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HALF-FACTORIALITY OF $D[S]$

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Abstract. In this note we discussed the half-factoriality of Krull monoid domain $D[S]$ whenever the monoid S has trivial divisor class group.

Throughout we mean by semigroup (resp. monoid), the additive commutative semigroup (resp. additive commutative monoid). Let S be a cancellative monoid with quotient group G. A non empty subset I of G is called a fractional ideal of S if $S + I \subseteq I$ and there exist $s \in S$ such that $s + I \subseteq S$. A fractional ideal is said to be principal if $I = x + S$ for some $x \in G$. Whereas $\mathcal{F}(S)$, the set of all fractional ideals of S is a monoid with zero element S under the binary operation addition, defined as; $I + J = \{i + j : i \in I, \ j \in J\}$, where $I, J \in \mathcal{F}(S)$.

If $I, J \in \mathcal{F}(S)$, then $I : J = \{x \in G : x + J \subseteq I\} \in \mathcal{F}(S)$. The fractional ideal $S : (S : I)$ is called the divisorial ideal associated with I and it is denoted by I_v. If $I = I_v$, then I is known as divisorial. If S is a cancellative monoid with quotient group G, then the v-operation induces an equivalence relation $\sim$ on $\mathcal{F}(S)$ defined by $I \sim J$ if $I_v = J_v$. For $I \in \mathcal{F}(S)$, $div(I)$ represents the equivalence class of I under $\sim$ and the set $\text{Đ}(S)$ of all divisor classes of S, is a monoid under the binary operation addition, defined as; $div(I) + div(J) = div(I + J)$. Moreover $\text{Þ}(S) = \{div(x + S) : x \in G\}$ is the subgroup of the group of invertible elements of $\text{Đ}(S)$.

The set $Cl(S) = \text{Đ}(S)/\text{Þ}(S)$ represent the divisor class monoid of S. If every fractional ideal of S is invertible, then $Cl(S)$ become a group and known as divisor class group of S. Also if S is cancellative and completely integrally closed (that is, let $t \in G$ is said to be almost integral over S if there exist $s \in S$ such that $s + nt \in S$ for some $n \in \mathbb{Z}^+$. If there does not exist any $t \in G - S$, almost integral over S, then S is said to be completely integrally closed.), then $Cl(S)$ becomes a group (cf. [7, Theorem 16.5]).

Following [7, p.190, 191], the torsion free cancellative monoid S with quotient group G, is a Krull monoid if there exists a family $(v_\alpha)_{\alpha \in A}$ of rank-one discrete valuations on G such that S is the intersection of the valuation semigroups of the $v_{\alpha,s}$ and for every $x \in S$, $\{\alpha \in A : v_\alpha(x) > 0\}$ is finite (see also [3, p.1460]).

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Let D be an integral domain with quotient field K, then D-submodule F of K is said to be fractional ideal of D if there exist a nonzero element $a \in D$ such that $aF \subseteq D$. A finitely generated D-submodule of K is a fractional ideal of D. A fractional ideal is finitely generated if it admits a finite set of generators in K and principal if it has a single generator in K. We represent $\mathcal{F}(D)$, the set of all fractional ideals of D in K. If $F \in \mathcal{F}(D)$, then we define $F^{-1} = D : F = \{x \in K : xF \subseteq D\}$ and $F_v = (F^{-1})^{-1} = D : (D : F) = \{x \in K : xF^{-1} \subseteq D\}$.

Equivalently F_v is the intersection of principal fractional ideals of D containing F. The map $F \longmapsto F_v$ is called the *v-operation* on $\mathcal{F}(D)$ and a fractional ideal F is called divisorial or *v-ideal* if $F = F_v$. Define an equivalence relation $\sim$ on $\mathcal{F}(D)$ as; $F \sim F'$ if and only if $F_v = F'_v$ and the equivalence classes under $\sim$ are called divisor classes of D. The class of $F \in \mathcal{F}(D)$ is denoted by $div(F)$. The set of all divisor classes of D is represented by Ð(D). The binary operation addition on Ð(D) is defined as; $div(F) + div(F') = div(FF')$.

Under this operation Ð(D) is a monoid with zero element $div(D)$ and Ð(D) is a group if and only if D is completely integrally closed (cf. [7, pp.208-209]). The set Þ$(D) = \{div(xD) : x \in K, x \neq 0\}$ is a subgroup of $Inv(D)$, the group of invertible elements of Ð(D). The set $Cl(D) =$ Ð$(D)/$Þ(D) represent the divisor class monoid of D and if D is completely integrally closed, then $Cl(D)$ is divisor class group of D(cf. [7, p.209]).

