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ON THE MULTIPLE VALUES AND UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING SMALL FUNCTIONS AS TARGETS

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ABSTRACT. The purpose of this article is to deal with the multiple values and uniqueness of meromorphic functions with small functions in the whole complex plane. We obtain a more general theorem which improves and extends strongly the results of R. Nevanlinna, Li-Qiao, Yao, Yi, and Thai-Tan.

1. Introduction and main results

Let h be a nonzero holomorphic function on the whole complex plane $\mathbb{C}$, expanding f as $h(z) = \sum_{i=0}^{\infty} b_i(z-z_0)^i$ around z_0, then we define $\nu_h(z_0) := \min\{i : b_i \neq 0\}$. Let k be a positive integer or $+\infty$. We set

$$\nu_{h,\leq k}(z) = \begin{cases} 0, & \text{if} \quad \nu_h(z) > k; \\ \nu_h(z), & \text{if} \quad \nu_h(z) \leq k. \end{cases}$$

Let φ be a nonconstant meromorphic function on $\mathbb{C}$ with reduced representation $\varphi = (\varphi_0 : \varphi_1)$, where φ_0, φ_1 are holomorphic functions on $\mathbb{C}$ having no common zeros and $\varphi = \frac{\varphi_0}{\varphi_1}$. We define $\nu_\varphi := \nu_{\varphi_0}$, $\nu_{\varphi,\leq k} := \nu_{\varphi_0,\leq k}$.

The characteristic function of φ is defined by

$$T_\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} \log||\varphi(re^{i\theta})||d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log||\varphi(e^{i\theta})||d\theta \quad (r > 1),$$

where $||\varphi|| = \left(|\varphi_0|^2 + |\varphi_1|^2\right)^{1/2}$.

For two meromorphic functions f and a on $\mathbb{C}$ with reduced representations $f = (f_0 : f_1)$, $a = (a_0 : a_1)$ respectively, we set $(f, a) = a_0 f_0 + a_1 f_1$. The meromorphic function a is said to be "small" with respect to f if $T_a(r) = o(T_f(r))$ as $r \to \infty$. Let $\mathcal{R}(f)$ be the set of meromorphic functions on $\mathbb{C}$ which are small with respect to f. Then $\mathcal{R}(f)$ is a field.

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In 1926, R. Nevanlinna [1] proved that for two nonconstant meromorphic functions f and g on $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values, then $f(z) \equiv g(z)$. After his very work, the uniqueness of meromorphic functions with shared values on $\mathbb{C}$ attracted many investigations (for references, see [8]).

It is very interesting to consider distinct small functions instead of distinct complex numbers on $\mathbb{C}$. In 1999, Li and Qiao [2] gave a generalization of the above Nevanlinna theorem that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ and five meromorphic functions $\{a_j\}_{j=1}^5$ in $\mathcal{R}(f) \cap \mathcal{R}(g)$ satisfy $\min\{\nu_{(f,a_j)}, 1\} = \min\{\nu_{(g,a_j)}, 1\}$ $(1 \le j \le 5)$, then $f(z) \equiv g(z)$. Recently, Thai and Tan [3] improved strongly the above-mentioned theorems and results of Yao [5] and Yi [6]. They obtained that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ and five meromorphic functions $\{a_j\}_{j=1}^5$ in $\mathcal{R}(f) \cap \mathcal{R}(g)$ satisfy $\min\{\nu_{(f,a_j),\le k}, 1\} = \min\{\nu_{(g,a_j),\le k}, 1\}$ $(1 \le j \le 5)$, then $f(z) \equiv g(z)$ for each $k \ge 3$.

In 1986, Yi [7] extended the Nevanlinna's very work and others' results, and obtained a general theorem on the multiple values and uniqueness of meromorphic functions as follows. The concepts of $\delta(a, \varphi)$ and $\Theta(a, \varphi)$ are defined as in section 2 below.

