

STRUCTURES OF IDEMPOTENT MATRICES OVER CHAIN SEMIRINGS

KYUNG-TAE KANG, SEOK-ZUN SONG, AND YOUNG-OH YANG

ABSTRACT. In this paper, we have characterizations of idempotent matrices over general Boolean algebras and chain semirings. As a consequence, we obtain that a fuzzy matrix $A = [a_{i,j}]$ is idempotent if and only if all $a_{i,j}$ -patterns of A are idempotent matrices over the binary Boolean algebra $\mathbb{B}_1 = \{0, 1\}$. Furthermore, it turns out that a binary Boolean matrix is idempotent if and only if it can be represented as a sum of line parts and rectangle parts of the matrix.

1. Introduction and preliminaries

It is well-known that over any field the structure of idempotent matrices is very simple, that is, each idempotent matrix is similar to a diagonal matrix with 0 and 1 on the main diagonal.

In general, the characterization of idempotent matrices in abstract algebraic systems is a vital problem that is crucial for the understanding the structure of these systems and in many other applications ([3, 6]). For matrices over algebraic systems that are not fields, this problem is far from being solved yet.

A *semiring* ([4]) is essentially a ring in which only the zero is required to have an additive inverse. Recently, there are many papers on the study of matrix theory over semirings. But there are few papers on the characterizations of idempotent matrices over semirings. Beasley and Pullman ([2]) characterized linear operators on the matrices over semirings strongly preserving idempotents (that map idempotents to idempotents and non-idempotents to non-idempotents). Bapat et al. ([1]) obtained characterizations of nonnegative real idempotent matrices.

For a fixed positive integer k , let \mathbb{B}_k be the (*general*) *Boolean algebra* of subsets of a k -element set \mathbb{S}_k and $\sigma_1, \sigma_2, \dots, \sigma_k$ denote the singleton subsets of \mathbb{S}_k . Union is denoted by $+$ and intersection by juxtaposition; 0 denotes the null set and 1 the set \mathbb{S}_k . Under these two operations, \mathbb{B}_k is a commutative semiring(that is, only 0 has an additive inverse); all of its elements, except 0

Received October 23, 2006; Revised November 12, 2007.

2000 *Mathematics Subject Classification.* 15A21, 15A33.

Key words and phrases. semiring, idempotent, frame, rectangle part, line part.

and 1, are zero-divisors. In particular, if $k = 1$, \mathbb{B}_1 is called the *binary Boolean algebra*.

Let \mathbb{K} be any set of two or more elements. If \mathbb{K} is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements $x, y \in \mathbb{K}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all $x, y \in \mathbb{K}$. If \mathbb{K} has a universal lower bound and a universal upper bound, then \mathbb{K} becomes a semiring, and called a *chain semiring*. The following are interesting examples of a chain semiring.

Let \mathbb{H} be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x \in \mathbb{H}} x$, and $1 = \bigcup_{x \in \mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H} \cup \{0, 1\}$ is a chain semiring.

Let α, w be real numbers with $\alpha < w$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq w\}$. Then \mathbb{S} is a chain semiring with $\alpha = 0$ and $w = 1$. It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq w\}$. Furthermore, if we choose the real numbers 0 and 1 as α and w in the previous example, then $\mathbb{F} \equiv \{\beta : 0 \leq \beta \leq 1\}$ is called *fuzzy semiring*.

In particular, if we take \mathbb{H} to be a singleton set, say $\{a\}$, and denote \emptyset by 0 and $\{a\}$ by 1, the resulting chain semiring becomes the binary Boolean algebra $\mathbb{B}_1 = \{0, 1\}$, and it is a subsemiring of every chain semiring. Since any general Boolean algebra \mathbb{B}_k ($k \geq 2$) is not totally ordered under inclusion, it is not a chain semiring.

Hereafter, unless otherwise specified, \mathbb{S} denote a semiring which is either a general Boolean algebra \mathbb{B}_k or a chain semiring \mathbb{K} .