Following Cohn [4], we say that D is an atomic domain if each nonzero nonunit element of D is a product of a finite number of irreducible elements (atoms) of D. In [10, 11] Zaks introduced the notion of half-factorial domain (HFD), which is defined as; an atomic domain D is a half-factorial domain if for each nonzero nonunit element $x \in D$, if $x = x_1x_2 \cdots x_m = y_1y_2 \cdots y_n$, with each x_i, y_j irreducible in D, then $m = n$. A UFD is an HFD but converse is not true. Obviously, if $D[X]$ is an HFD, then surely D is an HFD, but in general half-factorial domains do not behave very well under the polynomial extension, for example $D = \mathbb{R} + X\mathbb{C}[X]$, where $\mathbb{R}$ and $\mathbb{C}$ are real and complex fields respectively, is an HFD, but $D[Y]$ is not an HFD because $(X(1 + iY))(X(1 - iY)) = X^2(1 + Y^2)$ are factors of an element in $D[Y]$ into irreducibles with different size (cf. [1, p.121]).

In [11] Zaks established that, if D is a Krull domain with $|Cl(D)| \leq 2$, then $D[X]$ is an HFD (cf. [11, Theorem 2.4]). So it is natural to observe the examples of half-factorial Krull monoid domain $D[S]$ such that $|Cl(D)| \leq 2$. In first part of this discussion we restated a number of results from [7] regarding Krull monoid S and the Krull monoid domain $D[S]$ which are essential for the construction of half-factorial monoid domains. However we considered that the monoid S is not a group. In second phase of the discussion we considered S as a torsion free group G and then we have observed that which conditions on D and G assure the half-factoriality of the group ring $D[G]$.

1. The case $S \neq G$

When the class group of a monoid domain $D[S]$ is identical to the class group of its coefficient ring D? The following remark answer it.

Remark 1. Let S be a completely integrally closed cancellative monoid with quotient group G. If S has trivial class group, then for Krull domain $D[S]$, $Cl(D[S]) \simeq Cl(D)$ (cf.[7, Corollary 16.8]). Furthermore if $|Cl(D[S])| \leq 2$, then $|Cl(D)| \leq 2$.

Now we reconcile the notions and terminology to discuss the half-factoriality of Krull monoid domain $D[S]$.

Following [7, p.192], let $F = \sum_{\alpha \in A} \mathbb{Z}e_\alpha$ be a free abelian group with free basis $\{e_\alpha\}_{\alpha \in A}$. For $\beta \in A$, the mapping $\pi_\beta : F \to \mathbb{Z}$ defined by $\pi_\beta(\sum_{\alpha \in A} n_\alpha e_\alpha) = n_\beta$, is called the βth projection map on F. It is rank-one discrete valuation on F. The family $\{\pi_\alpha\}_{\alpha \in A}$ is of finite character, and we denote by F_+ the Krull monoid determined by this family; thus $F_+ = \{ \sum_{\alpha \in A} n_\alpha e_\alpha \in F : n_\alpha \geq 0$ for each $\alpha \in A\}$, the positive cone of F under the cardinal order.

In the following we restate [7, Theorem 15.2] as

Remark 2. Let H be the group of invertible elements of the monoid S. The following conditions are equivalent:

(i) S is a Krull monoid.

(ii) $S = H \oplus T$, where $T = M \cap F_+$ for some free group F and some subgroup M of F.

(iii) $S = H \oplus T$ with $T = G \cap F_+$, where F is a free group and G is the quotient group of T.

In remark 2, if 0 is the only invertible element in S, then $H = \{0\}$ and hence $S \simeq T$. This shows $T = M \cap F_+$ or $T = G \cap F_+$ is Krull monoid.

Following [7, p.205], if $T = M \cap F_+$, where $F = \sum_{\alpha\in A} \mathbb{Z}e_\alpha$ is a free abelian group on $\{e_\alpha\}_{\alpha\in A}$, then the monoid domain $D[T]$ can be regarded as a subring of the polynomial ring $D[\{X_\alpha\}_{\alpha\in A}]$ over D. Moreover $D[T]$ is generated as ring over D by pure monomials $X_{\alpha_1}^{e_1} X_{\alpha_2}^{e_2} \cdots X_{\alpha_n}^{e_n}$ with $e_{\alpha_i} \geq 0$ for each i. Conversely, each ring $D[\{m_\beta\}_{\beta\in B}]$, where each m_β is a pure monomial in the indeterminates X_α, is of the from $D[U]$, where U is the submonoid of F_+.

For the sake of better understanding and immediate reference we state [7, Corrolary 15.12] as

Remark 3. Assume that D is an integrally closed Noetherian domain and that $\{X_i\}_{i=1}^n$ is a finite set of indeterminates over D. Let $\{m_\alpha\}_{\alpha\in A}$ be a set of pure monomials in the indeterminates X_i , $1 \leq i \leq n$. Let T be the monoid generated by the pure monomials $\{m_\alpha\}_{\alpha\in A}$ and let $R = D[T]$. The following assertions are equivalent:

(i) T is finitely generated and integrally closed.

