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EXTENSIONS OF EXTENDED SYMMETRIC RINGS

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ABSTRACT. An endomorphism α of a ring R is called *right* (*left*) *symmetric* if whenever $abc = 0$ for $a, b, c \in R$, $ac\alpha(b) = 0$ ($\alpha(b)ac = 0$). A ring R is called *right* (*left*) *α-symmetric* if there exists a right (left) symmetric endomorphism α of R. The notion of an α-symmetric ring is a generalization of α-rigid rings as well as an extension of symmetric rings. We study characterizations of α-symmetric rings and their related properties including extensions. The relationship between α-symmetric rings and (extended) Armendariz rings is also investigated, consequently several known results relating to α-rigid and symmetric rings can be obtained as corollaries of our results.

1. Introduction

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Lambek called a ring R *symmetric* [13] provided $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. Every reduced ring is symmetric ([16, Lemma 1.1]) but the converse does not hold by [2, Example II.5]. Cohn called a ring R *reversible* [5] if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Historically, some of the earliest known results about *reversible* rings (although not so called at the time) were due to Habeb [6]. It is obvious that commutative rings are symmetric and symmetric rings are reversible; but the converses do not hold by [2, Examples I.5 and II.5] and [14, Examples 5 and 7].

Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia called a ring R *Armendariz* [15] if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. This nomenclature was used by them since it was Armendariz who initially showed that a reduced ring always satisfies this condition ([3, Lemma 1]).

For a ring R with a ring endomorphism $\alpha : R \to R$, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x; \alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

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The Armendariz property of a ring was extended to skew polynomial rings but with skewed scalar multiplication in [8, 9]: For an endomorphism α of a ring R, R is called *α-Armendariz* (resp. *α-skew Armendariz*) if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$, $pq = 0$ implies $a_i b_j = 0$ (resp. $a_i \alpha^i(b_j) = 0$) for all $0 \le i \le m$ and $0 \le j \le n$.

On the other hand, an endomorphism α of a ring R is called *rigid* [12] if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is an *α-rigid* ring [7] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings by [7, Proposition 5]. Any α-rigid ring is α-Armendariz [7, Proposition 6], but the converse is not true, in general; every α-Armendariz ring is α-skew Armendariz, but the converse does not hold by [9, Theorem 1.7 and Example 1.8]. In [8, Proposition 3], R is an α-rigid ring if and only if $R[x;\alpha]$ is reduced.

Motivated by the above, in this paper we introduce the notion of an α-symmetric ring for an endomorphism α of a ring R, as a generalization of α-rigid rings and an extension of symmetric rings, and study α-symmetric rings and their related properties. The relationship between α-symmetric rings and extended Armendariz rings is also investigated. Consequently, several known results are obtained as corollaries of our results.

Throughout this paper R denotes an associative ring with identity and α denotes a nonzero and non identity endomorphism, unless specified otherwise.

2. Properties of α-symmetric rings

We begin with the following definition.

Definition 2.1. An endomorphism α of a ring R is called *right* (*left*) *symmetric* if whenever $abc = 0$ for $a, b, c \in R$, $ac\alpha(b) = 0$ ($\alpha(b)ac = 0$). A ring R is called *right* (*left*) *α-symmetric* if there exists a right (left) symmetric endomorphism α of R. R is *α-symmetric* if it is both right and left α-symmetric.

Observe that every subring S with $\alpha(S) \subseteq S$ of a right α-symmetric ring is also right α-symmetric; and any domain R is α-symmetric for any endomorphism α of R, but the converse does not hold (see Example 2.7(1) below).

The next example shows that the concept of α-symmetric is not left-right symmetric.

Example 2.2. Let $\mathbb{Z}$ be the ring of integers. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Note that for $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB = O$ but $BA \neq O$. Thus R is not reversible, and so R is not symmetric.

