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SUBADDITIVE SEPARATING MAPS BETWEEN REGULAR BANACH FUNCTION ALGEBRAS

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ABSTRACT. In this note we extend the results of [3] concerning subadditive separating maps from $A = C(X)$ to $B = C(Y)$, for compact Hausdorff spaces X and Y, to the case where A and B are regular Banach function algebras (not necessarily unital) with A satisfying Ditkin's condition. In particular we describe the general form of these maps and get a result on continuity of separating linear functionals.

1. Introduction and preliminaries

Let A and B be two spaces of functions (or any arbitrary algebras). A map H between A and B is called *separating* (*or disjointness preserving*) if $f.g = 0$ implies $H(f) \cdot H(g) = 0$ for all $f, g \in A$. Clearly any algebra homomorphism between two algebras is separating. Weighted composition operators are important typical examples of separating maps between spaces of functions. Moreover, if A and B are both lattices then every lattice homomorphism is a separating map.

The study of separating maps between different spaces of functions (as well as operator algebras) has attracted a considerable interest in recent years, see for example [1, 2, 3, 4, 6, 8, 9]. The general form of linear separating maps between algebras of continuous functions on compact Hausdorff spaces were considered in [7]. Later on in [6] Font extended the results to certain regular Banach function algebras and considered automatic continuity problem on these linear maps. He proved that all separating linear maps between certain regular Banach function algebras are weighted composition operators (on a subset) and linear isometries between regular uniform algebras are automatically separating. For a survey on this topic one can refer to [10]. But very little is known about separating maps between non-normable topological algebras, see for instance [1] for a discussion on additive biseparating maps (bijective separating maps whose inverses are also separating) between $C(X)$ and $C(Y)$,

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where X and Y are completely regular spaces and [9] for characterizing linear separating maps on algebras of differentiable functions on an open subset of $\mathbb{R}^n$.

On the other hand, the well known results concerning linear separating maps from $C(X)$ to $C(Y)$, for compact Hausdorff spaces X and Y, have been extended to not necessarily linear case in [3]. In this paper we revisit the results of [3] and extend some of them to more general cases. In particular we describe the general form of subadditive separating maps between regular Banach function algebras and conclude a continuity result on separating linear functionals.

Let A be a commutative Banach algebra with or without identity. The maximal ideal space of A, which is denoted by $m(A)$, is a locally compact Hausdorff space with respect to Gelfand topology. For an element x in A, let $\hat{x} \in C_0(m(A))$ be its Gelfand transform. We denote the unitization of A by A_1, which is a commutative unital Banach algebra containing A as an ideal. Moreover, $m(A_1)$ is the one-point compactification of $m(A)$.

A commutative Banach algebra A is said to be *regular* if for any closed subset E of $m(A)$ and any point $\varphi \in m(A)\backslash E$, there exists an element $x \in A$ such that $\hat{x}(\varphi) = 1$ and $\hat{x} = 0$ on E.

By a *Banach function algebra* on a locally compact Hausdorff space X we mean a subalgebra A of $C_0(X)$ separating the points of X strongly such that A is a Banach algebra under some topology and the A-topology on X is the given topology. Here "separating strongly" means that A separates the points of X and for each $x \in X$ there exists an element f in A with $f(x) \neq 0$.

Clearly each Banach function algebra is commutative and semisimple. In fact, each commutative semisimple Banach algebra can be considered, through the Gelfand transform, as a Banach function algebra on its maximal ideal space. We note that if $(A, \|\cdot\|_A)$ is a Banach function algebra on a locally compact Hausdorff space X then $\|\cdot\|_\infty \leq \|\cdot\|_A$, where $\|\cdot\|_\infty$ is the supremum norm on X. For each $x \in X$ we use the notation δ_x, for the evaluation homomorphism at x defined on a Banach function algebra on X.

