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VIABILITY FOR SEMILINEAR DIFFERENTIAL EQUATIONS OF RETARDED TYPE

QIXIANG DONG AND GANG LI

ABSTRACT. Let X be a Banach space, $A : D(A) \subset X \to X$ the generator of a compact C_0-semigroup $S(t) : X \to X, t \geq 0$, D a locally closed subset in X, and $f : (a,b) \times C([-q,0];X) \to X$ a function of Caratheodory type. The main result of this paper is that a necessary and sufficient condition in order that D be a viable domain of the semilinear differential equation of retarded type

$$u'(t) = Au(t) + f(t, u_t), t \in [t_0, t_0 + T], u_{t_0} = \phi \in C([-q,0];X)$$

is the tangency condition

$$\liminf_{h \downarrow 0} h^{-1} d(S(h)v(0) + hf(t,v); D) = 0$$

for almost every $t \in (a,b)$ and every $v \in C([-q,0];X)$ with $v(0) \in D$.

1. Introduction

Let X be a real Banach space, $A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semigroup $S(t) : X \to X, t \geq 0$, D a nonempty subset in X. Let q and T be positive numbers and $-\infty \leq a < b \leq +\infty$. Given $t_0 \in (a,b)$, a function $x : [t_0 - q, t_0 + T] \to X$ and $t \in [t_0, t_0 + T]$, define $x_t : [-q, 0] \to X$ by $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-q, 0]$. In this paper we discuss the semilinear differential equation of retarded type:

$$u'(t) = Au(t) + f(t, u_t), \quad t \in [t_0, t_0 + T] \tag{1.1}$$

with the initial condition

$$u_{t_0} = \phi \in C([-q,0];X), \tag{1.2}$$

where $C([-q,0];X)$ denotes the Banach space of continuous X-valued functions on $[-q,0]$ with supermum norm, $f : (a,b) \times C([-q,0];X) \to X$ and $t_0 \in (a,b)$.

We say that D is viable domain for (1.1) if for each $t_0 \in (a,b)$, and $\phi \in C([-q,0];X)$ with $\phi(0) \in D$, there exists at least one mild solution $u : [t_0 -$

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$q, t_0+T] \to X$ to (1.1) and (1.2) with $T = T(t_0, \phi) > 0, t_0 + T < b$, such that $u(t) \in D$ for all $t \in [t_0, t_0+T]$. We recall that by mild solution to (1.1) and (1.2) we mean a continuous function $u : [t_0 - q, t_0 + T] \to X$, satisfying $u_{t_0} = \phi$, and

$$u(t) = S(t-t_0)\phi(0) + \int_{t_0}^{t} S(t-s)f(s, u_s)ds \tag{1.3}$$

for $t \in [t_0, t_0+T]$.

The viability problem for the differential equation

$$u'(t) = Au(t) + F(t, u(t)), t \in [t_0, t_0+T] \tag{1.4}$$

$$u(t_0) = x_0 \in D \tag{1.5}$$

has been studied by many authors by using various frameworks and techniques. In this respect it should be noted the pioneering work of Nagumo [15] who considered the finite dimensional case, $A = 0$ and F is continuous. In this context he showed that a necessary and sufficient condition in order that D be a viable domain for (1.3) is the following tangency condition:

$$\liminf_{h \downarrow 0} h^{-1} d(x + hF(t,x); D) = 0$$

for each $(t, x) \in (a, b) \times D$. It is interesting to note that Nagumo's result (or some variant of it) has been rediscovered several times among others by Brezis [4], Crandall [7], Hartman [9], and Martin [14]. For the development in this area, we refer the readers to Ursescu [22], Pavel [19], Cârjă and Marques [5], Cârjă and Vrabie [6]. Brief reviews of the main contributions in this area can be found in [5] and [6]. We emphasize Pavel's main contribution who was the first who formulated the corresponding tangency condition applying to the semilinear case. More precisely, Pavel [19] showed that, whenever A generates a compact C_0-semigroup and F is continuous on $(a, b) \times D$, where D is locally closed in X, a sufficient condition for viability is

$$\lim_{h \downarrow 0} h^{-1} d(S(h)x + hF(t,x); D) = 0$$

for each $(t, x) \in (a, b) \times D$.

