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# ON AN EXTENSION FORMULAS FOR THE TRIPLE HYPERGEOMETRIC SERIES $X_8$ DUE TO EXTON

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Abstract. The aim of this article is to derive twenty five transformation formulas in the form of a single result for the triple hypergeometric series $X_8$ introduced earlier by Exton. The results are derived with the help of generalized Watson's theorem obtained earlier by Lavoie et al. An interesting special cases are also pointed out.

## 1. Introduction

In 1982, Exton [2] introduced a triple hypergeometric function of the second order $X_8$, whose series representation is given by

$$(1.1)\qquad X_8\,(a,b,c;d,e,f;x,y,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p x^m y^n z^p}{(d)_m(e)_n(f)_p m!n!p!},$$

as usual, we write

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=a(a+1)\cdots(a+n-1)$$
$$(n\in\mathbb{N})\quad\text{and}\quad(a)_0=1,\ \mathbb{N}\text{ being the set of positive integers.}$$

The precise three-dimensional region of convergence of (1.1) is given by Srivastava and Karlsson [8, p. 101, Entry 41a]

$$4r=(s+t-1)^2,\qquad |x|<r,\quad |y|<s\quad\text{and}\quad |z|<t,$$

where the positive quantities $r, s$ and $t$ are associated radii of convergence. For details about this function and many other three-variables hypergeometric functions, we refer to Srivastava and Karlsson [8].

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Exton [2] also gave the following Laplace integral representation of (1.1):

$$\begin{aligned} &X_8(a,b,c;d,e,f;x,y,z) \\ &= \frac{1}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}{}_0F_1(-;d;u^2x){}_1F_1(b;e;uy){}_1F_1(c;f;uz)du \end{aligned} \tag{1.2}$$

provided $\Re e(a) > 0$.

It may be remarked in passing that $X_8$ reduces to Horn's function $H_4$ [9, p. 54] when $z \to 0$ and the Appell's function $F_2$ [9, p. 54] when $x \to 0$.

On the other hand, the well known Kampé de Fériet function is defined and represented as follows [9, p. 63, Eq. (16)]:

$$\begin{aligned} &F^{p:q;k}_{l:m;n}\left[(a_p):(b_q);\ (c_k);(\alpha_l):(\beta_m);(\gamma_n);x,y\right] \\ &= \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s x^r y^s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s r!s!}, \end{aligned} \tag{1.3}$$

where the several cases convergence condition are given in [9, p. 64].

The aim of this paper is to derive twenty five transformation formulas in the form of a single result for the triple hypergeometric series $X_8$ introduced by Exton [2].

We obtain 25 results for the series:

$$X_8\left(a,b,c;d,2b+c,2c+j:x,-x,x\right), \tag{1.4}$$

for $i,j = 0,\pm1,\pm2$.

The results are derived with the help of generalized Watson's theorem obtained earlier by Lavoie et al. [5]. The results easily derived in this paper are potentially useful.

## 2. Results required

In 1992, Lavoie, Grondin and Rathie [5] have given the generalizations of the Watson's theorem on the sum of a ${}_3F_2$ and obtained the following twenty five results, in the form of a single result

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}(a+b+i+1),2c+j;1\end{matrix}\right) \\ &= A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{a+b+i+1}{2})\Gamma\left(c+[\frac{j}{2}]+\frac{1}{2}\right)\Gamma(c-\frac{a+b-j-1}{2}-\frac{1}{2}|i+j|)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\ &\times\left\{B_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j}{4}(1-(-1)^i))\Gamma(c-\frac{1}{2}b+\frac{1}{2}+[\frac{j}{2}])}\right. \\ &\left.+C_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+[\frac{j+1}{2}]+\frac{(-1)^j}{4}(1-(-1)^j))\Gamma(c-\frac{1}{2}b+[\frac{j+1}{2}])}\right\} \end{aligned} \tag{2.1}$$

where $i,j = 0,\pm1,\pm2$.

As usual, $[x]$ is the greatest integer less than or equal to $x$, its modulus is $|x|$. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given respectively in [5].

The case $(i = j = 0)$ of (2.1) reduces immediately to well known Watson's theorem [6]:

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c; 1\end{matrix}\right] = \frac{2^{a+b-2}\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2}),} \tag{2.2}
$$

provided $\Re e(2c - a - b) > -1$.