A commutative semisimple Banach algebra $(A, \|.\|)$ is said to satisfy *Ditkin's condition*, if for any $\varphi \in m(A) \cup \{0\}$ and $x \in A$ with $\hat{x}(\varphi) = 0$ there exists a sequence $\{x_n\}$ in A such that each $\hat{x}_n$ is zero on a neighborhood of φ and

$$\|x_n x - x\| \longrightarrow 0.$$

Clearly for a locally compact Hausdorff space X, $C_0(X)$ is an example of a regular Banach function algebra on X satisfying Ditkin's condition, another examples are as follow :

- The Banach algebra $\mathrm{lip}(X, \alpha)$ of all Lipschitz functions f of order α ($\alpha \in (0,1)$) on a compact metric space (X, d) such that $\lim \frac{f(x)-f(y)}{d^\alpha(x,y)} = 0$ as $d(x,y) \longrightarrow 0$, under the usual Lipschitz norm [5, Theorems 4.4.24, 4.4.30].

- For any locally compact abelian group G, Fourier Algebra $A(G)$ on G [5, Theorem 4.5.18], and in general each Segal algebra on G [11].
- The algebras $AC(I)$ and $BVC(I)$ of all absolutely continuous functions and of continuous functions of bounded variations on unit interval $I = [0, 1]$ respectively which are endowed with the following norms [5, Theorem 4.4.35]:

$$\|f\|_{AC} = \|f\|_\infty + \int_0^1 |f'(s)|ds \qquad (f \in AC(I))$$
$$\|f\|_{var} = \|f\|_\infty + var(f) \qquad (f \in BVC(I))$$

2. Main results

Throughout this section $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ are Banach function algebras on their maximal ideal spaces X and Y respectively. Hence, for any $f \in A$ and $x \in X$ we may use $f(x)$ instead of $\hat{f}(x)$.

Let X_∞ and Y_∞ be the one-point compactifications of X and Y respectively. Given an element $f \in A \subseteq C_0(X)$ let $\mathrm{coz}(f)$ denote the cozero set of f, i.e., $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$. For a subset E in X, $cl_{X_\infty}(E)$ stands for the closure of E in X_∞.

For a separating map $H : A \longrightarrow B$ (without any linearity assumption) let us define the set $Y_0 = \{y \in Y : Hf(y) \neq 0 \text{ for some } f \in A\}$. Clearly Y_0 is an open subset of Y and since H is separating it is easily seen that $H0 = 0$ on Y_0.

The standard tools in dealing with separating maps are vanishing sets defined in the following. The definition in not necessarily linear case is the same as in linear case.

Definition 2.1. Let $H : A \longrightarrow B$ be a separating map and let $y \in Y_0$. An open subset U of X_∞ is called a *vanishing set* for $\delta_y \circ H$ if for each $f \in A$, $\mathrm{coz}(f) \subseteq U$ implies $\delta_y \circ H(f) = 0$. The *support* of $\delta_y \circ H$ is then defined by

$$\mathrm{supp}\ \delta_y \circ H = X_\infty \backslash \cup \{V \subseteq X_\infty : V \text{ is a vanishing set for } \delta_y \circ H\}.$$

Using the following standard proposition, we will see that for a certain separating map H, $\mathrm{supp}\ \delta_y \circ H$ is a singleton, for all $y \in Y_0$.

Proposition 2.2. *Let $\mathcal{A}$ be a regular commutative unital Banach algebra. Then for any finite open covering $\{U_i\}_{i=1}^n$ of $m(\mathcal{A})$ there exist elements $e_1, \ldots, e_n \in \mathcal{A}$ such that* $\mathrm{supp}\ \hat{e}_i \subseteq U_i$, $1 \leq i \leq n$, *and* $\sum_{i=1}^n \hat{e}_i = 1$.

Before stating our results we need the following definition borrowed from [3].