Concerning the differential equations of retarded type, the development was initialed about existence and stability by Travis and Webb [20], [21] and Webb [23], [24]. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors(see, e.g., [1], [2], [11], [13] and references therein). Iacob and Pavel [10] discussed viability problem for semilinear differential equations of retarded type. They proved that, whenever A generates a compact C_0-semigroup and f is continuous from $(a, b) \times C([-q, 0]; X)$ into X a necessary and sufficient condition for viability for (1.1) is

$$\lim_{h \downarrow 0} h^{-1} d(S(h)v(0) + hf(t, v); D) = 0$$

for each $t \in (a,b)$, each $v \in C([-q,0];X)$ with $v(0) \in D$, where D is a locally closed subset in X.

The aim of this paper is to discuss the viable problem of the semilinear differential equation of retarded type (1.1). We prove that a necessary and sufficient condition in order that D be a viable domain of (1.1) is the tangency condition. We only suppose that f is of Caratheodory type. Our result extends and improves that of Iacob and Pavel [10] who considered the case in which f is continuous, and also extends the well-known existence result of Hale [8] who considered the case in which X is finite dimensional and $A = 0$. Moreover, using a standard argument based on Zorn's Lemma, we get the existence of noncontinuable(saturated) mild solutions.

2. Preliminaries

Let X be a real Banach space, $A : D(A) \subset X \to X$ generates a C_0-semigroup $S(t) : X \to X, t \geq 0$. It is well known that in this case $S(t), t \geq 0$ is exponentially bounded, i.e., there are constants $C \geq 1$ and $\omega > 0$ such that

$$||S(t)|| \leq Ce^{\omega t}, \quad \forall t \geq 0.$$

Moreover, if $S(t), t \geq 0$ is a compact semigroup (i.e., $S(t)$ maps bounded subsets into relatively compact subsets for $t > 0$), then $S(t)$ is continuous in the uniformly operator topology for $t > 0$ (see Pazy [19]) and X is separable (see [5]). For more details of semigroups of linear operators, we refer the readers to Pazy [19].

For convenience of future reference, we list the following conditions:

(A1) for each $v \in C([-q,0];X)$, the function $f(\cdot,v) : (a,b) \to X$ is measurable on (a,b);

(A2) for almost every(a.e.) $t \in (a,b)$, $f(t,\cdot) : C([-q,0];X) \to X$ is continuous on $C([-q,0];X)$;

(A3) for every $r > 0$, there is a function $m_r \in L(a,b;X)$ such that $||f(t,v)|| \leq m_r(t)$ for a.e. $t \in (a,b)$ and every $v \in C([-q,0];X)$ with $||v|| \leq r$.

(T) (Tangency condition)

$$\liminf_{h\downarrow 0} h^{-1} d(S(h)v(0) + hf(t,v); D) = 0 \tag{2.1}$$

for a.e. $t \in (a,b)$ and all $v \in C([-q,0];X)$ with $v(0) \in D$, where $d(x,D)$ denotes the distance from $x \in X$ to the subset $D \subset X$.

Since the distance is non-expansive, i.e.,

$$|d(x,D) - d(y,D)| \leq ||x-y||, \qquad \forall x,y \in X,$$

by standard arguments(see [10], [17]), Condition (T) is equivalent to

$$\liminf_{h\downarrow 0} h^{-1} d(S(h)v(0) + h\int_t^{t+h} S(t+h-s)f(t,v)ds; D) = 0 \tag{2.2}$$

for a.e. $t \in (a,b)$ and all $v \in C([-q,0];X)$ with $v(0) \in D$.

We say that the function f is of Caratheodary type if f satisfies (A1)-(A3). A Caratheodory type function has the following Scorza Dragoni property which is nothing but the special case of [3], [12]. We denote by λ the Lebesgue measure on $\mathbb{R}$ and by $\mathcal{L}$, the collection of all Lebesgue measurable sets in $\mathbb{R}$.