(i) Let $\alpha : R \longrightarrow R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $ABC = O$ for $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$ and $C = \begin{pmatrix} a'' & b'' \\ 0 & c'' \end{pmatrix} \in R$, then we get $aa'a'' = 0$ and so $aa''a' = 0$. Thus this yields $AC\alpha(B) = O$, and hence R is right α-symmetric. However, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$ with $ABC = O$, we have $\alpha(B)AC \neq O$, and thus R is not left α-symmetric.

(ii) Let $\beta : R \longrightarrow R$ be an endomorphism defined by

$$\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By the similar method to (i), we can show that R is left β-symmetric. However, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ with $ABC = O$, we have $AC\beta(B) \neq O$, and thus R is not right β-symmetric.

Proposition 2.3. (1) *For a ring R, R is right α-symmetric if and only if $ABC = 0$ implies $AC\alpha(B) = 0$ for any three nonempty subsets A, B and C of R.*

(2) *Let R be a reversible ring. R is right α-symmetric if and only if R is left α-symmetric.*

Proof. (1) It suffices to show that $ABC = 0$ for any three nonempty subsets A, B and C of R implies $AC\alpha(B) = 0$, when R is right α-symmetric. Let $ABC = 0$. Then $abc = 0$ for $a \in A$, $b \in B$ and $c \in C$, and hence $ac\alpha(b) = 0$ by the condition. Thus $AC\alpha(B) = \sum_{a\in A, b\in B, c\in C} ac\alpha(b) = 0$.

(2) Let $abc = 0$ for $a, b, c \in R$. If R is right α-symmetric, then $ac\alpha(b) = 0$. Since R is reversible, we have $\alpha(b)ac = 0$ and hence R is left α-symmetric. The converse is similar. □

Example 2.2 shows that the condition "R is reversible" in Proposition 2.3(2) cannot be dropped as well as there exists a right symmetric endomorphism α of a ring R such that R is not symmetric. The next example provides that there exists a commutative reduced ring R which is not α-symmetric for some endomorphism α of R.

Example 2.4. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a commutative reduced ring, and so R is symmetric. Now, let $\alpha : R \longrightarrow R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R. For $a = (1, 0), b = (0, 1), c = (1, 1) \in R$, $abc = 0$ but $ac\alpha(b) = (1, 0) \neq 0$, and thus R is not right α-symmetric.

Recently, the reversible property of a ring is extended to a ring endomorphism in [4] as follows: An endomorphism α of a ring R is called *right reversible* if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$. A ring R is called *right α-reversible* if there exists a right reversible endomorphism α of R. The notion of an α-reversible ring is a generalization of α-rigid rings as well as an extension of reversible rings.

Theorem 2.5. *Let R be a right α-symmetric ring. Then we have the following.*

(1) *For $a, b, c \in R$, $abc = 0$ implies $ac\alpha^n(b) = 0$, $bc\alpha^n(a) = 0$, and $ab\alpha^n(c) = 0$ for any positive integer n, especially, R is a right α-reversible ring.*

(2) *Let α be a monomorphism of R. Then we have the following.*

(i) *R is a symmetric ring.*

(ii) *For $a, b, c \in R$ $abc = 0$ implies $\alpha^n(a)bc = 0$ and $a\alpha^n(b)c = 0$ for any positive integer n. Conversely, if $\alpha^m(a)bc = 0$, $a\alpha^m(b)c = 0$, or $ab\alpha^m(c) = 0$ for some positive integer m, then $abc = 0$.*

Proof. (1) Let $a, b, c \in R$ with $abc = 0$. Since R is right α-symmetric, $ac\alpha(b) = 0$. Then $0 = ac\alpha(b) = (ac)\alpha(b) \cdot 1$ implies $ac\alpha^2(b) = 0$. Continuing this process, we have $ac\alpha^n(b) = 0$ for any positive integer n. Similarly, $1 \cdot a(bc) = 0$ implies $bc\alpha(a) = 0$. By the same method as above, we obtain $bc\alpha^n(a) = 0$ for any positive integer n. Finally, $0 = abc = (ab)c \cdot 1$ implies $ab\alpha(c) = 0$, and thus $ab\alpha^n(c) = 0$ for any positive integer n.