Definition 2.3. A map $T : A \longrightarrow B$ is called *pointwise subadditive* if for each $f, g \in A$ and $y \in Y$,

$$|T(f+g)(y)| \leq |Tf(y)| + |Tg(y)|.$$

Theorem 2.4. *If A is regular and $H : A \longrightarrow B$ is a pointwise subadditive separating map, then for every $y \in Y_0$ the set* $\mathrm{supp}\delta_y \circ H$ *is a singleton.*

Proof. The proof is a modification of [3, Theorem 4.2]. Let $y \in Y_0$ and first assume that the set supp $\delta_y \circ H$ is empty. Then $X_\infty = \underset{\alpha}{\cup} U_\alpha$ where $\{U_\alpha\}_\alpha$ is the family of all vanishing sets for $\delta_y \circ H$ (we note that since $H0|_{Y_0} = 0$, the empty set is clearly a vanishing set). By compactness of X_∞ we can choose vanishing sets $U_1, \ldots, U_n$ for $\delta_y \circ H$ such that $X_\infty = \bigcup_{i=1}^n U_i$. Using Proposition 2.2, there exist elements $f_1, \ldots, f_n \in A_1$ with $cl_{X_\infty}(\mathrm{coz}(f_i)) \subseteq U_i$, $1 \leq i \leq n$, and $\sum_{i=1}^n f_i = 1$ on X_∞. In particular, for every $f \in A$, $f = \sum_{i=1}^n f f_i$. Note that $f f_i \in A$, $1 \leq i \leq n$, since A is an ideal in A_1. Obviously $\mathrm{coz}(f f_i) \subseteq U_i$, so that $H(f f_i)(y) = 0$, $1 \leq i \leq n$. Hence by pointwise subadditivity of H,

$$|Hf(y)| = |H(\sum_{i=1}^{n} f f_i)(y)| \leq \sum_{i=1}^{n} |H(f f_i)(y)| = 0$$

for all $f \in A$, which contradicts to y being in Y_0.

Now a similar argument to [3, Theorem 4.2] can be applied to show that, indeed, for each $y \in Y_0$, supp $\delta_y \circ H$ contains exactly one point, as desired. □

Definition 2.5. Under the hypothesis of the preceding theorem we can correspond to each $y \in Y_0$ an element $h(y) \in X_\infty$ which is the unique point of supp $\delta_y \circ H$. We call the mapping $h : Y_0 \longrightarrow X_\infty$ defined in this way the *support map* of H.

The following result can also be obtained with minor modifications of [3, Theorem 4.3], so we omit its proof.

Theorem 2.6. *Let A be regular and $H : A \longrightarrow B$ be a pointwise subadditive separating map. Then*

a) $h(\mathrm{coz}(Hf)) \subseteq cl_{X_\infty}(\mathrm{coz}(f))$, *for all* $f \in A$,

b) $\{h(y)\} = \bigcap_{\delta_y \circ H(f) \neq 0} cl_{X_\infty}(\mathrm{coz}(f))$, $y \in Y_0$.

Before stating more properties of the support map h of a pointwise subadditive separating map H, let we define the following concept introduced in a special case in [3].

Definition 2.7. Let A be regular and $H : A \longrightarrow B$ be a pointwise subadditive separating map with the support map h. Then we call H *strongly pointwise subadditive* if for each $y \in Y_0$ there exists $M_y > 0$ and for each scalar c there exists $\delta_{c,y}$ (depending on c and y) such that

$$|Hf(y) - Hg(y)| \leq M_y |H(f - g)(y)|$$

holds for all $f, g \in A$ with $f(h(y)) = c$ and $|f(h(y)) - g(h(y))| < \delta_{c,y}$.

Significant examples of pointwise and strongly pointwise subadditive separating maps can be found in [3]. In the following we shall show that under strongly pointwise subadditivity condition, the support map h will be continuous.

Lemma 2.8. *Let A be regular and $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. Then*

a) *if $f, g \in A$ and $f = g$ on a neighborhood U in X_∞, then $Hf = Hg$ on $h^{-1}(U)$.*
b) *the support map $h : Y_0 \longrightarrow X_\infty$ of H is continuous.*
c) *if H is injective then $h(Y_0)$ is dense in X_∞.*

