BOUNDEDNESS OF DISCRETE VOLTERRA SYSTEMS

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ABSTRACT. We investigate the representation of the solution of discrete linear Volterra difference systems by means of the resolvent matrix and fundamental matrix, respectively, and then study the boundedness of the solutions of discrete Volterra systems by improving the assumptions and the proofs of Medina's results in [6].

1. Introduction

Volterra difference system

$$x(n) = \sum_{r=0}^{n} B(n,r)x(r) + f(n), \ n \geq 0,$$

where $B : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}^{k \times k}$ and $f : \mathbb{Z}_+ \to \mathbb{R}^k$, can be regarded as a discrete analogue of classical Volterra integral equation

$$x(t) = \int_0^t K(t,s)x(s)ds + f(t), \ t \geq 0.$$

Similarly, linear Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r), \ n \geq 0$$

can be regarded as a discrete analogue of classical Volterra integro-differential equation

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds, \ t \geq 0.$$

Discrete Volterra systems arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. A property of crucial importance is the boundedness of the solution of a discrete Volterra system. In fact, error between the true and the numerical

solutions of a Volterra integral equation satisfies a discrete Volterra system and thus the boundedness of the solution of this discrete Volterra system assures the boundedness of the global error, that is, the stability of the considered numerical method [3].

In this paper, we investigate the representations of the solution of linear Volterra system by means of the resolvent matrix and fundamental matrix, respectively, and then study the boundedness of the solution of discrete Volterra system by improving the assumptions and the proofs of Medina's results in [6].

2. Resolvent and fundamental matrices

Let $\mathbb{R}^k$ denote the real k-dimensional Euclidean space with norm

$$|x| = \sum_{i=1}^{k} |x_i|, \; x \in \mathbb{R}^k,$$

and $\mathbb{Z}_+$ denotes the set of all nonnegative integers. Let $S = S(\mathbb{Z}_+, \mathbb{R}^k)$ be the space of all functions from $\mathbb{Z}_+$ into $\mathbb{R}^k$ and $S_1 = S_1(\mathbb{Z}_+, \mathbb{R}^k)$ be the Banach space in S of all bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^k$ with norm

$$|x|_{S_1} = \sup_{n \in \mathbb{Z}_+} |x(n)|.$$

Also, $S_2 = S_2(\mathbb{Z}_+, \mathbb{R}^k)$ denotes the space of all functions in S_1 having a limit at infinity ; $S_3 = S_3(\mathbb{Z}_+, \mathbb{R}^k)$ space of all functions in S_2 that have limit zero. S_2 and S_3 are Banach spaces with the supremum norm. For $1 \le p < \infty$, l_p is the space of sequences $y = (y_i)_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |y_i|^p < \infty$ with norm

$$|y|_p = (\sum_{i=1}^{\infty} |y_i|^p)^{\frac{1}{p}} < \infty.$$

We consider a linear Volterra difference system

$$(1) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + f(n), \; x(0) = x_0,$$

where $x(n) \in \mathbb{R}^k$; $A(n), B(n,r)$ are $k \times k$ real matrices on $\mathbb{Z}_+$ and $\mathbb{Z}_+ \times \mathbb{Z}_+$, respectively, and $f : \mathbb{Z}_+ \to \mathbb{R}^k$. Note that the matrix norm is given by

$$|A| = \max_{i=1,\dots,k} \sum_{j=1}^{k} |a_{ij}| \quad \text{for } A = (a_{ij}).$$

We define the *resolvent matrix* $R(n,m) = R_1$ of system (1) as the unique solution of the matrix equation

$$(2) \qquad R(n+1,m) = A(n)R(n,m) + \sum_{r=m}^{n} B(n,r)R(r,m), \; n \ge m,$$

$$R(m,m) = I \text{ (identity matrix)}.$$

We assume that $R(n, m)$ is nonsingular and note that

$$R^{-1}(n, m) \quad = \quad R(m, n),$$

and

$$R(n, r)R(r, m) \quad = \quad R(n, m), \; n, m, r \in \mathbb{Z}_+.$$

Elaydi [4] proved that system (1) has a unique solution $x(n)$ which can be expressed as

$$(3) \qquad x(n) = x(n, 0, x_0) = R(n, 0)x_0 + \sum_{r=0}^{n-1} R(n, r + 1)f(r).$$

For the completeness we prove in detail:

Proposition 2.1 (Variation of Constants Formula). *The unique solution*

$$x(n, 0, x_0)$$

of system (1) *satisfying* $x(0) = x_0$ *is given by* (3).