(2) Suppose that α is a monomorphism. (i): Let $a, b, c \in R$ with $abc = 0$. Then $ac\alpha(b) = 0$, and so $\alpha(b)\alpha(ac) = 0$ by (1). Since α is a monomorphism, $bac = 0$ and $acb = 0$. Thus R is symmetric. (ii): Note that R is symmetric and so reversible. Let $abc = 0$. Then $bc\alpha^n(a) = 0$ by (1). Since R is reversible, $\alpha^n(a)bc = 0$. Next, from $abc = 0$ we have $ac\alpha^n(b) = 0$ by (1). Since R is symmetric, $a\alpha^n(b)c = 0$. Conversely, if $\alpha^m(a)bc = 0$ for some positive integer m then $\alpha^m(a)\alpha^m(bc) = \alpha^m(abc) = 0$ by (i), and thus $abc = 0$, since α is a monomorphism. Similarly, if $a\alpha^m(b)c = 0$ then $ac\alpha^m(b) = 0$, since R is symmetric. Hence $\alpha^m(ac)\alpha^m(b) = 0$ by (i), and $acb = 0$ and so $abc = 0$. By the same method as above, we can obtain that $ab\alpha^m(c) = 0$ implies $abc = 0$. □

Corollary 2.6. *Every symmetric ring is reversible.*

Notice that for any positive integer n, "$a\alpha^n(b) = 0$" is equivalent to "$aR\alpha^n(b) = 0$", when R is a right α-symmetric ring with $ab = 0$ for $a, b \in R$: For, $abr = 0$ implies $ar\alpha(b) = 0$ for any $r \in R$. This shows that $ar\alpha^n(b) = 0$ for any positive integer n and any $r \in R$ from Theorem 2.5(1), and thus $aR\alpha^n(b) = 0$.

We remark that the converse of Theorem 2.5(1) does not hold. For example, the ring R with an endomorphism α in Example 2.2(1) is right α-symmetric. However, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = B \in R$, we have $A\alpha^n(B) = O = B\alpha^n(A)$ for any positive integer n but $AB \neq O$.

In the next example, part (1) shows that there exists a right α-symmetric ring R for an automorphism α, but R is not semiprime and so not α-rigid, and part (2) illuminates that there exists a commutative domain and an α-symmetric ring R, but R is not α-rigid where α is not a monomorphism.

Example 2.7. (1) Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Clearly, R is not semiprime and hence R is not α-rigid. Note that α is an automorphism. Moreover, R is right α-symmetric: Indeed, let $ABC = O$ for $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix}$ and $C = \begin{pmatrix} a'' & b'' \\ 0 & a'' \end{pmatrix} \in R$, then we get $aa'a'' = 0$ and $aa'b'' + ab'a'' + ba'a'' = 0$. If $a = 0$ then $ba'a'' = 0$, if $a' = 0$ then $ab'a'' = 0$, and $a'' = 0$ then $aa'b'' = 0$. These imply that $aa''a' = 0$ and $-aa''b' + ab''a' + ba''a' = 0$. Thus $AC\alpha(B) = O$, and hence R is right α-symmetric.

(2) Let $R = F[x]$ be the polynomial ring over a field F. Define $\alpha : R \to R$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. Then R is a commutative domain (and so reduced), but α is not a monomorphism. Since R is a domain, R is right α-symmetric for any endomorphism α of R. However, R is not α-rigid by [8, Example 5(2)].

The class of semiprime rings and the class of right α-symmetric rings do not depend on each other by Example 2.4 and Example 2.7(1). There exists a skew polynomial ring $R[x;\alpha]$ over a symmetric ring R which is not a symmetric ring. For example, consider the commutative ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the automorphism α of R defined by $\alpha((a, b)) = (b, a)$, as in Example 2.4. Then R is a symmetric ring, but $R[x;\alpha]$ is not reversible hence not symmetric: Indeed, for $p = (1, 0), q = (0, 1)x \in R[x;\alpha]$, we get $pq = 0$ but $0 \neq (0, 1)x = qp$.