Let $\mathbb{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices with entries in \mathbb{S} . The usual definitions for addition, multiplication by scalars, and the product of matrices over fields are applied to $\mathbb{M}_n(\mathbb{S})$ as well. The zero matrix is denoted by O_n , the identity matrix by I_n and the matrix with all entries equal to 1 is denoted by J_n . The matrix in $\mathbb{M}_n(\mathbb{S})$ all of whose entries are zero except its $(i, j)^{\text{th}}$, which is 1, is denoted by $E_{i,j}$. We call this a *cell*. When $i \neq j$, we say $E_{i,j}$ is an *off-diagonal* cell; $E_{i,i}$ is a *diagonal* cell.

A matrix $A \in \mathbb{M}_n(\mathbb{S})$ is called *idempotent* if $A^2 = A$. The matrices O_n, I_n and J_n are clearly idempotents in $\mathbb{M}_n(\mathbb{S})$. Furthermore we can easily show that all diagonal cells are idempotents, but all off-diagonal cells are not idempotents.

In this paper, we deal with idempotent matrices over Boolean algebras and chain semirings including fuzzy semiring. In Section 2, we characterize idempotent matrices over the binary Boolean algebra. Also in Section 3, we obtain characterizations of idempotent matrices over chain semirings.

2. The binary Boolean case

In this section, we shall characterize idempotent matrices over the binary Boolean algebra $\mathbb{B}_1 = \{0, 1\}$.

The following is an immediate consequence of the rules of matrix multiplication.

Proposition 2.1. *For any cells $E_{i,j}$ and $E_{u,v}$, we have $E_{i,j}E_{u,v} = E_{i,v}$ or O_n according as $j = u$ or $j \neq u$.*

For a matrix $A = [a_{ij}]$ in $\mathbb{M}_n(\mathbb{B}_1)$, A can be written uniquely as $\sum_{i,j=1}^n a_{ij}E_{i,j}$.

Since $a_{ij} \in \{0, 1\}$, the matrix A is a sum of cells.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are in $\mathbb{M}_n(\mathbb{B}_1)$, we say B dominates A (written by $A \sqsubseteq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all $i, j = 1, \dots, n$. This provides a reflexive and transitive relation on $\mathbb{M}_n(\mathbb{B}_1)$.

For a matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{B}_1)$, if $a_{ij} = 1$ for some i and j , then we say that the cell $E_{i,j}$ is in A . Thus the cell $E_{i,j}$ is in A if and only if $E_{i,j} \sqsubseteq A$.

Lemma 2.2. *Let E_1, \dots, E_m and F be cells in an idempotent matrix $X \in \mathbb{M}_n(\mathbb{B}_1)$, where $m \geq 2$. Then*

- (i) *if $E_1 \cdots E_m$ is not zero, then it is a cell in X ;*
- (ii) *if F is off-diagonal, then there exist distinct cells G and H in A such that $F = GH$. Moreover if both cells G and H are off-diagonal, then three cells F, G and H are mutually distinct.*

Proof. (i) Clearly $E_1 \cdots E_m$ is a cell by Proposition 2.1. Let A, B, C and D be matrices in $\mathbb{M}_n(\mathbb{B}_1)$. Then we can easily show that if $A \sqsubseteq B$ and $C \sqsubseteq D$, then $AC \sqsubseteq BD$. It follows from $E_i \sqsubseteq X$ for all $i = 1, \dots, m$ that $E_1 \cdots E_m \sqsubseteq X^m$. Since X is idempotent, we have $X^m = X$ for $m \geq 2$ so that $E_1 \cdots E_m$ is in X .

(ii) Let F_1, \dots, F_k be cells in X so that $X = \sum_{i=1}^k F_i$. Since X is idempotent, we have

$$\sum_{i=1}^k F_i^2 + \sum_{i,j=1, i \neq j}^k F_i F_j = X^2 = X = \sum_{i=1}^k F_i.$$

Since $F \sqsubseteq X$, we have either $F \sqsubseteq F_i^2$ or $F \sqsubseteq F_i F_j$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$. Since F is off-diagonal, it follows from Proposition 2.1 that $F \not\sqsubseteq F_i^2$. Thus we have $F \sqsubseteq F_i F_j$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$. If we let $G = F_i$ and $H = F_j$, then $F = GH$. Furthermore if G and H are off-diagonal, then F, G and H are mutually distinct by Proposition 2.1. □