Proof. It is enough to show that $x(n)$ given by (3) is a solution of (1).

$$x(n + 1) = R(n + 1, 0)x_0 + \sum_{r=0}^{n} R(n + 1, r + 1)f(r)$$

$$= [A(n)R(n, 0) + \sum_{r=0}^{n} B(n, r)R(r, 0)]x_0$$

$$+ \sum_{r=0}^{n-1}[A(n)R(n, r + 1) + \sum_{\tau=r+1}^{n} B(n, \tau)R(\tau, r + 1)]f(r) + f(n)$$

$$= A(n)[R(n, 0)x_0 + \sum_{r=0}^{n-1} R(n, r + 1)f(r)]$$

$$+ \sum_{r=0}^{n} B(n, r)R(r, 0)x_0 + \sum_{r=0}^{n-1}\sum_{\tau=r+1}^{n} B(n, \tau)R(\tau, r + 1)f(r) + f(n)$$

$$= A(n)x(n) + f(n) + \sum_{r=0}^{n-1} B(n, n)R(n, r + 1)f(r)$$

$$+ \sum_{r=0}^{n} B(n, r)R(r, 0)x_0 + \sum_{r=0}^{n-1}\sum_{\tau=r+1}^{n-1} B(n, \tau)R(\tau, r + 1)f(r).$$

Then, in view of Fubini's theorem [1, 2], we obtain

$$x(n+1) = A(n)x(n) + f(n) + B(n,n)\sum_{\tau=0}^{n-1} R(n,\tau+1)f(\tau)$$

$$+ \sum_{r=0}^{n} B(n,r)R(r,0)x_0 + \sum_{r=0}^{n-1}\sum_{\tau=0}^{r-1} B(n,r)R(r,\tau+1)f(\tau)$$

$$= A(n)x(n) + \sum_{r=0}^{n} B(n,r)\left(\sum_{\tau=0}^{r-1} R(r,\tau+1)f(\tau) + R(r,0)x_0\right) + f(n)$$

$$= A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + f(n).$$

This completes the proof. $\hspace{2cm}$ $\square$

Remark 2.2. There is another type for $R(n,m)$:

$$R(n,m) = R(n,m+1)A(m)$$

$$(2') \hspace{3cm} + \sum_{r=m}^{n-1} R(n,r+1)B(r,m), \; n \geq m.$$

From (2') we can derive the formula (3).

Proof. Let $p(l) = R(n,l)x(l), n \geq l$ for the solution $x(l)$ of (1). Then

$$\sum_{l=0}^{n-1} \Delta p(l) = p(n) - p(0)$$

$$= \sum_{l=0}^{n-1}[R(n,l+1)A(l) - R(n,l)]x(l)$$

$$+ \sum_{l=0}^{n-1}\sum_{r=0}^{l} R(n,l+1)B(l,r)x(r) + \sum_{l=0}^{n-1} R(n,l+1)f(l)$$

$$\equiv L + \sum_{l=0}^{n-1} R(n,l+1)f(l).$$

If $L = 0$, then we have

$$p(n) = p(0) + \sum_{l=0}^{n-1} R(n,l+1)f(l),$$

$$x(n) = R(n,0)x_0 + \sum_{l=0}^{n-1} R(n,l+1)f(l).$$

Thus we show that $L = 0$. By the Fubini's theorem we obtain

$$\sum_{l=0}^{n-1}\sum_{r=0}^{l} R(n,l+1)B(l,r)x(r) = \sum_{l=0}^{n-1}\sum_{r=l}^{n-1} R(n,r+1)B(r,l)x(l),$$

and from the equality

$$\sum_{l=0}^{n-1}[R(n,l+1)A(l) - R(n,l) + \sum_{r=l}^{n-1} R(n,r+1)B(r,l)]x(l) = 0$$

we have $L = 0$. □

In the case $A(n) = A$, a nonsingular constant matrix and $B(n,r) = B(n-r)$, we obtain

$$R(n,m) = R(n-m),$$

and thus, equation (2) reduces to

$$(4) \qquad R(n+1) = AR(n) + \sum_{r=0}^{n} B(n-r)R(r),$$

and $R(n,m)$ is denoted by R_2.