Theorem 2.1. *Let X, Y be separable metric spaces and $I = (a,b)$ or $I \in \mathcal{L}((a,b))$. Let $f : I \times X \to Y$ be a function such that $f(\cdot, x)$ is measurable for every $x \in X$ and $f(t, \cdot)$ is continuous for almost every $t \in I$. Then, for each $\varepsilon > 0$, there exists a compact subset $K \subset I$ such that $\lambda(I \setminus K) < \varepsilon$ and the restriction of f to $K \times X$ is continuous.*

Suppose that $u : (a-q, b) \to X$ is continuous. Then the mapping $t \mapsto u_t$, from (a,b) into $C([-q,0]; X)$ is also continuous. The following result is a kind of variance of Lebesgue derivative type, which is useful in the sequel.

Theorem 2.2. *Assume that D is a nonempty subset of a separable Banach space X, $S(t)$ is a C_0-semigroup on X and $f : (a,b) \times C([-q,0]; X) \to X$ is a function which satisfies the conditions* (A1), (A2) *and* (A3). *Then there exists a negligible subset Z of (a,b) such that, for every $t \in (a,b) \setminus Z$, one has*

$$\lim_{h \downarrow 0} h^{-1} \int_t^{t+h} S(t+h-s) f(s, u_s) ds = f(t, u_t) \tag{2.3}$$

for all continuous functions $u : (a,b) \to X$.

The proof of Theorem 2.2 is similar to that of [5] Theorem 2.3. So we omit it.

3. Main result

Now we are ready to state our main result of this paper.

Theorem 3.1. *Let $D \subset X$ be a locally closed subset in a general Banach space, $f : (a,b) \times C([-q,0]; X) \to X$ a function satisfying* (A1)-(A3), *and let $A : D(A) \to X$ be the infinitesimal generator of a compact C_0-semigroup $S(t) : X \to X, t \geq 0$. Then a necessary and sufficient condition in order that D be a viable domain of* (1.1) *is the tangency condition* (T).

Proof of necessity. Let Z be given by Theorem 2.2, let $t_0 \in (a,b) \setminus Z$. Let $v \in C([-q,0]; X)$ such that $v(0) \in D$. By hypothesis, there exists $T = T(t_0, v) > 0$ with $t_0 + T < b$ and a continuous function u satisfying (1.3) with $\phi = v$. Since $u(t_0 + h) \in D$ for all $h \in [0, T]$, we have

$$\begin{aligned} & h^{-1} d(S(h)v(0) + hf(t_0, v); D) \\ \leq \; & h^{-1} \|S(h)v(0) + hf(t_0, v) - u(t_0 + h)\| \\ \leq \; & \|f(t_0, v) - h^{-1} \textstyle\int_{t_0}^{t_0+h} S(t_0 + h - s) f(s, u_s) ds\|. \end{aligned} \tag{3.1}$$

Letting $h \downarrow 0$, one obtains the condition (T). □

In the proof of sufficiency, the following lemma is needed. We first note that, since D is locally closed, there is a real number $r > 0$ such that $D \cap B(\phi(0), r)$ is closed. On the basis of the continuity of ϕ on $[-q, 0]$, there is a real number $T > 0$ such that

$$(3.2) \qquad \|\phi(\theta_1) - \phi(\theta_2)\| \leq \frac{1}{2}r, \quad \forall \theta_1, \theta_2 \in [-q, 0], |\theta_1 - \theta_2| \leq T.$$

Set $R = r + \|\phi(0)\|$ and

$$(3.3) \qquad M = \int_{t_0}^{t_0+T} m_R(t)dt,$$

where m_R is the function appeared in (A3). Moreover, we may choose T small enough such that $t_0 + T < b$ and

$$(3.4) \qquad \max_{0 \leq t \leq T} \|S(t)\phi(0) - \phi(0)\| + N(M + T) \leq \frac{1}{2}r, \quad (N = Ce^{\omega T}).$$

Lemma 3.2. *Suppose that the hypotheses of Theorem 3.1 hold. Suppose further that $f : (a, b) \times C([-q, 0]; X) \to X$ satisfies the tangency condition* (T). *Then for each $t_0 \in (a, b)$, $\phi \in C([-q, 0]; X)$ with $\phi(0) \in D$, each positive integer n, and each open subset $L_n \subset \mathbb{R}$ with $Z \subset L_n$ and $\lambda(L_n) < \frac{1}{n}$, there exist a $\bar{t} \in [t_0, t_0 + T] \setminus Z$, an nondecreasing sequence $\{t_i^n\}_{i=1}^{\infty} \subset [t_0, t_0 + T]$, and an approximate solution u^n on $[t_0, t_0 + T]$ in the following sense:*

