | 6.15 | 5.08 | 7.12 | 6.02 | 10.38 | 8.96 | 23.37 |
| | 0.95 | 5.69 | 10.23 | 7.76 | 19.59 | 11.82 | 47.65 | 30.98 | 97.29 |
| | 0.90 | 5.51 | 15.33 | 9.80 | 41.82 | 19.25 | 85.11 | 76.93 | 99.99 |
| | 0.80 | 6.15 | 26.34 | 14.84 | 72.76 | 53.00 | 98.57 | 99.96 | 100.00 |
| | 0.70 | 7.01 | 39.24 | 27.70 | 87.10 | 89.84 | 99.74 | 100.00 | 100.00 |

NOTE : *Tests for* $H_0 : \rho = 1$ *in model* $y_t = \sum_{i=1}^{d} \mu_i \delta_{it} + \rho(y_{t-d} - \sum_{i=1}^{d} \mu_i \delta_{it}) + e_t$; $y_{-d+1} = \mu_1$, $y_{-d+2} = \mu_2, \ldots, y_0 = \mu_d$ *and* $\mu_i \sim N(0,1)$ *for* $i = 1, \ldots, d$; $\rho = 1.00, 0.99, 0.95, 0.90, 0.80, 0.70$; *Size of tests* = *5%;* $e_t \sim N(0, \sum_{i=1}^{d} \sigma_i^2 \delta_{it})$, $\sigma_i^2 \sim U(0.5, 1.5)$ *for* $i = 1, \ldots, d$; *Once* $\mu_i$ *and* $\sigma_i^2$ *are generated independently of* $e_t$, *they are fixed throughout replications; number of replications* = 10,000.

## APPENDIX

*Proof of the marginal likelihood and its derivatives*

In order to calculate score vector and the Fisher information matrix, we need to derive the first and second derivatives of $l_M^{(i)}$, $l_M^{(i)} = \ln L_M^{(i)}(\rho, \sigma^2)$. Each components of the marginal likelihood, formula (2.7), can be represented as the followings:

$$\hat{\mu}_i(\rho) = (X'V_i^{-1}X)^{-1}X'V_i^{-1}Y$$
$$= \frac{2\rho\left(\frac{y_i + y_{i+(T-1)d}}{2}\right) + (1-\rho)T\bar{y}_i}{2\rho + T(1-\rho)},$$
$$\frac{\partial \hat{\mu}_i}{\partial \rho}(\rho) = \frac{T(y_i + y_{i+(T-1)d} - 2\bar{y}_i)}{\{2\rho + T(1-\rho)\}^2},$$
$$\hat{\mu}_i(1) = \frac{(y_i + y_{i+(T-1)d})}{2},$$
$$\frac{\partial \hat{\mu}_i}{\partial \rho}(1) = \frac{1}{2}\left\{\frac{T}{2}(y_i + y_{i+(T-1)d}) - T\bar{y}_i\right\},$$
$$|V_i^{-1}| = \frac{(1-\rho^2)}{\sigma_i^{2T}},$$
$$|X'V_i^{-1}X| = \frac{(1-\rho)\{(T-2)(1-\rho)+2\}}{\sigma_i^2},$$
$$Y'(V_i^{-1} - V_i^{-1}X(X'V_i^{-1}X)^{-1}X'V_i^{-1})Y$$
$$= \frac{1}{\sigma_i^2}\left\{\sum_{t=2}^{T}(y_{i+(t-1)d} - \rho y_{i+(t-2)d})^2 + (1-\rho)A_i\right\},$$

where $A_i = (1+\rho)y_i^2 - (T-(T-2)\rho)\hat{\mu}_i^2$.

Then we have the first derivatives

$$\frac{\partial l_M^{(i)}}{\partial \rho} = \frac{1}{2(1+\rho)} + \frac{T-2}{2(T-(T-2)\rho)}$$
$$-\frac{1}{\sigma_i^2}\left\{-\sum_{t=2}^{T} y_{i+(t-2)d}(y_{i+(t-1)d} - \rho y_{i+(t-2)d}) - \frac{A_i}{2} + \left(\frac{1-\rho}{2}\right)\frac{\partial A_i}{\partial \rho}\right\}$$

and

$$\frac{\partial l_M^{(i)}}{\partial \sigma_i^2} = -\frac{T-1}{2\sigma_i^2} + \frac{1}{2\sigma_i^4}\left\{\sum_{t=2}^{T}(y_{i+(t-1)d} - \rho y_{i+(t-2)d})^2 + (1-\rho)A_i\right\}.$$

Let $\tilde{\theta}_i = [1, \tilde{\sigma}_i^2]'$ and $\tilde{\sigma}_i^2$ maximizes the marginal likelihood under $H_0: \rho = 1$, then $\tilde{\sigma}_i^2 = (1/T-1)\sum_{t=2}^{T}(\Delta_d y_{i+(t-1)d})^2$.