Corollary 2.3. *If all cells in $A \in \mathbb{M}_n(\mathbb{B}_1)$ are off-diagonal, then A is not idempotent.*

Proof. On the contrary, assume that A is idempotent. Let $\Phi = \{F_1, \dots, F_m\}$ be the set of all cells in A , where each F_i is off-diagonal. We shall show that there exists an infinite subset of cells in Φ , which is impossible. We proceed by induction on the number of cells in Φ . Since A is idempotent, by Lemma 2.2(ii), there exist indices $i, j \in \{1, \dots, m\}$ such that $F_i F_j = F_1$ and the three cells F_i, F_j and F_1 are mutually distinct. By Proposition 2.1, we can write $F_i = E_{a,x_1}, F_j = E_{x_1,b}$ and $F_1 = E_{a,b}$ for certain mutually distinct indices a, b and x_1 . Since $F_i = E_{a,x_1} \in \Phi$ and A is idempotent, it follows from Lemma 2.2(ii) that there exist two cells $E_{a,x_2}, E_{x_2,x_1} \in \Phi$ such that $E_{a,x_1} = E_{a,x_2} E_{x_2,x_1}$ for some index x_2 different from a and x_1 . Assume that for some $k \geq 2$, the set of distinct cells $\{E_{a,x_1}, \dots, E_{a,x_k}\} \subseteq \Phi$ was already

constructed. Then we may add a new element to this set as follows. Since A is idempotent, by Lemma 2.2(ii), there exist two cells $E_{a,x_{k+1}}, E_{x_{k+1},x_k} \in \Phi$ such that $E_{a,x_k} = E_{a,x_{k+1}}E_{x_{k+1},x_k}$ for some index x_{k+1} different from a and x_k . Assume that there exists an index $i \in \{1, \dots, k-1\}$ such that $x_i = x_{k+1}$. By Lemma 2.2(i), we have that

$$E_{x_i,x_i} = E_{x_{k+1},x_i} = E_{x_{k+1},x_k} \cdots E_{x_{i+1},x_i} \subseteq A.$$

This contradicts the assumption that all cells in A are off-diagonal. Thus $x_i \neq x_{k+1}$ for all $i = 1, \dots, k$. It follows that $E_{a,x_i} \in \Phi$ are distinct cells for $i = 1, \dots, k+1$. Hence Φ contains an infinite subset of distinct cells. This contradiction completes the proof that A is not idempotent. \square

Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_1)$. For $i = 1, \dots, n$, we define an i^{th} row matrix $r_i[A]$ of A as a matrix whose i^{th} row is the same as the i^{th} row of A and the other rows are zero. Similarly, we can define a j^{th} column matrix $c_j[A]$ of A for $j = 1, \dots, n$. A line matrix is an i^{th} row matrix or a j^{th} column matrix of a matrix; Cells E_1, \dots, E_k are called *collinear* if $\sum_{i=1}^k E_i$ is dominated by a line matrix of J_n .

Let $A \in \mathbb{M}_n(\mathbb{B}_1)$. For indices $i, j \in \{1, \dots, n\}$, $r_i[A]$ and $c_j[A]$ are said to be (i, j) -disjoint if $XY = O_n$ for all off-diagonal cells $X \subseteq r_i[A]$ and $Y \subseteq c_j[A]$.

Lemma 2.4. *Let A be idempotent in $\mathbb{M}_n(\mathbb{B}_1)$. If $r_i[A]$ and $c_j[A]$ are not (i, j) -disjoint, then $E_{i,j} \subseteq A$.*

Proof. If $r_i[A]$ and $c_j[A]$ are not (i, j) -disjoint, then there exist off-diagonal cells $X \subseteq r_i[A]$ and $Y \subseteq c_j[A]$ such that $XY \neq O_n$. Thus we may write that $X = E_{i,x}$ and $Y = E_{y,j}$ for some indices x and y . Since $XY \neq O_n$, it follows from Proposition 2.1 that $x = y$ and $XY = E_{i,j}$. Since A is idempotent, $XY = E_{i,j} \subseteq A$ by Lemma 2.2(i). \square