It is interesting to observe here that, if in (2.1), we take $b = -2n$ and replace $a$ by $a + 2n$ or we let $b = -2n - 1$ and replace $a$ by $a + 2n + 1$. Where $n$ is a nonnegative integer, then in each case, one of the two terms on the right-hand side of (2.1) will vanish and fifty summation formulas are obtained. Under similar conditions, they can be represented as

$$
{}_3F_2\left[\begin{matrix} -2n, a+2n, c \\ \frac{1}{2}(a+i+1), 2c+j; 1\end{matrix}\right] = D_{i,j}\frac{(\frac{1}{2})_n(\frac{1}{2}a - c + \frac{3}{4} - \frac{(-1)^i}{4} - [\frac{j}{2} + \frac{1}{4}(1-(-1)^i)])_n}{(c+\frac{1}{2}+[\frac{j}{2}])_n(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))_n} \tag{2.3}
$$

and

$$
{}_3F_2\left[\begin{matrix} -2n-1, a+2n+1, c \\ \frac{1}{2}(a+i+1), 2c+j; 1\end{matrix}\right] = E_{i,j}\frac{(\frac{3}{2})_n(\frac{1}{2}a - c + \frac{5}{4} + \frac{(-1)^i}{4} - [\frac{j}{2} + \frac{1}{4}(1+(-1)^i)])_n}{(c+\frac{1}{2}+[j+\frac{1}{2}])_n(\frac{1}{2}a+\frac{1}{4}(3-(-1)^i))_n}, \tag{2.4}
$$

where $i, j = 0, \pm 1, \pm 2$. The coefficients $D_{i,j}$ and $E_{i,j}$ are given in [5].

We also recall the following well-known identities involving the Pochhammer symbol in (1.1) (See [7, p. 6–8]):

$$
(\alpha)_{n-p} = \frac{(-1)^p(\alpha)_n}{(1-\alpha-n)_p} \text{ and } (n-p)! = \frac{(-1)^p n!}{(-n)_p} \quad (\alpha = 1); \tag{2.5}
$$

$$
(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n; \tag{2.6}
$$

$$
\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}; \tag{2.7}
$$

$$
\frac{\Gamma(\alpha-2n)}{\Gamma(\alpha)} = \frac{1}{2^{2n}\left(\frac{1}{2}\alpha-\frac{1}{2}\right)_n\left(1-\frac{1}{2}\alpha\right)_n}; \tag{2.8}
$$

$$
(\alpha)_m(\alpha+m)_n = (\alpha)_{m+n}. \tag{2.9}
$$

## 3. Main transformation formulas

The following twenty five transformation formulas in the form of a single result will be established.

$$
\begin{aligned}
&X_8(a,b,c;d,2b+i,2c+j;x,-x,x)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{i,j}\frac{(\frac{1}{2}a)_{m+n}(\frac{a+1}{2})_{m+n}}{(d)_m(c+\frac{1}{2}+[\frac{j}{2}])_n m!n!}\cdot\frac{(b+\frac{1+i}{2}-\frac{1}{4}(1+(-1)^i))_n}{(b+\frac{1}{2}i)_n(b+\frac{i+1}{2})_n}\\
&\times\frac{(\frac{1}{2}b)_n(\frac{b+1}{2})_n(\frac{b+c}{2}+\frac{i}{4}-\frac{1}{8}+\frac{(-1)^i}{8}+\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}{(\frac{1}{2}b+\frac{1}{4}+\frac{i}{4}-\frac{1}{8}(1+(-1)^i))_n(\frac{1}{2}b+\frac{3}{4}+\frac{i}{4}-\frac{1}{8}(1+(-1)^i))_n}\\
&\times\frac{(\frac{b+c}{2}+\frac{i}{4}+\frac{3}{8}+\frac{(-1)^i}{8}+\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n(4x)^m x^{2n}}{(b+c+\frac{i}{2}-\frac{1}{4}+\frac{(-1)^i}{4}+[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}\\
&-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{i,j}\left(\frac{ab}{2b+i}\right)\frac{(\frac{a+1}{2})_{m+n}(\frac{1}{2}a+1)_{m+n}}{(d)_m(c+\frac{1}{2}+[\frac{j+1}{2}])_n m!n!}\\
&\times\frac{(b+\frac{3+i}{2}-\frac{1}{4}(3-(-1)^i))_n}{(b+\frac{i+1}{2})_n(b+\frac{1}{2}i+1)_n}\\
&\times\frac{(\frac{b+1}{2})_n(\frac{1}{2}b+1)_n(\frac{b+c}{2}+\frac{i}{4}+\frac{1}{8}-\frac{(-1)^i}{8}-\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n}{(\frac{1}{2}b+\frac{3}{4}+\frac{i}{4}-\frac{1}{8}(3-(-1)^i))_n(\frac{1}{2}b+\frac{5}{4}+\frac{i}{4}-\frac{1}{8}(3-(-1)^i))_n}\\
&\times\frac{(\frac{1}{2}b+\frac{1}{2}c+\frac{i}{4}+\frac{5}{8}-\frac{(-1)^i}{8}-\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n(4x)^m x^{2n+1}}{(b+c+\frac{i}{2}+\frac{1}{4}-\frac{(-1)^i}{4}+[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n}
\end{aligned}
\tag{3.1}
$$

for $i,j=0,\pm1,\pm2$.