However, we have the following theorem.

Theorem 2.8. (1) *For a ring R, R is α-rigid if and only if R is semiprime and right α-symmetric and α is a monomorphism.*

(2) *If the skew polynomial ring $R[x;\alpha]$ of a ring R is a symmetric ring, then R is α-symmetric.*

Proof. (1) Let R be α-rigid. Note that any α-rigid ring is reduced and α is a monomorphism by [7, p. 218]. We show that R is right α-symmetric. Assume that $abc = 0$ for $a, b, c \in R$. Then we obtain $bac = 0$, since R is reduced (and so symmetric). Thus $ac\alpha(b)\alpha(ac\alpha(b)) = ac\alpha(bac)\alpha^2(b) = 0$. Since R is α-rigid, $ac\alpha(b) = 0$ and thus R is right α-symmetric.

The converse follows from [4, Proposition 2.5(3)] and Theorem 2.5(1).

(2) Suppose that $abc = 0$ for $a, b, c \in R$. Let $p = a$, $q = b$ and $h = cx$ in $R[x;\alpha]$. Then $pqh = abcx = 0 \in R[x;\alpha]$. Since $R[x;\alpha]$ is symmetric, we get $0 = phq = (ac)xb = ac\alpha(b)x$, and so $ac\alpha(b) = 0$. Thus R is right α-symmetric and therefore R is α-symmetric by Proposition 2.3(2). □

Corollary 2.9 ([10, Proposition 2.7(1)]). *A ring R is reduced if and only if R is a semiprime and symmetric ring.*

Observe that the class of right α-symmetric rings and the class of α-Armendariz rings do not depend on each other by Example 2.7(2) and [11, Example 14].

Theorem 2.10. *Let R be an α-Armendariz ring. The following statements are equivalent:*

(1) *$R[x;\alpha]$ is symmetric.*
(2) *R is α-symmetric.*
(3) *R is right α-symmetric.*
(4) *R is symmetric.*

Proof. (1)⇔(4) by [9, Theorem 3.6 (1)] and (1)⇒(2) by Theorem 2.8 (2). (2)⇒(3) is trivial. Now we show (3)⇒(4). Suppose $abc = 0$ for $a, b, c \in R$. Then $ac\alpha(b) = 0$, and so $acb = 0$ by [9, Proposition 1.3 (2)]. Thus R is symmetric. □

The next result is a direct consequence of Theorem 2.10.

Corollary 2.11 ([10, Proposition 3.4]). *Let R be an Armendariz ring. R is symmetric if and only if $R[x]$ is symmetric.*

Notice that the converse of Theorem 2.8(2) does not hold and the condition "R is an α-Armendariz ring" in Theorem 2.10 are not superfluous by Example 2.7(2): Indeed, consider $A = R[y;\alpha] = F[x][y;\alpha]$. Now, let $p = 1$, $q = xy$ and $h = x \in A$. Then $pqh = 0$, but $phq = x^2y \neq 0$. Hence A is not symmetric. Note that R is not α-Armendariz by [9, Example 1.9].

3. Extensions of α-symmetric rings

Given a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R,R)$ of R, $\bar{\alpha} : T(R,R) \longrightarrow T(R,R)$ defined by

$$\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$$

is an endomorphism of $T(R,R)$. Since $T(R,0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ on $T(R,0)$ to α.

Note that the trivial extension of a reduced ring is symmetric by [10, Corollary 2.4]. For a right α-symmetric ring R, $T(R,R)$ needs not to be an $\bar{\alpha}$-symmetric ring by the next example.

Example 3.1. Consider the right α-symmetric ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

in Example 2.7(1) where α is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

For

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \; B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T(R,R),$$

$ABC = O$ but $AC\bar{\alpha}(B) \neq O$. Thus R is not $\bar{\alpha}$-symmetric.