Let E_1, E_2, E_3 and E_4 be four distinct cells in $\mathbb{M}_n(\mathbb{B}_1)$. Then their sum is called a *frame* if the four 1's constitute a rectangle with at least one entry on diagonal. That is, there exist indices $i, j, k \in \{1, \dots, n\}$ with $i \neq j, k$ such that

$$\sum_{l=1}^4 E_l = E_{i,i} + E_{i,j} + E_{k,i} + E_{k,j}.$$

Proposition 2.5. *Let A be idempotent in $\mathbb{M}_n(\mathbb{B}_1)$. If F is an off-diagonal cell in A such that F is not collinear with any diagonal cell in A , then F is in a frame with one diagonal cell and two additional off-diagonal cells in A .*

Proof. Let $\Phi_1 = \{E_1, \dots, E_m\}$ be the set of all distinct diagonal cells in A and Φ_2 be the set of all off-diagonal cells in A . By Corollary 2.3, we have $m \geq 1$. Let us denote $E_i = E_{a_i,a_i}$ for all $i = 1, \dots, m$ and $F = E_{b,c}$. Since F and E_i are not collinear for all i , it follows that a_1, \dots, a_m, b , and c are mutually distinct indices. Assume that F is not in a frame with one diagonal cell in Φ_1

and two off-diagonal cells in Φ_2 . Then we can construct an infinite subset of cells in Φ_2 applying the induction process as in the proof of Corollary 2.3.

The base of induction. Since A is idempotent, by Lemma 2.2(ii), there exist two distinct cells $E_{b,x_1}, E_{x_1,c} \sqsubseteq A$ such that $E_{b,c} = E_{b,x_1}E_{x_1,c}$ for some index x_1 . If $E_{b,x_1} \in \Phi_1$ or $E_{x_1,c} \in \Phi_1$, then $F = E_{b,c}$ is collinear with a diagonal cell. This is a contradiction. Thus $E_{b,x_1}, E_{x_1,c} \in \Phi_2$. If $x_1 = a_i$ for some i , then we obtain that $E_{x_1,x_1} + E_{b,x_1} + E_{x_1,c} + E_{b,c}$ is a frame, a contradiction to the assumption. Hence, $x_1 \neq a_i$ for all i .

Since A is idempotent and $E_{b,x_1} \in \Phi_2$, by Lemma 2.2(ii), we can find two cells E_{b,x_2} and E_{x_2,x_1} in $\Phi_1 \cup \Phi_2$ such that $E_{b,x_1} = E_{b,x_2}E_{x_2,x_1}$ for some index x_2 . Then we have $E_{x_2,c} = E_{x_2,x_1}E_{x_1,c} \sqsubseteq A$ by Lemma 2.2(i). If $x_2 = x_1$, then the four cells $E_{x_2,x_1}, E_{b,x_2}, E_{x_1,c}$ and $F = E_{b,c}$ are in a frame, which contradicts the assumption. If $x_2 = a_i$ for some i , then $E_{x_2,x_2}, E_{b,x_2}, E_{x_2,c}$ and $F = E_{b,c}$ are in a frame, which also contradicts the assumption. Thus $x_2 \neq a_i$ for all i and $x_2 \neq x_1$.

The induction step. Assume that for certain $k \geq 2$, the set of cells

$$\{E_{b,x_1}, \dots, E_{b,x_k}, E_{x_2,x_1}, \dots, E_{x_k,x_{k-1}}\} \subseteq \Phi_2$$

was already constructed. Then we may add new elements to this set as follows. By Lemma 2.2(ii), since A is idempotent, there exist two cells $E_{b,x_{k+1}}$ and E_{x_{k+1},x_k} in $\Phi_1 \cup \Phi_2$ such that $E_{b,x_k} = E_{b,x_{k+1}}E_{x_{k+1},x_k}$ for some index x_{k+1} . Thus by Lemma 2.2(i),

