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Residuated Partially Ordered Semigroups

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Abstract

In this paper, we investigate the properties of residuated partially ordered sets as weak definitions of algebraic structures in many valued logics. We study the left(resp. right) residuated semigroups induced by right(resp. left) associated map . We give their examples.

Key words : left (right) residuated semi-groups (groupoids), right (left) associated map

1. Introduction and preliminaries

In many valued logics, the most frequent structures are Heyting algebras [13], MV-algebra [3] and residuated lattices [4]. A Heyting algebra [13] is a pseudo-Boolean algebra and relatively pseudo-complemented distributive lattice with 0. It is a algebraic model of intuitionistic propositional logic.

MV-algebras were introduced by Chang [3] for the purpose of analysing algebraically the many valued logics of Lukasiewicz. Dilworth and Ward [4] introduced a residuated lattice. From 1980's to present, these algebraic structures in many valued logics have been developed by many researchers [1-2, 5-12,14].

In this paper, we investigate the properties of residuated partially ordered sets as weak definitions of algebraic structures in many valued logics. We study the left-(resp. right-)residuated semigroups induced by RA-maps (resp. LA-map). We give their examples.

Let $(L, \leq)$ be a partially ordered set and $\top$ and $\bot$ are the greatest element and the least element, respectively. We define the weak definitions of algebraic structures in the sense of [11,12].

Definition 1.1. [1,11,12] A triple $(L, \leq, \odot)$ is called a partially ordered groupoid (for short, po-groupoid) iff it satisfies

(P) if $a_1 \leq a_2$ and $b_1 \leq b_2$, then $a_1 \odot b_1 \leq a_2 \odot b_2$.

A po-groupoid $(L, \leq, \odot)$ is called;

(1) commutative if $a \odot b = b \odot a$ for all $a, b \in L$,

(2) a po-semigroup if $(a \odot b) \odot c = a \odot (b \odot c)$ for all $a, b, c \in L$,

(3) a po-monoid if it is a po-semigroup and $a \odot e = e \odot a = a$ for all $a \in L$.

(4) right (resp. left) sided if $a \odot \top \leq a$ (resp. $\top \odot a \leq a$) for all $a \in L$,

(5) strictly right (resp. left) sided if $a \odot \top = a$ (resp. $\top \odot a = a$) for all $a \in L$,

(6) strictly two-sided if every element in L is strictly right and left-sided.

(7) right (resp. left) divisible if for each $a \leq b$, there exists $c \in L$ such that $a = b \odot c$ (resp. $a = c \odot b$).

(8) bounded if it have $\top$ and $\bot$.

Definition 1.2. [1,11,12] A quadruple $(L, \leq, \odot, \Rightarrow)$ is called right-residuated if it satisfies:

(R) $c \leq a \Rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in L$,

A quadruple $(L, \leq, \odot, \rightarrow_l)$ is called left-residuated if it satisfies:

(L) $c \leq a \rightarrow_l b$ iff $c \odot a \leq b$ for all $a, b, c \in L$.

A right-(resp. left) residuated partially ordered set $(L, \leq, \odot, \Rightarrow)$ with $\bot$ has called a right-negation $a^* = a \Rightarrow \bot$ (resp. $a^\circ = a \rightarrow \bot$).

A structure $(L, \leq, \odot, \Rightarrow, \rightarrow_l)$ is called residuated if it is both left and right residuated.

Definition 1.3. [6,8,11,12] A bounded, strictly two-sided, residuated po-semigroup lattice is called a pseudo-BL algebra if it satisfies:

(B1) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \Rightarrow y)$.

(B2) $(x \rightarrow y) \vee (y \rightarrow x) = (x \Rightarrow y) \vee (y \Rightarrow x) = \top$.

A pseudo-BL algebra is called:

(1) a pseudo MV-algebra if $x = x^{\circ *} = x^{* \circ}$.

(2) a BL-algebra if it is commutative.