The coefficients $D_{ij}$ and $E_{ij}$ can be obtained from the tables of $D_{ij}$ and $E_{ij}$ given in [5] by replacing $a$ by $1-2b-i-4n$ and $-1-2b-i-4n$, respectively.

## 4. Proof of (3.1)

In order to derive our main transformation formula (3.1), let us first consider some special cases of (2.3) and (2.4) which will be needed in our present investigation.

In (2.3), if we take $a=1-2b-i-4n,\ \ i,j=0,\pm1,\pm2$, then we have

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-2n,1-2b-i-2n,c\\ 1-b-n,2c+j;1\end{matrix}\right]\\
&=D_{i,j}\frac{(\frac{1}{2})_n(\frac{5}{4}-b-c-\frac{i}{2}-2n-\frac{(-1)^i}{4}-[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}{(c+\frac{1}{2}+[\frac{j}{2}])_n(\frac{1}{2}-\frac{i}{2}-b-2n+\frac{1}{4}(1+(-1)^i))_n},
\end{aligned}
$$

which can be written as

$$= D_{i,j}\frac{(\frac{1}{2})_n\Gamma(\frac{5}{4}-b-c-\frac{i}{2}-\frac{(-1)^i}{4}-[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)]-n)}{(c+\frac{1}{2}+[\frac{j}{2}])_n\Gamma(\frac{5}{4}-b-c-\frac{i}{2}-\frac{(-1)^i}{4}-[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)]-2n)}$$
$$\times\frac{\Gamma(\frac{1}{2}-\frac{i}{2}-b+\frac{1}{4}(1-(-1)^i)-2n)}{\Gamma(\frac{1}{2}-\frac{i}{2}-b+\frac{1}{4}(1-(-1)^i)-n)}.$$

Now if we use the identities (2.7) and (2.8) then, after some simplification, we finally have

(4.1)

$${}_3F_2\left[\begin{matrix}-2n, 1-2b-i-2n, c\\ 1-b-2n, 2c+j; 1\end{matrix}\right]$$
$$= D_{i,j}\frac{(\frac{1}{2})_n(\frac{b+c}{2}+\frac{i}{4}+\frac{1}{8}+\frac{(-1)i}{8}+\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}{(c+\frac{1}{2}+[\frac{j}{2}])_n(b+c+\frac{i}{2}-\frac{1}{4}+\frac{(-1)i}{4}+[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}$$
$$\times\frac{(b+\frac{1+i}{2}-\frac{1}{4}(1+(-1)^i))_n(\frac{b+c}{2}+\frac{i}{4}+\frac{3}{8}+\frac{(-1)^i}{8}+\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)])_n}{(\frac{b}{2}+\frac{1+i}{4}-\frac{1}{8}(1+(-1)^i))_n(\frac{b}{2}+\frac{3+i}{4}-\frac{1}{8}(1+(-1)^i))_n}$$

for $i, j = 0, \pm1, \pm2$.

In exactly the same manner, if in (2.4), we take $a = -1-2b-i-4n$, then we obtain the following form:

(4.2)

$${}_3F_2\left[\begin{matrix}-2n-1, -2b-i-2n, c\\ -b-2n, 2c+j; 1\end{matrix}\right]$$
$$= E_{i,j}\frac{(\frac{3}{2})_n(\frac{b+c}{2}+\frac{i}{4}+\frac{1}{8}-\frac{(-1)i}{8}-\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n}{(c+\frac{1}{2}+[j+\frac{1}{2}])_n(b+c+\frac{i}{2}+\frac{1}{4}-\frac{(-1)^i}{4}+[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n}$$
$$\times\frac{(b+\frac{3+i}{2}-\frac{1}{4}(3-(-1)^i))_n(\frac{b+c}{2}+\frac{i}{4}+\frac{5}{8}-\frac{(-1)^i}{8}-\frac{1}{2}[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)])_n}{(\frac{b}{2}+\frac{3+i}{4}-\frac{1}{8}(3-(-1)^i))_n(\frac{b}{2}+\frac{5+i}{4}-\frac{1}{8}(3-(-1)^i))_n}$$

for $i, j = 0, \pm1, \pm2$.