(i) $t_0^n = t_0, t_{i+1}^n - t_i^n = d_i^n \leq \frac{1}{n}, \lim_{i \to \infty} t_i^n = t_0 + T$;

(ii) $u_{t_0}^n = \phi, u^n(t_i^n) = x_i^n \in D \cap B(\phi(0), r)$;

(iii) $h_n(s) = f(t_i^n, u_{t_i^n}^n)$ *in case* $t_i^n \notin L_n$ *while* $h_n(s) = f(\bar{t}, u_{t_i^n}^n)$ *in case* $t_i^n \in L_n$ *for* $s \in [t_i^n, t_{i+1}^n)$;

(iv) $u^n(t) = S(t - t_i^n)x_i^n + \int_{t_i^n}^t S(t - s)h_n(s)ds + (t - t_i^n)p_i^n$ *for* $t \in [t_i^n, t_{i+1}^n]$, *where* $x_i^n \in D$ *and* $p_i^n \in X$ *with* $\|p_i^n\| \leq \frac{1}{n}$. *Moreover,* $u_{t_i^n}^n \in B(\phi, r) \cap C([-q, 0]; X)$.

Proof. Let $t_0 \in (a, b)$, $\phi \in C([-q, 0]; X)$ and $n \in \mathbb{N}$ be given. We may assume that (2.2) and (2.3) hold for each $t \in [t_0, t_0 + T] \setminus L_n$. Fix $\bar{t} \in [t_0, t_o + T] \setminus L_n$. We shall construct u^n and t_i^n by induction. Set $t_0^n = t_0, u^n(t_0^n) = \phi(0) = x_0^n, u_{t_0^n}^n = \phi$. To simplify notation, we drop n as a superscript for t_i, x_i, u, p_i etc. Suppose that u is constructed on $[t_0 - q, t_i]$. Then we define t_{i+1} in the following manner. If $t_i = t_0 + T$, set $t_{i+1} = t_0 + T$, and if $t_i < t_0 + T$, then we define t_{i+1} as the following two cases.

Case 1 : $t_i \in L_n$. Set

$$(3.5) \qquad \begin{aligned} \delta_i = \; & \sup\{h \in (0, \tfrac{1}{n}] : t_i + h \leq t_0 + T, [t_i, t_i + h) \subset L_n, \\ & d(S(h)x_i + \textstyle\int_{t_i}^{t_i+h} S(t_i + h - s)f(\bar{t}, u_{t_i})ds; D) \leq \tfrac{h}{2n}\}. \end{aligned}$$

In view of (2.1) and the fact that

$$\lim_{h \downarrow 0} h^{-1} \int_{t_i}^{t_i+h} S(t_i + h - s)f(\bar{t}, u_{t_i})ds = f(\bar{t}, u_{t_i}),$$

one can easily see that $\delta_i > 0$. Choose a number $d_i \in (\frac{1}{2}\delta_i, \delta_i]$, such that

$$d(S(d_i)x_i + \int_{t_i}^{t_i+d_i} S(t_i + d_i - s)f(\bar{t}, u_{t_i})ds; D) \le \frac{d_i}{2n}. \tag{3.6}$$

Define $t_{i+1} = t_i + d_i$. By (3.6), there is $x_{i+1} \in D$ such that

$$\|S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(\bar{t}, u_{t_i})ds - x_{i+1}\| \le \frac{d_i}{n}.$$

Consequently, x_{i+1} can be written as

$$x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(\bar{t}, u_{t_i})ds + (t_{i+1} - t_i)p_i \tag{3.7}$$

with $\|p_i\| \le \frac{1}{n}$. In this case we define u on $[t_i, t_{i+1}]$ as

$$u(t) = S(t - t_i)x_i + \int_{t_i}^{t} S(t - s)f(\bar{t}, u_{t_i})ds + (t - t_i)p_i. \tag{3.8}$$