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2. Residuated partially ordered semigroups

Theorem 2.1. Let $(L, \leq, \odot, \Rightarrow, \rightarrow)$ be a residuated partially ordered set. Then the following properties hold:

(1) $a \odot (a \Rightarrow b) \leq b$, $a \leq (a \Rightarrow b) \rightarrow b$, $b \leq a \Rightarrow (a \odot b)$.

(2) $(a \rightarrow b) \odot a \leq b$, $a \leq (a \rightarrow b) \Rightarrow b$, $a \leq b \rightarrow (a \odot b)$.

(3) If L has the least element $\perp$, then $a \leq a^{\circ *}$ and $a \leq a^{*\circ}$.

(4) $(L, \leq, \odot)$ is a po-groupoid.

(5) If $b \leq c$, then $a \Rightarrow b \leq a \Rightarrow c$, $c \Rightarrow a \leq b \Rightarrow a$, $a \rightarrow b \leq a \rightarrow c$ and $c \rightarrow a \leq b \rightarrow a$.

(6) $a \rightarrow b = ((a \rightarrow b) \Rightarrow b)) \rightarrow b$ and $a \Rightarrow b = ((a \Rightarrow b) \rightarrow b)) \Rightarrow b$.

(7) If L has the least element $\perp$, then $a^{\circ} = a^{\circ * \circ}$ and $a^{*} = a^{* \circ *}$.

(8) $a \odot (a \Rightarrow b) = b$ iff there exists $c \in L$ such that $a \odot c = b$.

(9) $(a \rightarrow b) \odot a = b$ iff there exists $c \in L$ such that $c \odot a = b$.

(10) $a \Rightarrow (a \odot b) = b$ iff there exists $c \in L$ such that $a \Rightarrow c = b$.

(11) $a \rightarrow (a \odot b) = b$ iff there exists $c \in L$ such that $a \rightarrow c = b$.

(12) $(b \rightarrow a) \Rightarrow a = b$ iff there exists $c \in L$ such that $c \Rightarrow a = b$.

(13) If $\Rightarrow = \rightarrow$, then L is commutative.

(14) If $(L, \odot)$ is a residuated po-semigroup, then $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$ and $a \Rightarrow (b \Rightarrow c) = (b \odot a) \Rightarrow c$.

Proof. (1) It follows $a \Rightarrow b \leq a \Rightarrow b$ iff $a \odot (a \Rightarrow b) \leq b$ iff $a \leq (a \Rightarrow b) \rightarrow b$. Moreover $a \odot b \leq a \odot b$ iff $b \leq a \Rightarrow (a \odot b)$.

(2) It is similar to (1).

(3) By (1) and (2), $a \leq (a \Rightarrow \perp) \rightarrow \perp = a^{*\circ}$ and $a \leq (a \rightarrow \perp) \Rightarrow \perp = a^{\circ *}$.

(4) If $a \leq b$, by (2), we have $a \leq b \leq c \rightarrow (b \odot c)$. Hence $a \odot c \leq b \odot c$. Moreover, $a \leq b \leq c \Rightarrow (c \odot b)$. Hence $c \odot a \leq c \odot b$.

(5) Let $b \leq c$ be given. Since $a \odot (a \Rightarrow b) \leq b \leq c$, we have $a \Rightarrow b \leq a \Rightarrow c$. Since $b \odot (c \Rightarrow a) \leq c \odot (c \Rightarrow a) \leq a$, we have $c \Rightarrow a \leq b \Rightarrow a$.

(6) and (7) $a \Rightarrow b \leq ((a \Rightarrow b) \rightarrow b) \Rightarrow b$.

Since $a \leq (a \Rightarrow b) \rightarrow b$, by (2), we have $a \Rightarrow b \geq ((a \Rightarrow b) \rightarrow b) \Rightarrow b$.

(8) and (9) It follows $b = a \odot c \leq a \odot (a \Rightarrow b) \leq b$ and $b = c \odot a \leq (a \rightarrow b) \odot a \leq b$.