The second derivatives are obtained as

$$\frac{\partial^2 l_M^{(i)}}{\partial\rho^2} = -\frac{1}{2(1+\rho)^2} + \frac{(T-2)^2}{2(2\rho+T(1-\rho))^2} - \frac{1}{\sigma^2}\left\{\sum_{t=2}^{T-1} y_{i+(t-1)d}^2 - (T-2)\hat{\mu}_i^2 + 2\hat{\mu}_i \frac{T(y_i + y_{i+(T-1)d} - 2\bar{y}_i)}{T-(T-2)\rho} + \left(\frac{1-\rho}{2}\right)\frac{\partial^2 A_i}{\partial\rho^2}\right\},$$

$$\frac{\partial^2 l_M^{(i)}}{\partial\rho\partial\sigma_i^2} = \frac{1}{\sigma^4}\left\{-\sum_{t=2}^{T} y_{i+(t-2)d}(y_{i+(t-1)d} - \rho y_{i+(t-2)d}) - \frac{A_i}{2} + \left(\frac{1-\rho}{2}\right)\frac{\partial A_i}{\partial\rho}\right\},$$

$$\frac{\partial^2 l_M^{(i)}}{\partial\sigma_i^4} = \frac{T-1}{2\sigma_i^4} - \frac{1}{\sigma_i^6}\left\{\sum_{t=2}^{T}(y_{i+(t-1)d} - \rho y_{i+(t-2)d})^2 + (1-\rho)A_i\right\},$$

$$\frac{\partial A_i}{\partial\rho} = y_i^2 + (T-2)\hat{\mu}_i^2 - 2\hat{\mu}_i \frac{T(y_i + y_{i+(T-1)d} - 2\bar{y}_i)}{T-(T-2)\rho}.$$

The expectation of the observed Fisher information is

$$E\left[-\frac{\partial^2 l_M^{(i)}}{\partial\rho^2}\right] = \frac{(T-1)^2}{8},\quad E\left[-\frac{\partial^2 l_M^{(i)}}{\partial\rho\partial\sigma_i^2}\right] = -\left(\frac{T-1}{4}\right)\frac{1}{\sigma_i^2},\quad E\left[-\frac{\partial^2 l_M^{(i)}}{\partial^2\sigma_i^2}\right] = \frac{T-1}{2\sigma_i^4},$$

since $\sum_{t=2}^{T-1} E(y_t^2) = \{(T+1)(T-2)/2\}\sigma^2$, $E(y_i + y_{i+(T-1)d}/2)^2 = (T+3)/4\sigma_i^2$, $E[T\bar{y}_i(y_i + y_{i+(T-1)d})] = T(T+3)/2\sigma_i^2$, $E[\sum_2^T y_{i+(t-1)d} y_{i+(t-2)d}] = T(T-1)/2\sigma_i^2$ and $E[\sum_{t=2}^{T}(\Delta_d y_{i+(t-1)d})^2] = E[\sum_2^T e_t^2] = (T-1)\sigma^2$.

*Proof of Theorem 2.1*

Since $\tilde{\sigma}_i^2/(T-1) = \sigma^2 + o_p(1)$ by the law of large numbers and $(y_{i+(T-1)d} - y_i)/(\sigma_i\sqrt{T-1}) \xrightarrow{d} W(1)$ by the central limit theorem, we have $t_M^{(i)} \xrightarrow{d} \{W(1)^2 - 1\}/\sqrt{2}$ as $T \to \infty$. This immediately results in the limiting null distribution of $t_M$ that $t_M = \sum_{i=1}^{d} t_M^{(i)}/\sqrt{d} \xrightarrow{d} (1/\sqrt{2d})\sum_{i=1}^{d}\{[W_i(1)]^2 - 1\}$ as $T \to \infty$.

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