Theorem A ([7]). *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$, let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ such that*

$$k_1 \ge k_2 \ge \cdots \ge k_q$$

and

$$\min\{\nu_{(f_1,a_j),\le k_j}, 1\} = \min\{\nu_{(f_2,a_j),\le k_j}, 1\} (j = 1, 2, \ldots, q).$$

Set

$$\Theta_{f_i} = \sum_a \Theta(a, f_i) - \sum_{j=1}^{q} \Theta(a_j, f_i), (i = 1, 2),$$

and

$$A_i = \frac{\delta(a_1, f_i) + \delta(a_2, f_i)}{k_3 + 1} + \sum_{j=3}^{q} \frac{k_j + \delta(a_j, f_i)}{k_{j+1}} + \Theta_{f_i} - 2, \quad (i = 1, 2).$$

If

$$\begin{aligned} \min\{A_1, A_2\} &\ge 0, \\ \max\{A_1, A_2\} &> 0. \end{aligned}$$

Then $f_1(z) \equiv f_2(z)$.

It is natural to ask the following:

Problem 1. *Does Theorem A still hold if $a_j (j = 1, 2, \ldots, q)$ are q distinct elements in $\mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ instead of distinct complex numbers ?*

The purpose of this article is to deal with this problem. In fact, by making use of a recent result of Yamanoi [4], we obtain a more general result as follows, which improves and extends strongly the results of R. Nevanlinna [1], Li-Qiao [2], Yao [5], Yi [6], [7], and Thai-Tan [3].

Theorem 1. *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$, $a_j(j=1,2,\ldots,q)$ be q distinct meromorphic functions in $\mathcal{R}(f_1)\cap\mathcal{R}(f_2)$, and $k_j(j=1,2,\ldots,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq \cdots \geq k_q \tag{1}$$

and

$$\min\{\nu_{(f_1,a_j),\leq k_j},1\} = \min\{\nu_{(f_2,a_j),\leq k_j},1\}(j=1,2,\ldots,q). \tag{2}$$

Set

$$\Theta_{f_i} = \sum_a \Theta(0,f_i-a) - \sum_{j=1}^{q}\Theta(0,f_i-a_j),(i=1,2),$$

and

$$\begin{aligned} A_1 &= \frac{\sum_{j=1}^{m-1}\delta(0,f_1-a_j)}{k_m+1} + \sum_{j=m}^{q}\frac{k_j+\delta(0,f_1-a_j)}{k_{j+1}} \\ &\quad + \frac{(m-2)k_m}{k_m+1} - \frac{k_n}{k_n+1} + \Theta_{f_1} - 2, \\ A_2 &= \frac{\sum_{j=1}^{n-1}\delta(0,f_2-a_j)}{k_n+1} + \sum_{j=n}^{q}\frac{k_j+\delta(0,f_2-a_j)}{k_{j+1}} \\ &\quad + \frac{(n-2)k_n}{k_n+1} - \frac{k_m}{k_m+1} + \Theta_{f_2} - 2, \end{aligned}$$

where m and n are positive integers in $\{1,2,\ldots,q\}$ and a is an arbitrary meromorphic function in $\mathcal{R}(f_i)$ $(i=1,2)$. If

$$\min\{A_1,A_2\} \geq 0, \tag{3}$$

$$\max\{A_1,A_2\} > 0. \tag{4}$$

Then $f_1(z)\equiv f_2(z)$.

From Theorem 1, we obtain the following corollaries.

Corollary 1. *Let f_1 and f_2 be two nonconstant meromorphic functions on $\mathbb{C}$, $a_j(j=1,2,\ldots,q)$ be q distinct meromorphic functions in $\mathcal{R}(f_1)\cap\mathcal{R}(f_2)$, and $k_j(j=1,2,\ldots,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq \cdots \geq k_q$$

and

$$\min\{\nu_{(f_1,a_j),\leq k_j},1\} = \min\{\nu_{(f_2,a_j),\leq k_j},1\}(j=1,2,\ldots,q).$$

Set

$$A_1 = \sum_{j=m}^{q} \frac{k_j}{k_{j+1}} + \frac{(m-2)k_m}{k_m+1} - \frac{k_n}{k_n+1} - 2,$$

$$A_2 = \sum_{j=n}^{q} \frac{k_j}{k_{j+1}} + \frac{(n-2)k_n}{k_n+1} - \frac{k_m}{k_m+1} - 2,$$

where m and n are positive integers in $\{1,2,\ldots,q\}$. If

$$\min\{A_1, A_2\} \geq 0, \quad \max\{A_1, A_2\} > 0.$$

Then $f_1(z) \equiv f_2(z)$.