(10) and (11) Since $a \Rightarrow (a \odot (a \Rightarrow c)) \leq a \Rightarrow c$, we have

$$\begin{aligned} b &\leq a \Rightarrow (a \odot b) \\ &= a \Rightarrow (a \odot (a \Rightarrow c)) \leq a \Rightarrow c = b. \end{aligned}$$

(12) Let $c \in L$ such that $c \Rightarrow a = b$. Then $z \odot b \leq a$. Thus,

$$b \leq (b \rightarrow a) \Rightarrow a \leq c \Rightarrow a = b$$

(13) It follows from:

$$\begin{aligned} a \odot b \leq a \odot b \quad &\text{iff } b \leq a \Rightarrow (a \odot b) \\ &\text{iff } b \leq a \rightarrow (a \odot b) \\ &\text{iff } b \odot a \leq a \odot b \end{aligned}$$

(14) Since $(b \odot a) \odot ((b \odot a) \Rightarrow c) \leq c$, we have $(b \odot a) \Rightarrow c \leq a \Rightarrow (b \Rightarrow c)$. Since

$$\begin{aligned} &(b \odot a) \odot (a \Rightarrow (b \Rightarrow c)) \\ &= b \odot (a \odot (a \Rightarrow (b \Rightarrow c))) \\ &= b \odot (b \Rightarrow c) \leq c, \end{aligned}$$

Hence $(b \odot a) \Rightarrow c \geq a \Rightarrow (b \Rightarrow c)$. Other case is similarly proved.

□

Example 2.2. Let Z be integers, respectively. Define an operation $\odot, \rightarrow, \Rightarrow$: $Z \times Z \rightarrow Z$ as $x \odot y = x + 2y$ and

$$a \rightarrow b = b - 2a, \quad a \Rightarrow b = [\frac{b-a}{2}]$$

where $[x] = n$ for $n \leq x < n+1$, $n \in Z$. Let $a \odot b = a + 2b \leq c$. Then $b \leq \frac{c-a}{2}$ implies $b \leq [\frac{c-a}{2}] = a \Rightarrow c$. Let $b \leq [\frac{c-a}{2}] \leq \frac{c-a}{2}$. Then $a \odot b \leq c$. Furthermore, $a \odot b = a + 2b \leq c$ iff $a \leq b \rightarrow c$. Thus, $(Z, \leq, \odot, \rightarrow, \Rightarrow)$ is residuated po-groupoid. Since $11 = ((-3) \odot 2) \odot 5) < (-3) \odot (2 \odot 5) = 21$, $(Z, \leq, \odot)$ is not a semi-group. Since $x + 0 = x$ for all $x \in Z$ where 0 is not a top element, $(Z, \leq, \odot)$ is not strictly right-sided. For $a \leq b$, we have $a = (b \rightarrow a) \odot b$. Hence $(Z, \leq, \odot)$ is left-divisible but not right-divisible because, for $2 < 3$, $2 \neq 3 \odot (3 \Rightarrow 2) = 1$.

Example 2.3. (1) Let $(L = \{0, a, b, c, d, 1\}, \leq, \wedge)$ be a lattice defined as

$$0 < a < b < d < 1, \ 0 < a < c < d < 1, \ b \wedge c = a, b \vee c = d$$

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	a	a
b	0	0	b	0	b	b
c	0	a	a	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\Rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	c	c	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	1	1	1
b	c	c	1	c	1	1
c	b	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \Rightarrow, \rightarrow, 0, 1)$ is a bounded strictly two-sided, residuated po-monoid lattice.

(1) Since $(b \Rightarrow c) \vee (c \Rightarrow b) = (b \rightarrow c) \vee (c \rightarrow b) = c \vee b = d \neq 1$, it does not satisfy the condition of Definition 1.3 (B2).

(2) It does not satisfy the condition of Definition 1.3 (B1) because

$$a = b \wedge c \neq b \odot (b \Rightarrow c) = 0$$

$$a = c \wedge b \neq (c \rightarrow b) \odot c = 0$$

(3) For $a \leq c$, there does not exist $x \in L$ such that $a = x \odot c$ and for $a \leq b$, there does not exist $y \in L$ such that $a = b \odot y$. Hence it is neither left-divisible nor right-divisible.