Recall that another generalization of a symmetric ring is a semicommutative ring. A ring R is *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about semicommutative rings (although not so called at the time) was due to Shin [16]. He proved that any symmetric ring is semicommutative ([16, Proposition 1.4]) but the converse does not hold ([16, Example 5.4(a)]). Semicommutative rings were also studied under the name *zero insertive* by Habeb [6].

Proposition 3.2. *Let R be a reduced ring. If R is an α-symmetric ring, then $T(R,R)$ is an $\bar{\alpha}$-symmetric ring.*

Proof. Let $ABC = O$ for

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \text{ and } C = \begin{pmatrix} a'' & b'' \\ 0 & a'' \end{pmatrix} \in T(R,R).$$

Then we have

(1) $aa'a'' = 0$; and

(2) $aa'b'' + ab'a'' + ba'a'' = 0$.

In the following, we freely use the fact that R is a reduced ring if and only if for any $a, b \in R$, $ab^2 = 0$ (or, $a^2b = 0$) implies $ab = 0$; and every reduced ring is semicommutative. From Eq.(1), we get $aa''a' = 0$ and $aRa'Ra'' = 0$. From Eq.(2)$\times a''$, we have $aa'b''a'' + ab'(a'')^2 + ba'(a'')^2 = 0$ and so $ab'a'' + ba'a'' = 0$. Then $0 = a(ab'a'' + ba'a'') = a^2b'a''$, and hence $ab'a'' = 0$. So Eq.(2) becomes

(3) $aa'b'' + ba'a'' = 0$.

If we multiply Eq.(3) on the left side by a, then $0 = a^2a'b'' = aa'b''$ and so $ba'a'' = 0$. Then $aa'a'' = 0, ab'a'' = 0, aa'b'' = 0$ and $ba'a'' = 0$, and hence we obtain $aa''\alpha(a') = 0, aa''\alpha(b') = 0, ab''\alpha(a') = 0$ and $ba''\alpha(a') = 0$, since R is α-symmetric. Thus $AC\bar{\alpha}(B) = O$ and therefore $T(R, R)$ is $\bar{\alpha}$-symmetric. □

Corollary 3.3 ([10, Corollary 2.4]). *Let R be a reduced ring, then $T(R, R)$ is a symmetric ring.*

The trivial extension $T(R, R)$ of a ring R is extended to a ring

$$T_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

for any $n \geq 3$ and an endomorphism α of a ring R is also extended to the endomorphism $\bar{\alpha} : T_n \to T_n$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

The following example shows that T_n cannot be $\bar{\alpha}$-symmetric for any $n \geq 3$, even if R is an α-rigid ring.

Example 3.4. Let α be an endomorphism of an α-rigid ring R. Note that if R is an α-rigid ring, then $\alpha(e) = e$ for $e^2 = e \in R$ by [7, Proposition 5]. In particular $\alpha(1) = 1$. First, we show that T_3 is not $\bar{\alpha}$-symmetric. For

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3,$$

$ABC = O$. But we have $AC\bar{\alpha}(B) = CB \neq O \in T_3$.

In case of $n \geq 4$, we can also prove that T_n is not $\bar{\alpha}$-symmetric by the same method as the above.

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha} : R[x] \to R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. The *Laurent polynomial ring* $R[x, x^{-1}]$ with an indeterminate x, consists of all formal sums $\sum_{i=k}^{n} a_i x^i$, where $a_i \in R$ and k, n are (possibly negative) integers. The map $\bar{\alpha} : R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ extends α and also is an endomorphism of $R[x, x^{-1}]$. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$.

The following results extend the class of right α-symmetric rings.