Proof. a) Let $y \in Y_0$ and let U be an open neighborhood of $h(y)$ in X_∞. We first consider the case where $f \in A$ and $f = 0$ on U. By definition of h, for each $x \in X_\infty \backslash U$ there exists a vanishing set U_x for $\delta_y \circ H$ containing x. Choose by compactness of $X_\infty \backslash U$, finitely many vanishing sets $U_1, \ldots, U_n$ such that $X_\infty \backslash U \subseteq \bigcup_{i=1}^n U_i$. Set $U_{n+1} = U$, by Proposition 2.2 there exist elements $e_1, \ldots, e_{n+1} \in A_1$, the unitization of A, such that $\text{coz}(e_i) \subseteq U_i$, $1 \leq i \leq n+1$, and $\sum_{i=1}^{n+1} e_i = 1$. Clearly, $fe_i \in A$, for all i, and $fe_{n+1} = 0$. Hence

$$|Hf(y)| = |H(\sum_{i=1}^{n+1} fe_i)(y)| \leq \sum_{i=1}^{n+1} |H(fe_i)(y)| = 0$$

that is $Hf = H0$ on $h^{-1}(U)$.

We now consider the general case. Let $f, g \in A$ and $f = g$ on a neighborhood U in X_∞. By the above argument $H(f-g) = H0$ on $h^{-1}(U)$ and since $H0 = 0$ on Y_0, this means that $H(f-g) = 0$ on $h^{-1}(U)$. For $y \in h^{-1}(U)$ let M_y and $\delta_{f(h(y)),y}$ be as in Definition 2.7. Then since by hypothesis $|f(h(y)) - g(h(y))| = 0 < \delta_{f(h(y)),y}$ we conclude that $|H(f(y) - Hg(y)| \leq M_y |H(f-g)(y)| = 0$ that is $Hf(y) = Hg(y)$ as claimed.

b) Let $y \in Y_0$ and let U be an open neighborhood of $h(y)$ in X_∞. By definition of Y_0 we can find an element $f \in A$ with $Hf(y) \neq 0$. Since X_∞ is compact, $cl_{X_\infty}(V) \subseteq U$ for some open neighborhood V of $h(y)$ in X_∞. Choose by regularity of A_1 an element $g \in A_1$ such that $g = 1$ on V and $g = 0$ on $X_\infty \backslash U$. Since $fg, f \in A$ and $fg = f$ on V, it follows from (a) that $H(fg)(y) = Hf(y) \neq 0$. Therefore $\text{coz}(H(fg))$ is an open neighborhood of y in Y_0. By Theorem 2.6(a), $h(\text{coz}(H(fg)) \subseteq cl_{X_\infty}\text{coz}(fg) \subseteq cl_{X_\infty}(U)$, that is h is continuous at y.

c) Suppose that U is a non-empty open neighborhood in X_∞ and choose by regularity of A a non-zero element $f \in A$ with $cl_{X_\infty}(\text{coz}(f)) \subseteq U$. Since H is assumed to be injective, $Hf(y) \neq 0$, for some $y \in Y_0$. Now Theorem 2.6(a) shows that $h(y) \in cl_{X_\infty}(\text{coz}(f))$, that is $h(Y_0) \cap U \neq \phi$. Hence $h(Y_0)$ is dense in X_∞. □

Theorem 2.9. *Let A be regular and $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. If A satisfies Ditkin's condition then $h(y) \in X$, when $y \in Y_0$ is such that $\delta_y \circ H$ is $\|\cdot\|_A$-continuous.*

Proof. Assume on the contrary that $y \in Y_0$ and $\delta_y \circ H$ is $\|\cdot\|_A$-continuous but $h(y) \notin X$, that is, $h(y) = \infty$. Hence for all $f \in A$, $f(h(y)) = 0$. Let $f \in A$ be chosen such that $Hf(y) \neq 0$, then since A satisfies Ditkin's condition, there exist a sequence $\{f_n\}$ in A and a sequence $\{U_n\}$ of open neighborhoods of $h(y)$ in X_∞ such that $f_n = 0$ on U_n, $n \in \mathbb{N}$, and $\|f_n f - f\|_A \longrightarrow 0$. By continuity of $\delta_y \circ H$ we conclude that $H(f_n f)(y) \longrightarrow H(f)(y)$. Note also for each $n \in \mathbb{N}$, $f_n f = 0$ on U_n, hence it follows from Lemma 2.8 that $H(f_n f)(y) = 0$, for all n, which is a contradiction. □

Now we are ready to present the general form of a strongly pointwise subadditive separating map $H : A \longrightarrow B$ as a generalized composition map (in the sense of the next theorem). Let us first fix some notations. Let A be regular and $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. Assume $y \in h^{-1}(X)$ and U is an open neighborhood of $h(y)$ in X with compact closure in X. By regularity of A we can choose an element $e_{y,U} \in A$ such that $e_{y,U} = 1$ on U. It follows from Lemma 2.8(a) that for each scalar α, $H(\alpha e_{y,U})(y)$ is independent on the choice of U and $e_{y,U}$.