Corollary 2. *Let f and g be two nonconstant meromorphic functions on $\mathbb{C}$, $a_j (j = 1,2,\ldots,q)$ be q distinct meromorphic functions in $\mathcal{R}(f) \cap \mathcal{R}(g)$, and $k_j (j = 1,2,\ldots,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq \cdots \geq k_q$$

and

$$\min\{\nu_{(f,a_j),\leq k_j}, 1\} = \min\{\nu_{(g,a_j),\leq k_j}, 1\} (j = 1,2,\ldots,q).$$

If

$$A = \sum_{j=m}^{q} \frac{k_j}{k_{j+1}} + \frac{(m-3)k_m}{k_m+1} - 2 > 0,$$

where m is a positive integers in $\{1,2,\ldots,q\}$. Then $f(z) \equiv g(z)$.

Corollary 3. *Let f and g be two nonconstant meromorphic functions on $\mathbb{C}$, $a_j (j = 1,2,\ldots,q)$ be q distinct meromorphic functions in $\mathcal{R}(f) \cap \mathcal{R}(g)$, and $k_j (j = 1,2,\ldots,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq \cdots \geq k_q$$

and

$$\min\{\nu_{(f,a_j),\leq k_j}, 1\} = \min\{\nu_{(g,a_j),\leq k_j}, 1\} (j = 1,2,\ldots,q).$$

If

$$\sum_{j=3}^{q} \frac{k_j}{k_{j+1}} > 2,$$

where m is a positive integers in $\{1,2,\ldots,q\}$. Then $f(z) \equiv g(z)$.

Corollary 4. *Let f and g be two nonconstant meromorphic functions on $\mathbb{C}$, $a_j (j = 1,2,\ldots,q)$ be q distinct meromorphic functions in $\mathcal{R}(f) \cap \mathcal{R}(g)$, and $k_j (j = 1,2,\ldots,q)$ be positive integers or ∞ such that*

$$k_1 \geq k_2 \geq \cdots \geq k_q$$

and

$$\min\{\nu_{(f,a_j),\leq k_j},1\} = \min\{\nu_{(g,a_j),\leq k_j},1\}(j=1,2,\ldots,q).$$

Then

(i) *if* $q=7$, *then* $f(z)\equiv g(z)$.
(ii) *if* $q=6$ *and* $k_3\geq 2$, *then* $f(z)\equiv g(z)$.
(iii) *if* $q=5$, $k_3\geq 3$ *and* $k_5\geq 2$, *then* $f(z)\equiv g(z)$.
(iv) *if* $q=5$ *and* $k_4\geq 4$, *then* $f(z)\equiv g(z)$.
(v) *if* $q=5$, $k_3\geq 5$ *and* $k_4\geq 3$, *then* $f(z)\equiv g(z)$.
(vi) *if* $q=5$, $k_3\geq 6$ *and* $k_4\geq 2$, *then* $f(z)\equiv g(z)$.

Remark. The above-mentioned result of Thai and Tan [3] is just the special case as $q=5$ and $k_1=k_2=\cdots=k_5=k\geq 3$. Thus Corollary 4(iii) is an improvement of it.

2. Basic notions in Nevanlinna theory

Let h be a nonzero holomorphic function on $\mathbb{C}$ and k be a positive integer or $k=\infty$. We define

$$N_{h,\leq k}(r) = \int_1^r \frac{n_{\leq k}(t)}{t}dt \quad \text{and} \quad \overline{N}_{h,\leq k}(r) = \int_1^r \frac{\overline{n}_{\leq k}(t)}{t}dt \quad (r>1),$$

where $n_{\leq k}(t)=\sum_{|z|\leq t}\nu_{h,\leq k}(z)$ and $\overline{n}_{\leq k}(t)=\sum_{|z|\leq t}\min\{\nu_{h,\leq k}(z),1\}$.