Case 2 : $t_i \notin L_n$. In this case we set

$$\begin{aligned} \delta_i = \ & \sup\{h \in (0, \tfrac{1}{n}] : t_i + h \le t_0 + T, \\ & d(S(h)x_i + \textstyle\int_{t_i}^{t_i+h} S(t_i + h - s)f(t_i, u_{t_i})ds; D) \le \frac{h}{2n}\}. \end{aligned} \tag{3.9}$$

By (2.2) we see that $\delta_i > 0$. Choose $d_i \in (\frac{1}{2}\delta_i, \delta_i]$, such that

$$d(S(d_i)x_i + \int_{t_i}^{t_i+d_i} S(t_i + d_i - s)f(t_i, u_{t_i})ds; D) \le \frac{d_i}{2n}. \tag{3.10}$$

Define $t_{i+1} = t_i + d_i$. By (3.10), there is $x_{i+1} \in D$ such that

$$\|S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(t_i, u_{t_i})ds - x_{i+1}\| \le \frac{d_i}{n}.$$

Consequently, x_{i+1} can be written as

$$x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)f(t_i, u_{t_i})ds + (t_{i+1} - t_i)p_i \tag{3.11}$$

with $\|p_i\| \le \frac{1}{n}$. In this case we define u on $[t_i, t_{i+1}]$ as

$$u(t) = S(t - t_i)x_i + \int_{t_i}^{t} S(t - s)f(t_i, u_{t_i})ds + (t - t_i)p_i. \tag{3.12}$$

Setting $h(s) = f(\bar{t}, u_{t_i})$ in case $t_i \in L_n$ and $h(s) = f(t_i, u_{t_i})$ in case $t_i \notin L_n$ for $s \in [t_i, t_{i+1}]$. Let us define the step functions α_n and β_n as $\alpha_n(s) = t_i$ in case $t_i \notin L_n$, $\alpha_n(s) = \bar{t}$ in case $t_i \in L_n$ and $\beta_n(s) = t_i$ for $s \in [t_i, t_{i+1})$. Then

h_n can be written as $h(s) = f(\alpha(s), u_{\beta(s)})$. By the induction hypotheses, u can be written in the form

$$(3.13)\quad \begin{aligned} u(t) = \quad & S(t-t_0)\phi(0) + \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} S(t-s)h(s)ds \\ & + \int_{t_i}^{t} S(t-s)h(s)ds + \sum_{m=0}^{i-1} (t_{m+1}-t_m)S(t-t_{m+1})p_m \\ & + (t-t_i)p_i. \end{aligned}$$

Let us check that $u_{t_{i+1}} \in B(\phi, r)$. To do this, we have to estimate $\|u_{t_{i+1}}(\theta) - \phi(\theta)\|$ for each $\theta \in [-q, 0]$. If $-q \le \theta \le t_0 - t_{i+1}$, then by (3.2),

$$\begin{aligned} \|u_{t_{i+1}}(\theta) - \phi(\theta)\| \quad & = \|u(t_0 + (t_{i+1} + \theta - t_0)) - \phi(0)\| \\ & = \|\phi(t_{i+1} + \theta - t_0) - \phi(0)\| \le \tfrac{1}{2}r < r \end{aligned}$$

since $t_{i+1} - t_0 \le T$. If $t_0 - t_{i+1} \le \theta \le 0$, then $t_{i+1} + \theta \ge t_0$, so by (3.13), (3.2) and (3.4), we have

$$\begin{aligned} & \|u_{t_{i+1}}(\theta) - \phi(\theta)\| \\ \le\ & \|u(t_{i+1} + \theta) - \phi(0)\| + \|\phi(0) - \phi(\theta)\| \\ \le\ & \|S(t_{i+1} + \theta - t_0)\phi(0) - \phi(0)\| + N \sum_{m=0}^{i} \int_{t_m}^{t_{m+1}} \|h(s)\|ds \\ & + \sum_{m=0}^{i} (t_{m+1} - t_m)N\|p_m\| + \|\phi(0) - \phi(\theta)\| \\ \le\ & \|S(t_{i+1} + \theta - t_0)\phi(0) - \phi(0)\| + N(M+T) + \|\phi(0) - \phi(\theta)\| \\ \le\ & \frac{1}{2}r + \frac{1}{2}r = r \end{aligned}$$

and hence $u_{t_{i+1}} \in B(\phi, r)$. Using again (3.13), we derive

$$\|u(t) - \phi(0)\| \le \|S(t-t_0)\phi(0) - \phi(0)\| + N(M+T) \le \frac{1}{2}r < r$$

for all $t \in [t_0, t_{i+1}]$, i.e., $u(t) \in B(\phi(0), r)$ for $t \in [t_0, t_{i+1}]$. This remark, along with the fact that $\phi \in C([-q, 0]; X)$, implies that $u_{t_{i+1}} \in B(\phi, r) \cap C([-q, 0]; X)$. Thus, properties (ii), (iii) and (iv) are verified.