(4) $a^{*\circ} = (a \Rightarrow 0) \rightarrow 0 = b \neq a$ and $a^{\circ *} = (a \rightarrow 0) \Rightarrow 0 = c \neq a$.

3. Partially ordered semigroups and RA-maps

Definition 3.1. Let $(L, \odot)$ be a po-semigroup. A map $j : L \rightarrow L$ is called a *right-associated map*(for short, RA-map) which satisfies the following conditions:

(J1) $j(a) \leq j(b)$ for $a \leq b$.

(J2) $j(a \odot j(b)) = j(a) \odot j(b)$.

A map $j : L \rightarrow L$ is called a *left-associated map*(for short, LA-map) which satisfies the condition (J1) and

(J3) $j(j(a) \odot b) = j(a) \odot j(b)$.

Theorem 3.2. Let $(L, \odot, \rightarrow)$ be a left-residuated po-semigroup and $j : L \rightarrow L$ an RA-map. Define two operations $*, \rightarrow_j$: $L \times L \rightarrow L$ as $x * y = x \odot j(y)$ and $x \rightarrow_j y = j(x) \rightarrow y$. Then the following properties hold:

(1) $(L, *, \rightarrow_j)$ is a left residuated po-semigroup.

(2) If $x \leq j(x)$ for all $x \in L$ and $(L, \odot, \rightarrow)$ is left divisible, then $(L, *, \rightarrow_j)$ be left divisible.

(3) If $j(e) = e$ with right identity e, then $(L, *)$ has the right identity e.

(4) If $j(\bigvee_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} j(x_i)$ for all $x_i \in L$ and $(L, \odot, \Rightarrow, \rightarrow)$ is residuated, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \Rightarrow_j y = \bigvee\{z \in L \mid x * z \leq y\}.$$

(5) If $j(x) = a \odot x$ for all $x \in L$ and $(L, \odot, \Rightarrow, \rightarrow)$ is residuated, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \rightarrow_j y = (a \odot x) \rightarrow y = a \rightarrow (x \rightarrow y),$$

$$x \Rightarrow_j y = a \Rightarrow (x \Rightarrow y) = (x \odot a) \Rightarrow y.$$

Proof. (1) If $y_1 \leq y_2$, then $x * y_1 = x \odot j(y_1) \leq x \odot j(y_2) = x * y_2$.

$(x * y) * z = x \odot (j(y) \odot j(z)) = x \odot j(y \odot j(z)) = x * (y * z)$.

Let $x * y = x \odot j(y) \leq z$. Then $x \leq j(y) \rightarrow z = y \rightarrow_j z$. Let $x \leq y \rightarrow_j z$. Then $x * y = x \odot j(y) \leq (y \rightarrow_j z) \odot j(y) = (j(y) \rightarrow z) \odot j(y) \leq z$.

(2) Since $(L, \odot)$ is left divisible, for $x \leq y \leq j(y)$, there exists $a \in L$ such that $x = a \odot j(y)$. So, $a \leq j(y) \rightarrow x$ implies

$$x = a \odot j(y) \leq (j(y) \rightarrow x) \odot j(y) \leq x.$$

Hence $x = a \odot j(y) = a * y$.

(3) $x * e = x \odot j(e) = x$.

(4) It follows from the definitions.

(5) Since $j(x \odot j(y)) = (a \odot x) \odot (a \odot y) = j(x) \odot j(y)$, j is an RA-map. By (1), $(L, *)$ is a po-semigroup. Since $x * y = x \odot a \odot y \leq z$ iff $x \leq a \rightarrow (y \rightarrow z)$ iff $y \leq a \Rightarrow (x \Rightarrow z)$, by Theorem 2.1 (14), the results hold. □

Corollary 3.3. Let $(L, \odot, \Rightarrow)$ be a right-residuated po-semigroup and $j : L \rightarrow L$ an LA-map. Define two operations $*, \rightarrow_j$: $L \times L \rightarrow L$ as $x * y = j(x) \odot y$ and $x \Rightarrow_j y = j(x) \Rightarrow y$. Then the following properties hold:

(1) $(L, *, \Rightarrow_j)$ is a right residuated po-semigroup.