We consider the linear homogeneous Volterra system of convolution type

$$(5) \qquad x(n+1) = Ax(n) + \sum_{r=0}^{n} B(n-r)x(r), \ x(0) = x_0.$$

There are k vector solutions $x_1(n), \ldots, x_k(n)$ of system (5) with $x_i(0) = e_i = (0, \ldots, 1, \ldots, 0)^T$, the standard ith unit vector in $\mathbb{R}^k$. The $k \times k$ matrix $X(n)$, whose ith column is $x_i(n)$, is called a *fundamental matrix* of system (5). Note that $X(0) = I$ and

$$X(n+1) = AX(n) + \sum_{r=0}^{n} B(n-r)X(r)$$

holds. Furthermore

$$x(n,0,x_0) = X(n)x_0$$

is the unique solution of (5) with $x(0,0,x_0) = x_0$. Moreover, by the result [6, Theorem 6], the unique solution of

$$(6) \qquad x(n+1) = Ax(n) + \sum_{r=0}^{n} B(n-r)x(r) + f(n), \ x(0) = x_0$$

is given by

$$(7) \qquad x(n,0,x_0) = X(n)x_0 + \sum_{r=0}^{n-1} X(n-r-1)f(r).$$

Now, every solution of system (1) can be represented in terms of fundamental matrix:

Proposition 2.3. *The unique solution of system (1) is given by*

$$
(8) \qquad x(n, 0, x_0) = X(n, 0)x_0 + \sum_{r=0}^{n-1} X(n, r+1)f(r),
$$

where $X(n, 0, x_0)$ *is a fundamental matrix of unperturbed linear system*

$$
(9) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n, r)x(r), \ x(0) = x_0.
$$

Proof. It is clear that $x(n, 0, x_0)$ satisfies (8) by the Fubini's theorem. Conversely (8) satisfies (1) by the following calculation :

$$
\begin{aligned}
x(n+1) &= X(n+1, 0)x_0 + \sum_{r=0}^{n-1} X(n+1, r+1)f(r) + f(n) \\
&= A(n)\Big[X(n, 0)x_0 + \sum_{r=0}^{n-1} X(n, r+1)f(r)\Big] + f(n) \\
&\quad + \sum_{r=0}^{n} B(n, r)X(r, 0)x_0 + \sum_{r=0}^{n-1} \sum_{s=r+1}^{n} B(n, s)X(s, r+1)f(r) \\
&= A(n)x(n) + f(n) + \sum_{r=0}^{n} B(n, r)X(r, 0)x_0 \\
&\quad + \sum_{r=0}^{n-1}\sum_{s=0}^{l-1} B(n, r)X(r, s+1)f(s) + \sum_{r=0}^{n-1} B(n, n)X(n, r+1)f(r) \\
&= A(n)x(n) + f(n) + \sum_{r=0}^{n} B(n, r)\Big[X(r, 0)x_0 + \sum_{s=0}^{r-1} X(r, s+1)f(s)\Big] \\
&= A(n)x(n) + \sum_{r=0}^{n} B(n, r)x(r) + f(n).
\end{aligned}
$$

This completes the proof. $\qquad \square$

Example 2.4 ([1]). For the linear Volterra difference equation

$$
x(n+1) = 2x(n) + \sum_{s=n_0}^{n} 2^{n-s}x(s), \ x(n_0) = x_0, \ n \ge n_0 \ge 0.
$$

We see that $R(n, n_0) = X(n, n_0) = \frac{1}{3}[1 + 2 \cdot 4^{n-n_0}]$ since

$$
R(n, m) = R(n, m+1)2 + \sum_{r=m}^{n-1} R(n, r+1)2^{r-m}, n-1 \ge m \ge n_0,
$$

and

$$
\frac{\partial x(n, n_0, x_0)}{\partial x_0} = X(n, x_0).
$$

Note that the solution of the above equation is given by

$$x(n, n_0, x_0) = \frac{x_0}{3}[1 + 2 \cdot 4^{n-n_0}], \ n \geq n_0.$$

3. Main results

Firstly, we examine the boundedness of Volterra system

$$(1) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + f(n), \ x(0) = x_0.$$

System (1) is a perturbation

$$(9) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r), \ x(0) = x_0.$$

Theorem 3.1. *If we assume that*
   (i) *every solution of* (9) *is bounded,*
   (ii) $\sum_{n=0}^{\infty} |f(n)| < \infty$,
then the solution $x(n)$ *of system* (1) *is bounded.*