To prove property (i), we first note that $\lim_{i\to\infty} t_i$ exists, since $\{t_i\}_{i=1}^{\infty}$ is increasing and $t_i \le t_0 + T$ for all $i = 1, 2, \ldots$. Suppose that $\lim_{i\to\infty} t_i = t^*$, then $t^* \le t_0 + T$. We have to prove $t^* = t_0 + T$. To do this, we first show that $\lim_{i\to\infty} x_i$ also exists. In fact, let $j \ge i$. Using (3.13) for $t = t_i$ and $t = t_j$, we derive

$$
\begin{aligned}
\|x_j - x_i\| \leq\ & \|S(t_i - t_0)(S(t_j - t_i)\phi(0) - \phi(0))\| \\
& + \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_i - s)(S(t_j - t_i)h(s) - h(s))\| ds \\
& + \sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_i - t_{m+1})(S(t_j - t_i)p_m - p_m)\| \\
& + \sum_{m=i}^{j-1} \| \int_{t_m}^{t_{m+1}} S(t_j - s)h(s)ds\| \\
& + \sum_{m=i}^{j-1} (t_{m+1} - t_m)\|S(t_j - t_{m+1})p_m\| \\
\leq\ & N\|S(t_j - t_i)\phi(0) - \phi(0)\| \\
& + N \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\| ds \\
& + N \sum_{m=0}^{i-1} (t_{m+1} - t_m)\|S(t_j - t_i)p_m - p_m\|. \\
& + N \int_{t_i}^{t_j} m_R(s)ds + N(t_j - t_i)\tfrac{1}{n}.
\end{aligned} \tag{3.14}
$$

Now given $\varepsilon > 0$. Since $m_R \in L(a, b; X)$, there is $\eta > 0$ such that $\int_{t'}^{t''} m_R(s)ds \leq \varepsilon/(5N)$ for $t', t'' \in (a, b)$ with $|t'' - t'| < \eta$. By the existence of $\lim_{i\to\infty} t_i = t^*$, there is a positive integer k_0 such that

$$
t_j - t_i < \min\left\{\frac{\varepsilon}{10N(N+1)M}, \frac{\varepsilon}{10(N+1)}, \eta\right\} \tag{3.15}
$$

for all $j > i \geq k_0$. Choose $k_1 > k_0$ with the properties: for $j > i \geq k_1$,

- $\|S(t_j - t_i)\phi(0) - \phi(0)\| \leq \varepsilon/(5N)$;
- $\|S(t_j - t_i)p_m - p_m\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$;
- $\|S(t_j - t_i)f(t_m, u_{t_m}) - f(t_m, u_{t_m})\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$ with $t_m \notin L_n$;
- $\|S(t_j - t_i)f(\bar{t}, u_{t_m}) - f(\bar{t}, u_{t_m})\| \leq \varepsilon/(10NT), \quad 1 \leq m \leq k_0 - 1$ with $t_m \in L_n$.

Then we have

$$
N\|S(t_j - t_i)\phi(0) - \phi(0)\| \leq N\frac{\varepsilon}{5N} = \frac{\varepsilon}{5}; \tag{3.16}
$$

$$
\begin{aligned}
& N \sum_{m=0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\| ds \\
\leq\ & N\Big(\sum_{m=0}^{k_0-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\| ds \\
& + \sum_{m=k_0}^{i-1} \int_{t_m}^{t_{m+1}} \|S(t_j - t_i)h(s) - h(s)\| ds\Big) \\
\leq\ & N(t_{k_0} - t_0)\tfrac{\varepsilon}{10NT} + N(t_i - t_{k_0})(N+1)M \\
\leq\ & \tfrac{\varepsilon}{5};
\end{aligned} \tag{3.17}
$$