Now we come to the derivation of (3.1). For this, if we denote the left-hand side of (3.1) by $X_8$, then replacing $e$ by $2b+i$, $f$ by $2c+j$, $y$ by $-x$, and $z$ by $x$ in (1.1), we have

$$\begin{aligned} X_8 &= X_8(a; b, c; d, 2b+i, 2c+j; x, -x, x)\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p x^m(-x)^n x^p}{(d)_m(2b+i)_n(2c+j)_p m!n!p!},\end{aligned}$$

which, upon using $(a)_{2m+n+p} = (a+2m)_{n+p}(a)_{2m}$, becomes

$$X_8 = \sum_{m=0}^{\infty}\frac{(a)_{2m}x^m}{(d)_m m!}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+2m)_{n+p}(b)_n(c)_p(-1)^n x^{n+p}}{(2b+i)_n(2c+j)_p n!p!}. \tag{4.3}$$

By making use of a simple formal manipulation for the double series [6, p. 56]:

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A(p,n) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} A(p,n-p)$$

in (4.3), we obtain

$$X_8 = \sum_{m=0}^{\infty} \frac{(a)_{2m} x^m}{(d)_m m!} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(a+2m)_n (b)_{n-p} (c)_p (-1)^{n-p} x^n}{(2b+i)_{n-p} (2c+j)_p (n-p)! p!}. \tag{4.4}$$

Applying (2.7) to (4.4), we get

(4.5)

$$\begin{aligned} X_8 &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}(a+2m)_n (b)_n (-1)^n x^{m+n}}{(d)_m (2b+i)_n m! n!} \sum_{p=0}^{n} \frac{(-n)_p (1-2b-i-n)_p (c)_p}{(1-b-n)_p (2c+j)_p p!} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}(a+2m)_n (b)_n (-1)^n x^{m+n}}{(d)_m (2b+i)_n m! n!} {}_3F_2\left[\begin{matrix} -n, 1-2b-i-n, c \\ 1-b-n, 2c+j; 1 \end{matrix}\right]. \end{aligned}$$

Separating into even and odd powers in (4.5), we have

(4.6)

$$\begin{aligned} X_8 = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}(a+2m)_{2n} (b)_{2n} x^{m+2n}}{(d)_m (2b+i)_{2n} m! (2n)!} {}_3F_2\left[\begin{matrix} -2n, 1-2b-i-2n, c \\ 1-b-2n, 2c+j; 1 \end{matrix}\right] \\ &- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2m}(a+2m)_{2n+1} (b)_{2n+1} x^{m+2n+1}}{(d)_m (2b+i)_{2n+1} m! (2n+1)!} \\ &\qquad\qquad \times {}_3F_2\left[\begin{matrix} -2n-1, -2b-i-2n, c \\ -b-2n, 2c+j; 1 \end{matrix}\right]. \end{aligned}$$

Applying the identities (2.5)–(2.9) to (4.6)

$$\begin{aligned} &(a)_{2m} = 2^{2m}\left(\frac{a}{2}\right)_m \left(\frac{a}{2}+\frac{1}{2}\right)_m, \\ &(a+2m)_{2n} = 2^{2n}\frac{(\frac{1}{2}a)_{m+n}(\frac{1}{2}a+\frac{1}{2})_{m+n}}{(\frac{1}{2}a)_m(\frac{1}{2}a+\frac{1}{2})_m}, \\ &(a+2m)_{2n+1} = \frac{a2^{2n}(\frac{1}{2}a+\frac{1}{2})_{m+n}(\frac{1}{2}a+1)_{m+n}}{\left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}a\right)_m}, \\ &(2n)! = 2^{2n} n! \left(\frac{1}{2}\right)_n, \\ &(2n+1)! = 2^{2n} n! \left(\frac{3}{2}\right)_n, \end{aligned}$$

we have

(4.7)

$$X_8=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}a)_{m+n}(\frac{1}{2}a+\frac{1}{2})_{m+n}(\frac{1}{2}b)_n(\frac{1}{2}b+\frac{1}{2})_n(4x)^m x^{2n}}{(d)_m(\frac{1}{2})_n(b+\frac{i}{2})_n(b+\frac{i}{2}+\frac{1}{2})_n m!n!}$$

$$\times\,{}_3F_2\left[\begin{matrix}-2n,1-2b-i-2n,c\\1-b-2n,2c+j;1\end{matrix}\right]$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{ab}{2b+i}\right)$$

$$\times\frac{(\frac{1}{2}a+\frac{1}{2})_{m+n}(\frac{1}{2}a+1)_{m+n}(\frac{1}{2}b+\frac{1}{2})_n(\frac{1}{2}b+1)_n(4x)^m x^{2n+1}}{(d)_m(\frac{3}{2})_n(b+\frac{i}{2}+\frac{1}{2})_n(b+\frac{i}{2}+1)_n m!n!}$$

$$\times\,{}_3F_2\left[\begin{matrix}-2n-1,-2b-i-2n,c\\-b-2n,2c+j;1\end{matrix}\right].$$

Finally, if we use the results (4.1) and (4.2) for ${}_3F_2(1)$ series and simplify, we arrive at the right hand side of (3.1). This completes the proof of (3.1).