(2) If $x \leq j(x)$ for all $x \in L$ and $(L, \odot, \Rightarrow)$ is right divisible, then $(L, *, \Rightarrow_j)$ be right divisible.

(3) If $j(e) = e$ with left identity e, then $(L, *)$ has the left identity e.

(4) If $j(\bigvee_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} j(x_i)$ for all $x_i \in L$ and $(L, \odot, \Rightarrow, \rightarrow)$ is residuated, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \rightarrow_j y = \bigvee\{z \in L \mid z * x \leq y\}.$$

(5) If $j(x) = x \odot a$ for all $x \in L$ and $(L, \odot, \Rightarrow, \rightarrow)$ is residuated, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \rightarrow_j y = (x \odot a) \rightarrow y = x \rightarrow (a \rightarrow y),$$

$$x \Rightarrow_j y = x \Rightarrow (a \Rightarrow y) = (a \odot x) \Rightarrow y.$$

Example 3.4. (1) Let $(L, \leq, \wedge)$ be a lattice in Example 2.3(1). Let $j(x) = c \odot x$ be given. We obtain:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	a	a	c	c	c
d	0	a	a	c	c	c
1	0	a	a	c	c	c

$\Rightarrow_j$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	1	1	1
b	1	1	1	1	1	1
c	b	b	b	1	1	1
d	b	b	b	1	1	1
1	b	b	b	1	1	1

$\rightarrow_j$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	1	1	1
b	b	1	1	1	1	1
c	b	b	b	1	1	1
d	b	b	b	1	1	1
1	b	b	b	1	1	1

Then $(L, \wedge, \vee, *, \Rightarrow_j, \rightarrow_j, 0, 1)$ is a bounded, residuated po-semigroup lattice.

Example 3.5. Let $L = \{x \in R \mid -4 \leq x \leq 0\}$ be a set and we define an operation $\odot, \Rightarrow = \rightarrow : L \times L \rightarrow L$ as follows:

$$x \odot y = (x+y) \vee (-4), \quad x \Rightarrow y = x \rightarrow y = ((-x)+y) \wedge 0.$$

For $x < y$, we have $(y \rightarrow x) \odot y = y \odot (y \Rightarrow x) = x$. We easily show that $(L, \odot, \Rightarrow, \rightarrow)$ is a commutative, divisible, residuated po-monoid with identity 0.

(1) If $j(x) = n$ for $n \leq x < n+1, n \in Z$, then

$$j(x \odot j(y)) = j(x) \odot j(y)$$

Put $x * y = x \odot j(y)$ and $x \rightarrow_j y = (y - j(x)) \wedge 0$. By Theorem 3.2 (1) and (3), $(L, *, \rightarrow_j)$ is a left-residuated po-semigroup with right identity 0; i.e., $x * 0 = x$ for all $x \in X$. It does not satisfy the condition $x \leq j(x)$ of Theorem 3.2 (2). For $-0.25 < -0.21$, there does not exist $a \in L$ such that $-0.25 = a * (-0.21) = a \odot (-3)$. Furthermore, we cannot define $(-1) \Rightarrow_j (-2)$ because $0 = j(\bigvee_{n \in N}(-\frac{1}{n})) \neq \bigvee_{i \in \Gamma} j(-\frac{1}{n}) = -1$ from the following statements. Suppose $(-1) \Rightarrow_j (-2) = a < 0$. Then there exists $n \in N$ such that $(-1) \Rightarrow_j (-2) = a < -\frac{1}{n} < 0$ but $(-1) * (-\frac{1}{n}) = -2$. It is a contradiction. Suppose $(-1) \Rightarrow_j (-2) = 0$. Then $(-1) * 0 = (-1) \not\leq (-2)$. It is a contradiction.