$$N\int_{t_i}^{t_j} m_R(s)ds \le N\frac{\varepsilon}{5N} = \frac{\varepsilon}{5}; \tag{3.18}$$

$$\begin{aligned} & N\sum_{m=0}^{i-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m - p_m\| \\ \le\ & N\sum_{m=0}^{k_0-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m - p_m\| \\ & +N\sum_{m=k_0}^{i-1}(t_{m+1}-t_m)\|S(t_j-t_i)p_m - p_m\| \\ \le\ & N(t_{k_0}-t_0)\tfrac{\varepsilon}{10NT} + (t_i - t_{k_0})N(N+1) \\ \le\ & \tfrac{\varepsilon}{5}; \end{aligned} \tag{3.19}$$

$$N(t_j - t_i) < \frac{\varepsilon}{5}. \tag{3.20}$$

From (3.14) to (3.20), we obtain that

$$\|x_j - x_i\| \le \varepsilon \tag{3.21}$$

for all $j > i \ge k_1$, i.e., $\{x_i\}$ is a Cauchy sequence. Therefore $\lim_{i\to\infty} x_i = x^*$ exists, and $x^* \in B(\phi(0), r) \cap D$ since $B(\phi(0), r) \cap D$ is closed. We define $u(t^*) = x^*$. By (iv) we have

$$\|u(t) - x_i\| \le \|S(t - t_i)x_i - x_i\| + (t_i - t)(M + 1)$$

and therefore $\lim_{t\uparrow t^*} u(t) = x^* = u(t^*)$. Accordingly, u is continuous on $[t - q, t^*]$, and hence $\lim_{i\to\infty} u_{t_i} = u_{t^*} \in C([-q, 0]; X) \cap B(\phi, r)$.

We assert that $t^* \notin L_n$ for sufficiently large n. Indeed, if $t^* \in L_n$, then there are only finite many $t_i \notin L_n$ since $[t_0, t^*] \setminus L_n$ is closed. Therefore there is a positive integer i_0 such that $t_i \in L_n$ for all $i \ge i_0$. But then $[t_{i_0}, t^*] \subset L_n$ by (3.5), which contradicts the fact that $\lambda(L_n) < \frac{1}{n}$ for sufficiently large n.

We now assume by contradiction that $t^* < t_0 + T$. We choose $h^* \in (0, \frac{1}{n}]$ such that

$$d(S(h^*)x^* + \int_{t^*}^{t^*+h^*} S(t^* + h^* - s)f(t^*, u_{t^*})ds; D) \le \frac{h^*}{4n}. \tag{3.22}$$

Since $\frac{1}{2}\delta_i < d_i$ and $d_i = t_{i+1} - t_i \to 0$ as $i \to \infty$, there is a positive integer i_0 such that $\delta_i < h^*$ for all $i > i_0$. On the basis of (3.9), we have

$$d(S(h^*)x^* + \int_{t_i}^{t_i+h^*} S(t_i + h^* - s)f(t^*, u_{t^*})ds; D) > \frac{h^*}{2n} \tag{3.23}$$

for $i > i_0$ and $t_i \notin L_n$. Letting $i \to \infty$ in (3.23), one obtains an inequality which contradicts (3.22). Hence $t^* = t_0 + T$, which concludes the proof. □

Proof of sufficiency. Let $\{L_n\}$ be a sequence of open subsets of $\mathbb{R}$ such that $Z \subset L_n$ and $\lambda(L_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Take $L = \cap_{n\ge 1} L_n$ and a sequence of

n-approximate solutions $\{u^n\}$ and $\{t_i^n\}$ obtained in Lemma 3.2. Let us define

$$g_n(t) = \sum_{m=0}^{i-1}(t_{m+1}^n - t_m^n)S(t - t_{m+1}^n)p_m^n + (t - t_i^n)p_i^n$$

for $t \in [t_i, t_{i+1}]$. Then $\|g_n(t)\| \leq \frac{NT}{n}$ for all $t \in [t_0, t_0 + T]$ and u^n can be written in the form

$$u^n(t) = S(t - t_0)\phi(0) + \int_{t_0}^{t} S(t - s)h_n(s)ds + g_n(t) \tag{3.24}$$