## 5. Special cases

In this section, we shall mention some of the interesting special cases of our main transformation formula.

(i) In (3.1), if we take $i=j=0$, then, we have, after some simplification, the following transformation formula:

(5.1)
$$X_8(a,b,c;d,2b,2c;x,-x,x)$$
$$=F^{2:2;0}_{0:3;1}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}:\frac{1}{2}b+\frac{1}{2}c,\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2};-;\\-:b+\frac{1}{2},c+\frac{1}{2},b+c;d;x^2,4x\end{matrix}\right]$$

which, upon taking $c=b$, yields

(5.2)
$$X_8(a,b,b;d,2b,2b;x,-x,x)$$
$$=F^{2:1;0}_{0:2;1}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}:b;-;\\-:b+\frac{1}{2},2b;d;x^2,4x\end{matrix}\right].$$

(ii) In (3.1), if we take $i=1,j=0$, then we have, after some simplification, the following transformation formula:

(5.3)
$$X_8(a,b,c;d,2b+1,2c;x,-x,x)$$
$$=F^{2:2;0}_{0:3;1}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}:\frac{1}{2}b+\frac{1}{2}c,\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2};-;\\-:b+\frac{1}{2},c+\frac{1}{2},b+c;d;x^2,4x\end{matrix}\right]$$
$$+\frac{a}{2(2b+1)}F^{2:2;0}_{0:3;1}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1:\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2},\frac{1}{2}b+\frac{1}{2}c+1;-;\\-:b+\frac{3}{2},c+\frac{1}{2},b+c+1;d;x^2,4x\end{matrix}\right]$$

which, upon taking $c = b$, yields

$$\begin{aligned} &X_8(a,b,b;d,2b+1,2b;x,-x,x) \\ &= F^{2:1;0}_{0:2;1}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : b; -; \\ - : b+\frac{1}{2}, 2b; d; \quad x^2, 4x \end{matrix}\right] \\ &+ \frac{a}{2(2b+1)} F^{2:1;0}_{0:2;1}\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : b+1; -; \\ - : b+\frac{3}{2}, 2b+1; d; \quad x^2, 4x \end{matrix}\right]. \end{aligned} \tag{5.4}$$

(iii) In (3.1), if we take $i = -1, j = 0$, then we have, after some simplification, the following transformation formula:

$$\begin{aligned} &X_8(a,b,c;d,2b-1,2c;x,-x,x) \\ &= F^{2:2;0}_{0:3;1}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : \frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}, \frac{1}{2}b+\frac{1}{2}c; -; \\ - : b-\frac{1}{2}, c+\frac{1}{2}, b+c-1; d; \quad x^2, 4x \end{matrix}\right] \\ &- \frac{a}{2(2b-1)} F^{2:2;0}_{0:3;1}\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \frac{1}{2}b+\frac{1}{2}c, \frac{1}{2}b+\frac{1}{2}c+\frac{1}{2}; -; \\ - : b+\frac{1}{2}, c+\frac{1}{2}, b+c; d; \quad x^2, 4x \end{matrix}\right] \end{aligned} \tag{5.5}$$

which, upon taking $c = b$, yields

$$\begin{aligned} &X_8(a,;b,b;d,2b-1,2b;x,-x,x) \\ &= F^{2:1;0}_{0:2;1}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : b; -; \\ - : b+\frac{1}{2}, 2b-1; d; \quad x^2, 4x \end{matrix}\right] \\ &- \frac{a}{2(2b-1)} F^{2:1;0}_{0:2;1}\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : b; -; \\ - : b+\frac{1}{2}, 2b; d; \quad x^2, 4x \end{matrix}\right]. \end{aligned} \tag{5.6}$$

Similarly other results can also be obtained.

*Remark.* In our main transformation formula (3.1), if we take $c = b$, then we can again obtain 25 results for the series

$$X_8(a,b,b;d,2b+i,2b+j;x,-x,x)$$

for $i, j = 0, \pm 1, \pm 2$, which are (presumably) new.

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