Theorem 3.5. *Let R be a ring.*

(1) *$R[x]$ is right $\bar{\alpha}$-symmetric if and only if $R[x; x^{-1}]$ is right $\bar{\alpha}$-symmetric.*

(2) *If R is an Armendariz ring, then R is right α-symmetric if and only if $R[x]$ is right $\bar{\alpha}$-symmetric.*

Proof. (1) It is sufficient to show necessity. Let $f(x), g(x)$ and $h(x) \in R[x;x^{-1}]$ with $f(x)g(x)h(x) = 0$. Then there exists a positive integer n such that $f_1(x) = f(x)x^n, g_1(x) = g(x)x^n$ and $h_1(x) = h(x)x^n \in R[x]$, and so $f_1(x)g_1(x)h_1(x) = 0$. Since $R[x]$ is right $\bar{\alpha}$-symmetric, we obtain $f_1(x)h_1(x)\bar{\alpha}(g_1(x)) = 0$. Hence $f(x)h(x)\bar{\alpha}(g(x)) = x^{-3n}f_1(x)h_1(x)\bar{\alpha}(g_1(x)) = 0$. Thus $R[x;x^{-1}]$ is right $\bar{\alpha}$-symmetric.

(2) It also suffices to establish necessity. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$ and $h(x) = \sum_{k=0}^{l} c_i x^i \in R[x]$ with $f(x)g(x)h(x) = 0$. By [1, Proposition 1], $a_i b_j c_k = 0$ for all i, j and k, and so $a_i c_k \alpha(b_j) = 0$ since R is Armendariz and right α-symmetric. This yields $f(x)h(x)\bar{\alpha}(g(x)) = 0$, and thus $R[x]$ is right $\bar{\alpha}$-symmetric. □

Corollary 3.6. (1) [10, Lemma 3.2(2)] *For a ring R, $R[x]$ is symmetric if and only if so is $R[x;x^{-1}]$.*

(2) [10, Proposition 3.4] *Let R be an Armendariz ring. R is symmetric if and only if $R[x]$ is symmetric.*

Note that Example 2.2(i) and Example 2.4 show that Armendariz rings and right α-symmetric rings do not depend on each other.

For an ideal I of R, if $\alpha(I) \subseteq I$ then $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring R/I. The homomorphic image of a symmetric ring may not necessarily be symmetric by [10, p.163]. One may conjecture that R is α-symmetric if for any right α-symmetric nonzero proper ideal I of R, R/I is $\bar{\alpha}$-symmetric, where I is considered as a ring without identity. However the next example erases the possibility.

Example 3.7. For a field F, consider a ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and an endomorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

For a right ideal $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of R, it can be easily checked that I is right α-symmetric and the factor ring

$$R/I = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \mid a, c \in F\right\}$$

is reduced. Observe that R/I is $\bar{\alpha}$-symmetric, where $\bar{\alpha}$ is an identity map on R/I.

However, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AC\alpha(B) \neq O$ and $ABC = O$. Thus R is not right α-symmetric.

Theorem 3.8. *Let R be a reduced ring and n be any positive integer. If R is right α-symmetric with $\alpha(1)=1$, then $R[x]/\langle x^n\rangle$ is a right $\bar{\alpha}$-symmetric ring, where $\langle x^n\rangle$ is the ideal generated by x^n.*

Proof. Let $S=R[x]/\langle x^n\rangle$. If $n=1$, then $S\cong R$. If $n=2$, then S is $\bar{\alpha}$-symmetric by Proposition 3.2, since $S\cong T(R,R)$. Now, we assume $n\geq 3$. Let $f=a_0+a_1\bar{x}+\cdots+a_{n-1}\bar{x}^{n-1}, g=b_0+b_1\bar{x}+\cdots+b_{n-1}\bar{x}^{n-1}$ and $h=c_0+c_1\bar{x}+\cdots+c_{n-1}\bar{x}^{n-1}\in S$ with $fgh=0$, where $\bar{x}=x+\langle x^n\rangle$. Note that $a_ib_jc_k\bar{x}^{i+j+k}=0$ for all i,j and k with $i+j+k\geq n$. Hence it suffices to show the cases $i+j+k\leq n-1$. Since $fgh=0$, we have the following equations:

(1) $a_0b_0c_0=0$.