for all $t \in [t_0, t_0 + T], u_{t_0}^n = \phi$. Set

$$y^n(t) = \int_{t_0}^{t} S(t - s)h_n(s)ds, \quad t \in [t_0, t_0 + T].$$

Since the semigroup $S(t) : X \to X, t \leq 0$, is compact and $\{h_n\}$ is uniformly integrable on $[t_0, t_0 + T]$, by a standard argument involving a compactness result, it follows that there is a $y \in C([t_0, t_0 + T]; X)$ such that at least on a subsequence we have

$$\lim_{n\to\infty} y^n(t) = y(t)$$

uniformly in $t \in [t_0, t_0+T]$. Since $\|g_n(t)\| \leq \frac{NT}{n}$ for all $t \in [t_0, t_0+T]$, it follows that

$$\lim_{n\to\infty} u^n(t) = S(t - t_0)\phi(0) + y(t) \equiv u(t) \tag{3.25}$$

uniformly in $t \in [t_0, t_0 + T]$. Let us observe that if $s \notin L$, then $s \notin L_n$ for sufficiently large n, and then we have $\alpha_n(s) \to s$ as $n \to \infty$. Also we have $\beta_n(s) \to s$ as $n \to \infty$ for all $s \in [t_0, t_0 + T]$. Therefore $h_n(s) \to f(s, u_s)$ as $n \to \infty$ for a.e. $s \in [t_0, t_0 + T]$. Moreover, $u^n(\alpha_n(s)) \in D \cap B(\phi(0), r)$ implies $u(s) \in D \cap B(\phi(0), r)$(which is closed). Finally, passing to limit in (3.24), one obtains (1.3), which completes the proof. □

Concerning the continuation of the solution to (1.1) satisfying (1.2). Recall that a solution $v : [t_0, t_0 + T_1] \to X$ of (1.1), with $T_1 \geq T$ is said to be a continuation to the right of the solution $u : [t_0, t_0 + T] \to X$ to (1.1), if $v(t) = u(t)$ for all $t \in [t_0, t_0 + T]$. A solution u is said to be noncontinuable if it has no proper continuation. Using a standard argument based on Zorn's Lemma, one can easily verify that, if the hypotheses of Theorem 3.1 hold, and $u : [t_0, b_0) \to X$ is a noncontinuable mild solution to (1.1) satisfying (1.2), then either $b_0 = b$ or $\lim_{t\uparrow b_0} \|u(t)\| = +\infty$. Moreover, the tangency condition (T) is also necessary. Precisely, we have

Theorem 3.3. *Under the hypotheses of Theorem 3.1, a necessary and sufficient condition in order that for each $t_0 \in (a, b)$, and each $\phi \in C([-q, 0]; X)$ with $\phi(0) \in D$, there is a noncontinuable mild solution $u(t) \in D$ to (1.1) satisfying (1.2) is the tangency condition (T).*

Remark 3.4. If, in addition to the hypotheses of Theorem 3.1, we suppose that $\phi(\theta) \in D$ for all $\theta \in [-q, 0]$, then there exists a solution to (1.1) and (1.2) with $u(t) \in D$ for all $t \in [t_0 - q, t_0 + T]$.

Remark 3.5. If D is open, then the tangency condition (T) is automatically satisfied. In this case, by Theorem 3.1, one obtains the locally existence result of problem (1.1) and (1.2), which extends the well-known result of J. K. Hale [8], who considered the case in which X is finite dimensional (i.e., $X = \mathbb{R}^n$) and $A = 0$.

Theorem 3.6. *Let X be a real Banach space X, $f : (a, b) \times C([-q, 0]; X) \to X$ a function satisfying* (A1)-(A3), *and let A be the infinitesimal generator of a compact C_0-semigroup $S(t) : t \geq 0$. Then for each $t_0 \in (a, b)$, and each $\phi \in C([-q, 0]; X)$ with $\phi(0) \in D$, the problem* (1.1) *and* (1.2) *has a locally mild solution, for some $T = T(t_0, \phi) > 0$, with $T < b - t_0$.*

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Qixiang Dong
School of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China
E-mail address: qxdongyz@yahoo.com.cn

Gang Li
School of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China
E-mail address: gli@yzu.edu.cn