Theorem 2.10. *Let A be regular satisfying Ditkin's condition and let $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. Set $Y_c = \{y \in Y_0 : \delta_y \circ H \text{ is } \|\cdot\|_A - \text{ continuous}\}$. If $y \in Y_c$ then $Hf(y) = H(f(h(y)).e_{y,U})(y)$ for all $f \in A$.*

Proof. Assume first that $y \in Y_c$ and $f \in A$ such that $f(h(y)) = 0$. Since A satisfies Ditkin's condition, there exists a sequence $\{f_n\}$ in A such that each f_n is zero on a neighborhood U_n of $h(y)$ and $\|ff_n - f\|_A \longrightarrow 0$. Hence $ff_n = 0$ on U_n, $n \in \mathbb{N}$, and by Lemma 2.8(a) $H(ff_n)(y) = 0$, for all n. Consequently $Hf(y) = \delta_y \circ H(f) = \lim \delta_y \circ H(ff_n) = 0 = H0(y)$.

We now pass to the general case. Let $y \in Y_c$ and $f \in A$. By preceding theorem $h(y) \in X$. Consider an open neighborhood U of $h(y)$ in X with compact closure in X and then $e_{y,U}$ as above. Obviously $(f - f(h(y))e_{y,U})(h(y)) = 0 < \delta_{f(h(y)),y}$, where $\delta_{f(h(y)),y}$ is chosen by strongly pointwise subadditivity of H. The above argument shows that $H(f - f(h(y))e_{y,U})(y) = 0$. Let M_y be as in Definition 2.7, then

$$|Hf(y) - H(f(h(y))e_{y,U})(y)| \leq M_y |H(f - f(h(y))e_{y,U})(y)| = 0$$

that is $Hf(y) = H(f(h(y))e_{y,U})(y)$ as desired. □

Remark. The above result has been proved in [6, Proposition 5] when A is just regular, H is linear and the set Y_c is defined by

$$Y_c = \{y \in Y : \delta_y \circ H \text{ is } \|\cdot\|_\infty\text{-continuous}\}.$$

But in the proof of Proposition 4 in [6] there is a small gap. In this proof, Font first associates to each $n \in \mathbb{N}$ and $f \in A$ a pair of closed subsets U_n and K_n of X_∞ (where K_n is indeed compact) and then using the regularity of A he gives a sequence $\{g_n\}$ in A such that $g_n|_{K_n} = 1$ and $g_n|_{U_n} = 0$, $n \in \mathbb{N}$. Since we do not know whether the absolute values of all g_n's are less than 1 (or any common bound), the convergence $\|fg_n - f\|_\infty \longrightarrow 0$ is not clear. Indeed, we do not know nothing about the values of g_n at the points belonging to $U_n \backslash K_n$. The same problem exists in the proof of Proposition 5 in [6]. Any way the above theorem shows that if A satisfies Ditkin's condition and if we define Y_c as the set of all $y \in Y_0$ such that $\delta \circ H$ is $\|.\|_A$-continuous (instead of $\|.\|_\infty$-continuous) then the same conclusion holds. But, on the other hand Corollary 2.13 below shows, in particular, that if A is regular and satisfies Ditkin's condition then for any linear separating map $H : A \longrightarrow B$ and each $y \in Y_0$, $\delta_y \circ H$ is $\|.\|_\infty$-continuous if and only if it is $\|.\|_A$-continuous.