(2) $a_0b_0c_1+a_0b_1c_0+a_1b_0c_0=0$.

(3) $a_0b_0c_2+a_0b_1c_1+a_0b_2c_0+a_1b_0c_1+a_1b_1c_0+a_2b_0c_0=0$.

$\vdots$

(n-2) $a_0b_0c_{n-2}+a_0b_1c_{n-3}+\cdots+a_{n-3}b_1c_0+a_{n-2}b_0c_0=0$.

(n-1) $a_0b_0c_{n-1}+a_0b_1c_{n-2}+\cdots+a_{n-2}b_0c_1+a_{n-2}b_1c_0+a_{n-1}b_0c_0=0$.

Recall that R is a reduced ring if and only if for any $a,b\in R$, $ab^2=0$ implies $ab=0$, and every reduced ring is semicommutative. We use these facts in the following.

Eq.(1) and Eq.(2)$\times b_0c_0$ give $a_1(b_0c_0)^2=0$, and so $a_1b_0c_0=0$ and $a_0b_0c_1+a_0b_1c_0=0$; multiplying b_1c_0 gives $0=a_0b_1(c_0)^2=a_0b_1c_0$, so we have

(2)$'$ $a_0b_0c_1=0, a_0b_1c_0=0$ and $a_1b_0c_0=0$.

From Eqs.(1), (2)$'$ and (3)$\times b_0c_0$, we get $a_2b_0c_0=0$ and

(3)$'$ $a_0b_0c_2+a_0b_1c_1+a_0b_2c_0+a_1b_0c_1+a_1b_1c_0=0$,

in a similar way. If we multiply Eq.(3)$'$ on the right side by b_1c_0, b_0c_1, b_2c_0 and b_1c_1 respectively, then we obtain $a_1b_1c_0=0, a_1b_0c_1=0, a_0b_2c_0=0, a_0b_1c_1=0$, and $a_0b_0c_2=0$ in turn.

Inductively we assume that $a_ib_jc_k=0$ for $i+j+k=0,1,\ldots,(n-2)$. We apply the above method to Eq.(n-1). First, the induction hypotheses and Eq.(n-1)$\times b_0c_0$ give $a_{n-1}b_0c_0=0$ and

(n-1)$'$ $a_0b_0c_{n-1}+a_0b_1c_{n-2}+\cdots+a_{n-2}b_0c_1+a_{n-2}b_1c_0=0$.

If we multiply Eq.(n-1)$'$ on the right side by $b_1c_0, b_0c_1,\ldots,$ and b_1c_{n-2} respectively, then we obtain $a_{n-2}b_1c_0=0, a_{n-2}b_0c_1=0,\ldots,a_0b_1c_{n-2}=0$ and so $a_0b_0c_{n-1}=0$, in turn. This shows that $a_ib_jc_k=0$ for all i, j and k with $i+j+k=n-1$. Consequently, $a_ib_jc_k=0$ for all i,j and k with $i+j\leq n-1$, and thus $a_ic_k\alpha^t(b_j)=0$ for any positive integer t by Theorem 2.5(1). This yields $fh\bar{\alpha}(g)=0$, and therefore S is right $\bar{\alpha}$-symmetric. □

Corollary 3.9 ([10, Theorem 2.3]). *If R is a reduced ring, then $R[x]/\langle x^n\rangle$ is a symmetric ring for any positive integer n.*

Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the ring $D=R\times S$ with operations $(r_1,s_1)+(r_2,s_2)=(r_1+r_2,s_1+s_2)$ and $(r_1,s_1)(r_2,s_2)=(r_1r_2+s_1r_2+s_2r_1,s_1s_2)$, where $r_i\in R$

and $s_i \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \longrightarrow D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

In the following, we give some other example of right α-symmetric rings.