(2) If $j(x) = n$ for $n - 1 < x \leq n, n \in Z$, then

$$j(x \odot j(y)) = j(x) \odot j(y)$$

Put $x * y = x \odot j(y)$ and $x \rightarrow_j y = (y - j(x)) \wedge 0$. By Theorem 3.2 (1),(2) and (3), $(L, *, \rightarrow_j)$ is a left divisible and left-residuated po-semigroup with right identity 0; i.e. $x * 0 = x$ for all $x \in X$. Since $j(\bigvee_{i \in \Gamma} x_i) = \bigvee_{i \in \Gamma} j(x_i)$ for all $x_i \in L$, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \Rightarrow_j y = \bigvee \{z \in L \mid x * z \leq y\}.$$

(3) If $j(x) = 2n$ for $2n - 2 < x \leq 2n, n \in Z$, then

$$j(x \odot j(y)) = j(x) \odot j(y).$$

Put $x * y = x \odot j(y)$ and $x \rightarrow_j y = (y - j(x)) \wedge 0$. By Theorem 3.2 (1) and (3), $(L, *, \rightarrow_j)$ is a left divisible and left-residuated po-semigroup with right identity 0; i.e. $x * 0 = x$ for all $x \in X$. Since $j(\bigvee_{i \in \Gamma} x_i) = \bigvee_{i \in \Gamma} j(x_i)$ for all $x_i \in L$, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \Rightarrow_j y = \bigvee \{z \in L \mid x * z \leq y\}.$$

(4) If $j(x) = 0$ for $x \in L$, then

$$j(x \odot j(y)) = j(x) \odot j(y) = 0.$$

Put $x * y = x \odot j(y) = x$ and $x \rightarrow_j y = y$. Since $x = (y \rightarrow_j x) * y$ for $x \leq y$, $(L, *, \rightarrow_j)$ is a left divisible and left-residuated po-semigroup with right identity 0; i.e. $x * 0 = x$ for all $x \in X$. Since $j(\bigvee_{i \in \Gamma} x_i) = \bigvee_{i \in \Gamma} j(x_i)$ for all $x_i \in L$, then $(L, *, \rightarrow_j, \Rightarrow_j)$ is a residuated po-semigroup with

$$x \Rightarrow_j y = \bigvee \{z \in L \mid x * z \leq y\}.$$

Then

$$x \Rightarrow_j y = \begin{cases} 0, & \text{if } x \leq y, \\ -4, & \text{otherwise.} \end{cases}$$

Theorem 3.6. Let $(L, \odot, \Rightarrow, \rightarrow)$ be a residuated po-semigroup. Then the following properties hold:

(1) If $j : L \rightarrow L$ an RA-map with $j(j(x)) = j(x)$ for all $x \in L$ and we define an operation $\otimes : L \times L \rightarrow L$ as $x \otimes y = j(x) \odot j(y)$, then $(L, \otimes)$ is a po-semigroup.

(2) If $k : L \rightarrow L$ an LA-map with $k(k(x)) = k(x)$ for all $x \in L$ and we define an operation $\oplus : L \times L \rightarrow L$ as $x \oplus y = k(x) \odot k(y)$, then $(L, \oplus)$ is a po-semigroup.

Proof. (1)

$$\begin{aligned} (x \otimes y) \otimes z &= (j(x) \odot j(y)) \otimes z \\ &= j(j(x) \odot j(y)) \odot j(z) \\ &= j(j(x \odot j(y))) \odot j(z) \\ &= j(x \odot j(y)) \odot j(z) \\ &= j(x \odot j(y) \odot j(z)), \end{aligned}$$

$$\begin{aligned} x \otimes (y \otimes z) &= x \otimes (j(y) \odot j(z)) \\ &= j(x) \odot j(j(y) \odot j(z)) \\ &= j(x) \odot j(j(y \odot j(z))) \\ &= j(x \odot j(y) \odot j(z)). \end{aligned}$$

Hence $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.

□

Example 3.7. Let $(L, \odot, \Rightarrow, \rightarrow)$ be a commutative divisible residuated po-monoid in Example 3.6. In (1), (2) and (3) of Example 3.6, $j(j(x)) = j(x)$ for all $x \in L$. Define

$$x \otimes y = j(x) \odot j(y).$$

Then $(L, \otimes)$ is a po-semigroup.

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