Proof. In view of Proposition 2.3, the solution $x(n)$ of (1) is given by

$$x(n) = X(n,0)x_0 + \sum_{r=0}^{n-1} X(n,r+1)f(r).$$

Note that there exists $M > 0$ such that $|X(n,0)| \leq M$ from the condition (i). Thus we have

$$
\begin{aligned}
|x(n)| &\leq |X(n,0)||x_0| + \sum_{r=0}^{n-1} |X(n,r+1)||f(r)| \\
&\leq M \left[|x_0| + \sum_{r=0}^{n-1} |f(r)| \right].
\end{aligned}
$$

This implies that $x(n)$ is bounded by the condition (ii). $\qquad \square$

Next, we can obtain the following result for the boundedness of Volterra system

$$(10) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n} [B(n,r) + D(n,r)]x(r), \ x(0) = x_0,$$

where B and D are $k \times k$ matrix functions defined on $\mathbb{Z}_+ \times \mathbb{Z}_+$. System (10) is a perturbation of system (9).

Theorem 3.2. *Assume that in system* (10)
   (i) *every solution of* (9) *is bounded,*
   (ii) $\sum_{r=0}^{\infty} \sum_{j=r}^{\infty} |D(j,r)| < \infty$,
then the solution $x(n)$ *of system* (10) *is bounded.*

Proof. The solution $x(n)$ of system (10) can be represented by

$$x(n) = X(n,0)x_0 + \sum_{r=0}^{n-1} X(n,r+1) \sum_{j=0}^{r} D(r,j)x(j)$$

in view of Proposition 2.3. From (i) we have $|X(n,0)| \leq M$ for some $M > 0$. Then we have

$$
\begin{aligned}
|x(n)| &\leq M \left[|x_0| + \sum_{r=0}^{n-1} \sum_{j=0}^{r} |D(r,j)||x(j)| \right] \\
&= M|x_0| + \sum_{r=0}^{n-1} \left[|D(r,r)|x(r)| + \sum_{j=0}^{r-1} |D(r,j)||x(j)| \right].
\end{aligned}
$$

By the discrete Gronwall inequality [1] and Fubini's theorem, we obtain

$$
\begin{aligned}
|x(n)| &\leq M|x_0| \prod_{r=0}^{n-1} \left[1 + M|D(r,r)| + M \sum_{j=0}^{r-1} |D(r,j)| \right] \\
&\leq M|x_0| \exp \left(M \sum_{r=0}^{n-1} \sum_{j=0}^{r} |D(r,j)| \right) \\
&= M|x_0| \exp \left(M \sum_{r=0}^{n-1} \sum_{j=r}^{n-1} |D(j,r)| \right) \\
&\leq M|x_0| \exp \left(M \sum_{r=0}^{\infty} \sum_{j=r}^{\infty} |D(j,r)| \right).
\end{aligned}
$$

It follows from this inequality that $x(n)$ is bounded. $\qquad\square$

Now, we investigate the boundedness of solutions of the Volterra system

$$(11) \quad x(n+1) = Ax(n) + \sum_{r=0}^{n} [B(n-r) + D(n-r)]x(r) + f(n), \ x(0) = x_0,$$

where $x(n)$ and $f(n)$ are column vectors ; A, B and D are square matrices, by improving Medina's results [6].

Medina obtained the following result [6, Theorem 1] by imposing the condition

(iv) $C(n) = \sum_{r=0}^{n} D(n-r) \in S_3$.

However, we consider the condition

$$|C|_{S_1}|R_2|_1 < 1$$

instead of (iv).

Theorem 3.3. *Assume that in system* (11)

(i) $f \in S_1$,

(ii) $B, D \in l_p$ for some $1 \le p < \infty$,

(iii) $R_2 \in l_1$,

(iv) $|C|_{S_1}|R_2|_1 < 1$.

Then the solution $x(n)$ *of system* (11) *is in* S_1.

Proof. The solution $x(n)$ of (11) can be represented by

$$x(n) = R_2(n)x_0 + \sum_{r=0}^{n-1} R_2(n-r-1) \sum_{j=0}^{r} D(r-j)x(j) + \sum_{r=0}^{n-1} R_2(n-r-1)f(r)$$

by the variation of constants formula (3).