The converse of the preceding theorem has been proved in [3] when X and Y are compact Hausdorff spaces, $A = C(X)$, $B = C(Y)$ and $H : C(X) \longrightarrow C(Y)$ is a strongly pointwise subadditive separating map which is 1-bounded in the sense that there exists a scalar $D > 0$ such that

$$\|H(a1)\|_\infty \leq D|a|.\|H1\|_\infty$$

for all scalars a.

The following definition is an extension of the above concept for not necessarily unital Banach function algebras.

Definition 2.11. Let A be regular. We call a strongly pointwise subadditive separating map $H : A \longrightarrow B$ *locally 1-bounded*, if there exists a scalar $D > 0$ such that for each $y \in Y$ we can choose a neighborhood U of $h(y)$ and an appropriate element $e_{y,U}$ in A with $e_{y,U} = 1$ on U such that $\|H(ae_{y,U})\|_B \leq D|a|.\|H(e_{y,U})\|_B$ for all scalars a.

Theorem 2.12. *Let A, B and H be as in Theorem 2.10. If H is locally 1-bounded and $y \in Y_0$ then $y \in Y_c$ if and only if*

$$Hf(y) = H(f(h(y)).e_{y,U})(y) \tag{1}$$

holds for all $f \in A$.

Proof. We first recall that as it was noted earlier, for each $y \in Y_0$, the value $H(f(h(y)).e_{y,U})(y)$ in (1) is independent on the choice of U and $e_{y,U}$.

The "only if" part is a consequence of Theorem 2.10.

Conversely assume that $y \in Y_0$ and that (1) holds for all $f \in A$. Let $\{f_n\}$ be a sequence in A converging to $f \in A$ and let $\delta_{f(h(y)),y}$ and M_y be as in Definition 2.7 and D be as in Definition 2.11. Then $\|f_n - f\|_\infty \leq \|f_n - f\|_A \leq \delta_{f(h(y)),y}$ for all sufficiently large n. Hence, for a suitable choice of $e_{y,U}$ and sufficiently large n we have

$$\begin{aligned}
|\delta_y \circ H(f_n) - \delta_y \circ H(f)| &= |Hf_n(y) - Hf(y)| \leq M_y|H(f_n - f)(y)| \\
&\leq M_y|H((f_n - f)(h(y)).e_{y,U})(y)| \\
&\leq M_y\|H((f_n - f)(h(y)).e_{y,U})\|_B \\
&\leq M_yD|f_n(h(y)) - f(h(y))|.\|H(e_{y,U})\|_B \\
&\leq M_yD\|f_n - f\|_A\|H(e_{y,U})\|_B \longrightarrow 0.
\end{aligned}$$

This shows that $\delta_y \circ H$ is $\|.\|_A$-continuous, i.e., $y \in Y_c$. □

Corollary 2.13. *Let A, B and H be as in Theorem 2.10. If H is locally 1-bounded then for each $y \in Y_0$, $\delta_y \circ H$ is $\|.\|_A$-continuous iff $\delta_y \circ H$ is $\|.\|_\infty$-continuous.*

Proof. If $y \in Y_c$ then by the preceding theorem relation (1) in this theorem holds, for all $f \in A$. Now let $\{f_n\}$ be a sequence in A converging to $f \in A$ in sup-norm. Then for sufficiently large n we have

$$\begin{aligned}|Hf(y) - Hf_n(y)| &\leq M_y|H(f - f_n)(y)| \\ &= |H((f(h(y)) - f_n(h(y))).e_{y,U})(y)| \\ &\leq \|H((f(h(y)) - f_n(h(y))).e_{y,U})\|_B \\ &\leq D|(f - f_n)(h(y))|.\|H(e_{y,U})\|_B\end{aligned}$$

where D and M_y are as in Definitions 2.11 and 2.7. This relation obviously shows that $\delta_y \circ H(f_n) \longrightarrow \delta_y \circ H(f)$, that is $\delta_y \circ H$ is $\|.\|_\infty$-continuous.

Conversely if $y \in Y_0$ is such that $\delta_y \circ H$ is $\|.\|_\infty$-continuous then since $\|\cdot\|_A \geq \|\cdot\|_\infty$ we conclude that $y \in Y_c$. □

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