(ii) R is Noetherian and integrally closed.
(iii) R is a Krull domain.

The following theorem extend a part of [11, Theorem 2.4] for monoid domain.

Theorem 1. *Let D be an integrally closed Noetherian domain and $D[\{m_\beta\}_{\beta\in B}]$, where each m_β is a pure monomial in the indeterminates $\{X_i\}_{i=1}^n$, is of the from $D[S]$, where S is a finitely generated submonoid of F_+. If $Cl(S) \simeq 0$ and $|Cl(D)| \leq 2$. Then $D[S]$ is an HFD.*

Proof. By remark 3, $D[S]$ is a Krull domain and therefore $Cl(D[S]) \simeq Cl(D) \oplus Cl(S)$, by [7, Corollary 16.8]. As $Cl(S) \simeq 0$, so $Cl(D[S]) \simeq Cl(D)$. This implies $|Cl(D[S])| \leq 2$ and by [10, Theorem 8] or [11, Theorem 1.4] $D[S]$ is an HFD. □

Example 1. By [3, Example 2], the submonoid S of free abelian group $F = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ generated by $(1,0,1)$ and $(0,5,2)$ has trivial class group, whereas $S = G \cap F_+$ such that G is the quotient group of S. So if D is an HFD with $Cl[D) \simeq \mathbb{Z}_2$, then by Theorem 1 $D[S]$ is an HFD.

Remark 4. (i) Ofcourse the half-factoriality of $D[S]$ in Theorem 1 implies the integral closedness of D, which generalizes the case of polynomial ring of Coykendall's [5, Theorem 2.2].

(ii) Since for an abelian group G, there exist a Dedekind domain D such that $G \simeq Cl(D)$ (cf. [6, Theorem 14.10]), therefore $Cl(D[S]) \simeq G \oplus Cl(S)$, where S is a Krull monoid. If $Cl(D[S]) \simeq \mathbb{Z}_2$ and $G = \{0\}$, then $Cl(S) \simeq \mathbb{Z}_2$. So $D[S]$ will also be an HFD.

(iii) As a Krull monoid S is factorial if and only if $Cl(S) \simeq \{0\}$. So we let S be the factorial Krull monoid. If D is a Krull domain with $Cl(D) \simeq \mathbb{Z}_2$, then $D[S]$ is a half-factorial domain.

(iv) Suppose $D[X_1, X_2, \ldots, X_n] \simeq D[Z_0^n]$ be a UFD and S be a Krull monoid such that $Cl(S) \simeq \mathbb{Z}_2$, then $D[Z_0^n \oplus S]$ is an HFD.

(v) Let S be a monoid generated by pure monomials $\{X^2, XY, Y^2\}$. By [2, Example 4.7(1)], for a Dedekind domain $D = \mathbb{Z}[\sqrt{-5}]$, $Cl(D[S]) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $D[S] = \mathbb{Z}[\sqrt{-5}][X^2, XY, Y^2]$ is a Krull domain. Hence it is not necessary that the integral closedness of D (or D to be a Krull domain) implies the half-factorality of monoid domain $D[S]$ for any monoid S. In addition if D is a UFD, then $D[S] = D[X^2, XY, Y^2]$ is a Krull domain and $Cl(D[S]) \simeq \{0\} \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2$. Hence $D[S]$ is an HFD.

2. The case $S = G$

In [9] Kang showed that if G is a torsion free group of type $(0,0,0,\ldots)$ (or cyclic subgroups of G satisfies ACC) and D be a completely integrally closed domain, then $Cl(D) \simeq Cl(D[X;G])$ as groups (cf. [9, Corollary 1]). Moreover D is a Krull domain and G is of type $(0,0,0,\ldots)$ if and only if $D[X;G]$ is a Krull domain (see [9, Corollary 3]). If D is a Krull domain with $Cl(D) \simeq \mathbb{Z}_2$

and G is a torsion free group of type $(0,0,0,\ldots)$, then $D[X;G]$ is a Krull domain with $Cl(D[X;G]) \simeq \mathbb{Z}_2$ and hence an HFD.

If D is a UFD, and G is a torsion free group and each element of G is of type $(0,0,0,\ldots)$, then $D[G]$ is an HFD, in fact it is a UFD (cf. [8, Theorem 7.13]). Obviously $Cl(D[G]) \simeq Cl(D) \simeq 0$. Let D be a field and the torsion free cancellative monoid S is either Z_0 or $\mathbb{Z}$. It follows by [8, Theorem 8.4] that $D[S]$ is a Dedekind domain and in both situations it is an HFD.

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