Let $(0, N) = \{0, 1, \ldots, N\}, N \in \mathbb{Z}_+$, be a discrete interval. Then we have

$$
\begin{aligned}
|x|_{(0,N)} &= \sup_{n \in (0,N)} |x(n)| = |x|_{S_1} \\
&\le |R_2 x_0|_{(0,N)} + |f|_{(0,N)}|R_2|_1 \\
&\quad + \sup_{n \in (0,N)} \left(\sum_{r=0}^{n-1} |R_2(n-r-1)| |\sum_{j=0}^{r} D(r-j)x(j)| \right) \\
&\le |R_2 x_0|_{(0,N)} + |f|_{(0,N)}|R_2|_1 + |R_2|_1 |C|_{S_1} |x|_{(0,N)} \\
&\le |R_2 x_0|_{S_1} + |R_2|_1 |f|_{S_1} + |C|_{S_1}|R_2|_1|x|_{(0,N)}.
\end{aligned}
$$

From the condition (iv), we obtain

$$|x|_{(0,N)} \le \frac{|R_2 x_0|_{S_1} + |R_2|_1 |f|_{S_1}}{1 - |C|_{S_1}|R_2|_1}$$

and we conclude that $x \in S_1$ by (i), (ii), and (iii).

Now, we consider the case $C \in S_3$, i.e., $\lim_{n \to \infty} C(n) = 0$, where $C(n) = \sum_{r=0}^{n} D(n-r)$. Then there exists $N > 0$ such that $C(n)$ is small for all $n > N$. If we denote $x(n+N) = x_N(n)$, then we have

$$
\begin{aligned}
x_N(n+1) &= Ax_N(n) + \sum_{r=0}^{n+N} [B(n+N-r) + D(n+N-r)]x(r) + f_N(n), \\
x_N(0) &= x(N).
\end{aligned}
$$

By performing the change of variable $r - N = u$, we obtain

$$x_N(n+1) = Ax_N(n) + \sum_{r=0}^{n+N} [B(n-u) + D(n-u)]x_N(u) + F(n),$$

where

$$F(n) = \sum_{u=-N}^{-1} [B(n-u) + D(n-u)]x_N(u) + f_N(n).$$

Note that $F(n) \in S_1$ since

$$\sup_{n \in \mathbb{Z}_+} |F(n)| \leq |f_N|_{S_1} + |x|_{(0,N)}[|B|_{S_1} + |D|_{S_1}].$$

Hence

$$|x|_{(N+1,\infty)} \leq \frac{|R_2 x_0|_{S_1} + |R_2|_1 |F|_{S_1}}{1 - |C|_{S_1} |R_2|_1}.$$

Therefore $|x|_{S_1} = \max\{|x|_{(0,N)}, |x|_{(N+1,\infty)}\}$ is bounded. This completes the proof.   □

The proof of the following theorem [6, Theorem 2] depends on the discrete Gronwall inequality [5] : $|x(n)| \leq M + \sum_{r=0}^{n-1} g(r)|x(r)|$, $g(r) \geq 0$ implies

$$|x(n)| \leq M \exp\left(\sum_{r=0}^{n-1} g(r)\right).$$

But Medina's proof has a defect that it can not apply to the above discrete Gronwall inequality directly. So we give a proof that depends on the another type of the discrete inequality.

Lemma 3.4. *Let for all $n \in \mathbb{Z}_+$ the following inequality be satisfied*

$$u(n) \leq a + \sum_{s=n_0}^{n-1} \left[b(s)u(s) + \sum_{j=n_0}^{s-1} C(s,j)u(j) \right].$$

Then, for all $n \in \mathbb{Z}_+$,

$$
\begin{aligned}
u(n) &\leq a \prod_{s=n_0}^{n-1} \left[1 + b(s) + \sum_{j=n_0}^{s-1} C(s,j) \right] \\
&\leq a \prod_{s=n_0}^{n-1} \left[\exp(b(s) + \sum_{j=n_0}^{s-1} C(s,j)) \right] \\
&\leq a \exp\left(\sum_{s=n_0}^{n-1} [b(s) + \sum_{j=n_0}^{s-1} C(s,j)] \right).
\end{aligned}
$$

Proof. Let $y(n) = a + \sum_{s=n_0}^{n-1}[b(s)u(s) + \sum_{j=n_0}^{s-1} C(s,j)u(j)]$. Then $u(n) \leq y(n)$ and $u(n_0) \leq y(n_0) = a$. We have

$$
\begin{aligned}
\Delta y(n) &= y(n+1) - y(n) = b(n)u(n) + \sum_{j=n_0}^{n-1} C(n,j)u(j) \\
&\leq \left[b(n) + \sum_{j=n_0}^{n-1} C(n,j) \right] y(n).
\end{aligned}
$$

Thus we get

$$y(n+1) \leq \left[1 + b(n) + \sum_{j=n_0}^{n-1} C(n,j)\right] y(n),$$

so we obtain

$$y(n) \leq a \prod_{s=n_0}^{n-1} \left[1 + b(s) + \sum_{j=n_0}^{s-1} C(s,j)\right].$$

$\square$

Theorem 3.5. *For system* (11), *assume that*
   (i) $f \in S_1$,
   (ii) $R_2 \in l_1$,
   (iii) $C(n) = \sum_{r=0}^{n} D(n-r) \in l_1$.
Then the solution $x(n)$ *of system* (11) *is in* S_1.