$$E_{x_{k+1},c} = E_{x_{k+1},x_k} \cdots E_{x_2,x_1}E_{x_1,c} \sqsubseteq A.$$

Now, we show that x_{k+1} is neither b nor x_i for all $i = 1, \dots, k$. Note that $x_{k+1} \neq b$ since $F = E_{b,c}$ is not collinear with any diagonal cell. Assume that $x_{k+1} = x_i$ for some $i \in \{1, \dots, k\}$. Then $E_{x_i,c} = E_{x_{k+1},c} \sqsubseteq A$ and by Lemma 2.2(i), $E_{x_i,x_i} = E_{x_{k+1},x_i} = E_{x_{k+1},x_k} \cdots E_{x_{i+1},x_i} \sqsubseteq A$. Therefore, the four cells $E_{x_i,x_i}, E_{b,x_i}, E_{x_i,c}$ and $F = E_{b,c}$ are in a frame, which contradicts the assumption. Thus $x_{k+1} \neq x_i$ for all $i = 1, \dots, k$ and we have constructed the set

$$\{E_{b,x_1}, \dots, E_{b,x_{k+1}}, E_{x_2,x_1}, \dots, E_{x_{k+1},x_k}\} \subseteq \Phi_2.$$

Therefore we obtain an infinite set of off-diagonal cells $\{E_{b,x_i} \mid i \in \mathbb{N}\}$ on the b^{th} row, which is impossible. This contradiction completes the proof. \square

The *weight* of a matrix A in $\mathbb{M}_n(\mathbb{B}_1)$ is the number of nonzero entries of A and will be denoted by $w(A)$. Note that $r_i[A]$ is the i^{th} row matrix of A , and $c_j[A]$ is the j^{th} column matrix of A .

Lemma 2.6. *Let A be an idempotent matrix in $\mathbb{M}_n(\mathbb{B}_1)$ with $E_{i,i} \sqsubseteq A$. If $w(r_i[A]) = s + 1$ and $w(c_i[A]) = t + 1$, then there exist exactly $s \cdot t$ frames in A dominating $E_{i,i}$.*

Proof. If $s = 0$ or $t = 0$, then the result is straightforward. Thus we assume $s, t \geq 1$. Since A is idempotent, Lemma 2.2(i) and Proposition 2.1 implies that for any cells $E_{k,i} \sqsubseteq A$ and $E_{i,l} \sqsubseteq A$, A dominates their product $E_{k,l}$. Therefore, the four cells $E_{i,i}, E_{k,i}, E_{i,l}$ and $E_{k,l}$ are in a frame in A for each k, l such that $E_{k,i} \sqsubseteq A$ and $E_{i,l} \sqsubseteq A$. Thus A has at least $s \cdot t$ frames such that each frame dominates $E_{i,i}$. It follows from the definition of frame that A has at most $s \cdot t$ frames dominating $E_{i,i}$. \square

For a matrix $A \in \mathbb{M}_n(\mathbb{B}_1)$, let

$$\{E_{i,i}, E_{j_1,i}, \dots, E_{j_s,i}\} \quad \text{and} \quad \{E_{i,i}, E_{i,i_1}, \dots, E_{i,i_t}\}$$

be the sets of cells in $\mathbf{c}_i[A]$ and $\mathbf{r}_i[A]$, respectively, where $s, t \geq 1$. If $E_{j_k,i_l} \sqsubseteq A$ for all $k = 1, \dots, s$ and $l = 1, \dots, t$, then

$$\sum_{k=1}^s \sum_{l=1}^t (E_{i,i} + E_{j_k,i} + E_{i,i_l} + E_{j_k,i_l})$$

is called an i^{th} *rectangle part* of A , and denoted by $RP_i[A]$.

Let A be idempotent in $\mathbb{M}_n(\mathbb{B}_1)$ with $E_{i,i} \sqsubseteq A$. If $w(\mathbf{r}_i[A]) > 1$ and $w(\mathbf{c}_i[A]) > 1$, then Lemma 2.6 shows that the i^{th} rectangle part $RP_i[A]$ of A exists.

Example 2.7. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{M}_4(\mathbb{B}_1)$. Then

$$RP_1[A] = RP_3[A] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

But there is neither $RP_2[A]$ nor $RP_4[A]$.

Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_1)$ with $E_{i,i} \sqsubseteq A$. If $w(\mathbf{r}_i[A]) = 1$ or $w(\mathbf{c}_i[A]) = 1$, then $\mathbf{r}_i[A] + \mathbf{c}_i[A]$ is called an i^{th} *line part* of A , and denoted by $LP_i[A]$. That is, $A \in \mathbb{M}_n(\mathbb{B}_1)$ has the i^{th} line part $LP_i[A] = \mathbf{r}_i[A] + \mathbf{c}_i[A]$ if and only if $\mathbf{r}_i[A] = E_{i,i}$ or $\mathbf{c}_i[A] = E_{i,i}$.

Example 2.8. Let $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be matrix in $\mathbb{M}_4(\mathbb{B}_1)$. Then B has

1st and 4th line parts, which are $LP_1[B] = E_{1,1} + E_{1,3} + E_{1,4}$ and $LP_4[B] = E_{4,4} + E_{1,4}$, respectively. But B has neither $LP_2[B]$ nor $LP_3[B]$.

Corollary 2.9. *If A is idempotent in $\mathbb{M}_n(\mathbb{B}_1)$, then every cell dominated by A is either in a rectangle part or in a line part of A .*

Proof. It follows directly from Proposition 2.5 and Lemma 2.6. □

Suppose that $A \in \mathbb{M}_n(\mathbb{B}_1)$ has i^{th} and j^{th} rectangle parts $RP_i[A]$ and $RP_j[A]$ for some i and j with $i \neq j$. We say that $RP_i[A]$ and $RP_j[A]$ are *disjoint* if $r_i[A]$ and $c_j[A]$ are (i, j) -disjoint, or $r_j[A]$ and $c_i[A]$ are (j, i) -disjoint or both.

Proposition 2.10. *Let A be idempotent in $\mathbb{M}_n(\mathbb{B}_1)$. Then any two rectangle parts of A are either disjoint or identical.*

Proof. It is straightforward that disjoint parts are not identical. Suppose that the i^{th} and j^{th} rectangle parts of A are not disjoint. By definition, we have $r_i[A]$ and $c_j[A]$ are not (i, j) -disjoint, and $r_j[A]$ and $c_i[A]$ are not (j, i) -disjoint. Therefore, $E_{i,j} \subseteq A$ and $E_{j,i} \subseteq A$ by Lemma 2.4. Then in this case we claim that for any cell E , we have $E \subseteq RP_i[A]$ if and only if $E \subseteq RP_j[A]$. It is straightforward that the four cells $E_{i,i}, E_{j,j}, E_{i,j}, E_{j,i} \subseteq RP_t[A]$ for $t = i, j$. Suppose $E \subseteq RP_i[A]$. We first consider the case $E \subseteq r_i[A]$, say $E = E_{i,a}$. Then we have $E_{j,a} = E_{j,i}E_{i,a} \subseteq A$ by Lemma 2.2(i), and the four cells

$$E_{i,a}, E_{i,j}, E_{j,a} \quad \text{and} \quad E_{j,j}$$

form a frame. Therefore, $E = E_{i,a} \subseteq RP_j[A]$. Similarly for the case $E \subseteq c_i[A]$ it follows that $E \subseteq RP_j[A]$.

Next, consider the case $E \not\subseteq r_i[A]$ and $E \not\subseteq c_i[A]$, say $E = E_{c,d}$. Since $E \subseteq RP_i[A]$, there exist two off-diagonal cells $E_{i,x} \subseteq r_i[A]$ and $E_{y,i} \subseteq c_i[A]$ such that $E_{c,d} = E_{y,i}E_{i,x}$. Therefore, we have that $c = y$ and $d = x$ by Proposition 2.1. Since A is idempotent, we obtain by Lemma 2.2(i) that

$$E_{c,j} = E_{y,j} = E_{y,i}E_{i,j} \subseteq A \quad \text{and} \quad E_{j,d} = E_{j,x} = E_{j,i}E_{i,x} \subseteq A.$$

Hence the four cells

$$E_{c,d}, E_{c,j}, E_{j,d} \quad \text{and} \quad E_{j,j}$$

form a frame. Therefore, we have $E = E_{c,d} \subseteq RP_j[A]$.