Proposition 3.10. (1) *If e is a central idempotent of a ring R with $\alpha(e) = e$ and $\alpha(1-e) = 1-e$, then eR and $(1-e)R$ are right α-symmetric if and only if R is right α-symmetric.*

(2) *If R is a right α-symmetric ring with $\alpha(1) = 1$ and S is a domain, then the Dorroh extension D of R by S is $\bar{\alpha}$-symmetric.*

Proof. (1) It is enough to show the necessity. Suppose that eR and $(1-e)R$ are right α-symmetric. Let $abc = 0$ for $a, b, c \in R$. Then $0 = eabc = a(eb)c$ and $0 = (1-e)abc = a((1-e)b)c$. By hypothesis, we get $0 = ac\alpha(eb) = ace\alpha(a) = eac\alpha(b)$ and $0 = ac\alpha((1-e)b) = ac(1-e)\alpha(b) = (1-e)ac\alpha(b)$. Thus $ac\alpha(b) = eac\alpha(b) + (1-e)ac\alpha(b) = 0$, and therefore R is right α-symmetric.

(2) Let $(r_1, s_1), (r_2, s_2), (r_3, s_3) \in D$ with $(r_1, s_1)(r_2, s_2)(r_3, s_3) = 0$. Then $r_1r_2r_3 + s_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_1s_2r_3 + s_1s_3r_2 + s_2s_3r_1 = 0$ and $s_1s_2s_3 = 0$. Since S is a domain, we get $s_1 = 0$, $s_2 = 0$ or $s_3 = 0$. In the following computations, we freely use the assumption that R is right α-symmetric with $\alpha(1) = 1$. If $s_1 = 0$, then $0 = r_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_2s_3r_1$ and so $0 = r_1(r_3+s_3)\alpha(r_2+s_2) = r_1r_3\alpha(r_2) + r_1s_3\alpha(r_2) + r_1r_3s_2 + r_1s_3s_2$. This yields $(r_1, s_1)(r_3, s_3)\bar{\alpha}((r_2, s_2)) = 0$. Similarly, let $s_2 = 0$. Then $(r_1+s_1)r_2(r_3+s_3) = 0$, and so $(r_1 + s_1)(r_3 + s_3)\alpha(r_2) = 0$, and hence $r_1r_3\alpha(r_2) + s_1s_2\alpha(r_2) + s_3r_1\alpha(r_2) + s_1s_3\alpha(r_2) = 0$. Thus we have $(r_1, s_1)(r_3, s_3)\bar{\alpha}((r_2, s_2)) = 0$. Finally, let $s_3 = 0$. Then $(r_1+s_1)(r_2+s_2)r_3 = 0$, and so $0 = (r_1+s_1)r_3(\alpha(r_2)+s_2) = (r_1r_3 + s_1s_3)\alpha(r_2) + s_2(r_1r_3 + s_1r_3)$. This imply $(r_1, s_1)(r_3, s_3)\bar{\alpha}((r_2, s_2)) = 0$. Therefore the Dorroh extension D is $\bar{\alpha}$-symmetric. □

Corollary 3.11. (1) [10, Proposition 3.6(2)] *For an abelian ring R, R is symmetric if and only if eR and $(1-e)R$ are symmetric for every idempotent e of R if and only if eR and $(1-e)R$ are symmetric for some idempotent e of R*

(2) [10, Proposition 4.2(1)] *Let R be an algebra over a commutative ring S, and D be the Dorroh extension of R by S. If R is symmetric and S is a domain, then D is symmetric.*

Note that the condition "$\alpha(1) = 1$" in Proposition 3.10(2) cannot be dropped by the next example.

Example 3.12. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\alpha : R \longrightarrow R$ be defined by $\alpha((a, b)) = (0, b)$. Consider the Dorroh extension D of R by the ring of integers $\mathbb{Z}$. Then we have

$$((1,0),0)((1,0),-1)((1,0),0) = 0$$

in D, but

$$((1,0),0)((1,0),0)\bar{\alpha}((1,0),-1) = (-(1,0),0) \neq 0$$

in D.

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