Proof. We write the solution $x(n)$ as

$$x(n) = R_2(n)x_0 + \sum_{r=0}^{n-1} R_2(n-r-1)f(r) + \sum_{r=0}^{n-1} R_2(n-r-1) \sum_{j=0}^{r} D(r-j)x(j).$$

Then we obtain

$$u(n) = |x(n)|$$
$$= |R_2 x_0|_{S_1} + |f|_{S_1}|R_2|_1 + |R_2|_{S_1} \sum_{r=0}^{n-1} \sum_{j=0}^{r} |D(r-j)||x(j)|.$$

Putting $a = |R_2 x_0|_{S_1} + |f|_{S_1}|R_2|_1$, we have, by Lemma 3.4,

$$u(n) \leq a + |R_2|_{S_1} \sum_{r=0}^{n-1} \left[|D(0)|u(r) + \sum_{j=0}^{r-1} |D(r-j)|u(j)\right]$$

$$\leq a \exp\left(|R_2|_{S_1} \sum_{r=0}^{n-1}[|D(0)| + \sum_{j=0}^{r-1} |D(r-j)|]\right)$$

$$= a \exp\left(|R_2|_{S_1} \sum_{r=0}^{n-1} \sum_{j=0}^{r} |D(r-j)|\right),$$

where $b(r) = |D(0)|$. Consequently, we have

$$|x(n)| \leq [|R_2 x_0|_{S_1} + |f|_{S_1}|R_2|_1] \exp\left(|R_2|_{S_1} \sum_{r=0}^{n-1} \sum_{j=0}^{r} |D(r-j)|\right)$$

$$\leq M$$

for some M by the assumptions. This completes the proof.   □

Furthermore, Medina in Theorem 4 [6] proved the boundedness of solutions of system

$$(10) \qquad x(n+1) = A(n)x(n) + \sum_{r=0}^{n}[B(n,r) + D(n,r)]x(r), \; x(0) = x_0,$$

under the assumptions

$$(12) \qquad \sum_{r=0}^{\infty}|D(r,j)\gamma^{-r} < \infty \text{ and } \lim_{j\to\infty}|D(r,j)|\gamma^{r-j} = 0, \; 0 < \gamma < 1.$$

But we impose the condition

$$\sum_{r=0}^{\infty}\sum_{j=0}^{r}|D(r,j)\gamma^{j-r-1} < \infty, \; 0 < \gamma < 1$$

instead of (12). Also, the discrete Gronwall-type inequality (Lemma 3.4) is needed to get the result. Let R_3 denote the resolvent matrix associated with system (10).

Theorem 3.6. *For system* (10), *assume that*
   (i) $|R_3(n,m)| \le \lambda\gamma^{n-m}$ *for all* $n \ge m$ *and some* $\lambda > 0$ *and* $\gamma \in (0,1)$,
   (ii) $\sum_{r=0}^{\infty}\sum_{j=0}^{r}|D(r,j)|\gamma^{j-r-1} < \infty$.
Then the solution $x(n)$ *of system* (10) *is bounded.*

Proof. The solution $x(n)$ of (10) is given by

$$x(n) = R_3(n,0)x_0 + \sum_{r=0}^{n-1}R_3(n,r+1)\sum_{j=0}^{r}D(r,j)x(j).$$

Then, from (i), we have

$$|x(n)| \quad \le \quad \lambda\gamma^n|x_0| + \lambda\sum_{r=0}^{n-1}\gamma^{n-r-1}\sum_{j=0}^{r}|D(r,j)||x(j)|.$$

Thus

$$u(n) = \gamma^{-n}|x(n)| \le \lambda|x_0| + \lambda\sum_{r=0}^{n-1}\sum_{j=0}^{r}|D(r,j)|\gamma^{j-r-1}u(j).$$

In view of Lemma 3.4, we obtain

$$u(n) \le \lambda|x_0|\exp(\lambda\sum_{r=0}^{n-1}\sum_{j=0}^{r}|D(r,j)|\gamma^{j-r-1}).$$

It follows that $x(n)$ is bounded. This completes the proof.   □

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