Similarly if E is a cell with $E \subseteq RP_j[A]$, then we have that $E \subseteq RP_i[A]$. Therefore, the two rectangle parts $RP_i[A]$ and $RP_j[A]$ are identical. □

Theorem 2.11. *Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_1)$. Then A is idempotent if and only if the following two conditions are satisfied:*

- (1) *there exist integers $r, l \geq 0$ such that A is a sum of r disjoint rectangle parts and l line parts,*
- (2) *if for some $i \neq j$ $r_i[A]$ and $c_j[A]$ are not (i, j) -disjoint, then $E_{i,j}$ is a cell in A .*

Proof. Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_1)$. It is routine to check that a matrix satisfying the two conditions is idempotent. To show the opposite implication, without loss of generality, we can assume that A has r rectangle parts and l line parts, where $r, l \geq 0$. Let F be an off-diagonal cell in A . By Corollary 2.9, F is in some rectangle part or some line part of A . Therefore, A is the sum of r rectangle parts and l line parts of A . It follows from Proposition 2.10 that

r rectangle parts of A are disjoint. By Lemma 2.4, if $\mathbf{r}_i[A]$ and $\mathbf{c}_j[A]$ are not (i, j) -disjoint, then $E_{i,j}$ is a cell in A . \square

Furthermore, Song et al. characterized the structures of general Boolean idempotent matrices as follow:

Theorem 2.12 ([8]). *Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Then A is idempotent if and only if all p^{th} constituents of A are idempotent in $\mathbb{M}_n(\mathbb{B}_1)$, where the p^{th} constituent of A is the matrix in $\mathbb{M}_n(\mathbb{B}_1)$ whose $(i, j)^{\text{th}}$ entry is 1 if and only if $a_{ij} \supseteq \sigma_p$.*

3. The chain semiring case

In this section, we characterize idempotent matrices over chain semirings \mathbb{K} including the fuzzy semiring $\mathbb{F} = [0, 1]$. We remind that for all $x, y \in \mathbb{K}$,

$$x + y = \max(x, y) \quad \text{and} \quad xy = \min(x, y).$$

Let α be a fixed member of \mathbb{K} , other than 1. For each $x \in \mathbb{K}$, define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \rightarrow x^\alpha$ is a homomorphism of \mathbb{K} onto \mathbb{B}_1 . Its entrywise extension to a mapping $A \rightarrow A^\alpha$ of $\mathbb{M}_n(\mathbb{K})$ onto $\mathbb{M}_n(\mathbb{B}_1)$ preserves matrix sums and products and multiplication by scalars. We call A^α the α -pattern of A .

Let $A = [a_{ij}]$ be a matrix in $\mathbb{M}_n(\mathbb{K})$. Then an a_{ij} -pattern of A may be a key to determine whether A is idempotent or not. For example, let

$$A = [a_{ij}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \in \mathbb{M}_2(\mathbb{F}),$$

where $\mathbb{F} = [0, 1]$ is the fuzzy semiring. Then the $a_{2,2}(= \frac{1}{4})$ -pattern of A , $A^{\frac{1}{4}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, is not idempotent in $\mathbb{M}_2(\mathbb{B}_1)$ by Lemma 2.6. Theorem 3.1(below) shows that A is not idempotent in $\mathbb{M}_2(\mathbb{F})$.

Theorem 3.1. *Let $A = [a_{i,j}]$ be a matrix in $\mathbb{M}_n(\mathbb{K})$. Then A is idempotent if and only if all $a_{i,j}$ -patterns of A are idempotent in $\mathbb{M}_n(\mathbb{B}_1)$.*

Proof. Let A be idempotent in $\mathbb{M}_n(\mathbb{K})$. Then all a_{ij} -patterns of A are idempotent in $\mathbb{M}_n(\mathbb{B}_1)$ because each a_{ij} -pattern of A is a homomorphism of $\mathbb{M}_n(\mathbb{K})$ onto $\mathbb{M}_n(\mathbb{B}_1)$.

Conversely, assume that each a_{ij} -pattern $A^{a_{ij}}$ of A is idempotent in $\mathbb{M}_n(\mathbb{B}_1)$. If $A^2 \neq A$, then for some $(i, j)^{\text{th}}$ entries of A and A^2 , we have

$$(3.1) \quad a_{ij} \neq \sum_{k=1}^n a_{ik}a_{kj}.$$

If $a_{ij} < \sum_{k=1}^n a_{ik}a_{kj}$, then the $(i, j)^{\text{th}}$ entry of $A^{a_{ij}}$ is 0, but that of $(A^{a_{ij}})^2$ is 1, a contradiction to the fact that a_{ij} -pattern of A is idempotent in $\mathbb{M}_n(\mathbb{B}_1)$.

Hence we have $a_{ij} > \sum_{k=1}^n a_{ik}a_{kj}$. We notice that the right side of (3.1) is just $a_{ik}a_{kj}$ for some $k \in \{1, \dots, n\}$. Furthermore we have $a_{ik}a_{kj} = a_{ik}$ or a_{kj} . If $a_{ik}a_{kj} = a_{ik}$, then $a_{ij} > \sum_{k=1}^n a_{ik}a_{kj} = a_{ik}$, and hence the $(i, j)^{\text{th}}$ entry of $A^{a_{ik}}$ is 1, but that of $(A^{a_{ik}})^2$ is 0, a contradiction. Similarly if $a_{ik}a_{kj} = a_{kj}$, then we have $(A^{a_{kj}})^2 \neq A^{a_{kj}}$, a contradiction. Therefore A is idempotent in $M_n(\mathbb{K})$. \square

References

- [1] R. B. Bapat, S. K. Jain, and L. E. Snyder, *Nonnegative idempotent matrices and minus partial order*, Linear Algebra Appl. **261** (1997), 143–154.
- [2] L. B. Beasley and N. J. Pullman, *Linear operators strongly preserving idempotent matrices over semirings*, Linear Algebra Appl. **160** (1992), 217–229.
- [3] J. S. Golan, *Semirings and their applications, updated and expanded version of the theory of semirings*, with applications to mathematics and theoretical computer science, Kluwer Academic Publishers, Dordrecht, 1999.
- [4] D. A. Gregory and N. J. Pullman, *Semiring rank : Boolean rank and nonnegative rank factorizations*, J. Combin. Inform. System Sci. **8** (1983), no. 3, 223–233.
- [5] S. Kirkland and N. J. Pullman, *Linear operators preserving invariants of non-binary matrices*, Linear and Multilinear Algebra **33** (1993), no. 3-4, 295–300.
- [6] V. N. Kolokoltsov and V. P. Maslov, *Idempotent Analysis and its Applications*, Mathematics and its Applications, **401**, Dordrecht: Kluwer Academic Publishers, 1997.
- [7] S. Z. Song and K. T. Kang, *Types and enumeration of idempotent matrices*, Far East J. Math. Sci. **3** (2001), no. 6, 1029–1042.
- [8] S. Z. Song, K. T. Kang, and L. B. Beasley, *Idempotent matrix preservers over Boolean algebras*, J. Korean Math. Soc. **44** (2007), no. 1, 169–178.

KYUNG-TAE KANG
 DEPARTMENT OF MATHEMATICS
 CHEJU NATIONAL UNIVERSITY
 JEJU 690-756, KOREA
 AND
 DEPARTMENT OF MATHEMATICS AND STATISTICS
 UTAH STATE UNIVERSITY
 LOGAN, UTAH 84322-3900, U.S.A.
E-mail address: kangkt@cheju.ac.kr

SEOK-ZUN SONG
 DEPARTMENT OF MATHEMATICS
 CHEJU NATIONAL UNIVERSITY
 JEJU 690-756, KOREA
E-mail address: szsong@cheju.ac.kr

YOUNG-OH YANG
 DEPARTMENT OF MATHEMATICS
 CHEJU NATIONAL UNIVERSITY
 JEJU 690-756, KOREA
E-mail address: yangyo@cheju.ac.kr