Let φ be a nonconstant meromorphic function on $\mathbb{C}$ with reduced representation $\varphi=(\varphi_0:\varphi_1)$. We define $N_{\varphi,\leq k}(r):=N_{\varphi_0,\leq k}(r)$ and $\overline{N}_{\varphi,\leq k}(r):=\overline{N}_{\varphi_0,\leq k}(r)$. For brevity we write $N_{\varphi,\leq\infty}(r)$ as $N_\varphi(r)$ or $N(r,\nu_\varphi)$; write

$$\overline{N}_{\varphi,\leq\infty}(r)$$

as $\overline{N}_\varphi(r)$ or $\overline{N}(r,\nu_\varphi)$; and write $N_{\varphi,\leq k}(r)$ as $N_{\leq k}(r,\nu_\varphi)$. Set

$$\nu_{h,\geq k+1}(z) = \begin{cases} 0, & \text{if} \quad \nu_h(z)<k; \\ \nu_h(z), & \text{if} \quad \nu_h(z)\geq k+1. \end{cases}$$

Similarly, we can get the corresponding definitions of $N_{\varphi,\geq k+1}(r)$, $\overline{N}_{\varphi,\geq k+1}(r)$, etc.

Let $\{a_j\}_{j=0}^q$ be meromorphic functions on $\mathbb{C}$ with reduced representations $a_j=(a_{j0}:a_{j1})$ $(0\leq j\leq q)$. For each $0\leq j\leq q$, we fix an index $k_j\in\{0,1\}$ such that $a_{jk_j}\not\equiv 0$ and set $a_j^*:=(a_{j1}:-a_{j0})$, $\tilde{a}_j:=\left(\frac{a_{j0}}{a_{jk_j}}:\frac{a_{j1}}{a_{jk_j}}\right)$, $\tilde{a}_j^*:=\left(\frac{a_{j1}}{a_{jk_j}}:-\frac{a_{j0}}{a_{jk_j}}\right)$.

Let f be a meromorphic function on $\mathbb{C}$ with reduced representation $f=(f_0:f_1)$. For each $0\leq j\leq q$, we set $(f,\tilde{a}_j)=\frac{a_{j0}f_0+a_{j1}f_1}{a_{jk_j}}$, $(f,\tilde{a}_j^*)=\frac{a_{j1}f_0-a_{j0}f_1}{a_{jk_j}}$.

For a meromorphic function f on $\mathbb{C}$, we define the proximity function of f by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$. Then

$$T_f(r) = N(r, \nu_{1/f}) + m(r, f) + O(1).$$

Let a be an arbitrary complex number. We denote the deficiency of a with respect to f by

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T_f(r)} = 1 - \limsup_{r\to\infty} \frac{N(r, \nu_{(f,a)})}{T_f(r)},$$

and denote the Valiron's deficiency by

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \nu_{(f,a)})}{T_f(r)}.$$

As usual, by the notation "$\|P$" we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

Theorem B ([4]). *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let a_1, a_2, $\ldots$, a_q be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i are small functions with respect to f for all $1 \leq i \leq q$. Then for each $\varepsilon > 0$, the following holds*

$$\|(q - 2 - \varepsilon)T_f(r) \leq \Sigma_{i=1}^q \overline{N}_{(f,a_i)}(r) + o(T_f(r)).$$

3. Proofs

For the proof of Theorem 1, we need give the following lemmas.

Lemma 1 ([3]). *Let f be a nonconstant meromorphic function on $\mathbb{C}$ and a_1, a_2 be two distinct small functions with respect to f. Then*

$$T_{\frac{(f,\tilde{a}_1)}{(f,\tilde{a}_2)}}(r) = T_f(r) + o(T_f(r)).$$

Lemma 2. *Let f be a nonconstant meromorphic function on $\mathbb{C}$, a be a small function with respect to f, and k be a positive integer. Then*

$$\overline{N}_{(f,a)}(r) \leq \frac{k}{k+1}\overline{N}_{(f,a),\leq k}(r) + \frac{1}{k+1}N_{(f,a)}(r);$$

and

$$\overline{N}_{(f,a)}(r) \leq \frac{k}{k+1}\overline{N}_{(f,a),\leq k}(r) + \frac{1}{k+1}T_f(r) + o(T_f(r)).$$

Proof. From

$$\overline{N}_{(f,a)}(r) = \overline{N}_{(f,a),\leq k}(r) + \overline{N}_{(f,a),\geq k+1}(r)$$

and

$$\overline{N}_{(f,a),\geq k+1}(r) \leq \frac{1}{k+1}N_{(f,a),\geq k+1}(r),$$

we deduce that

$$\begin{aligned}\overline{N}_{(f,a)}(r) &\le \frac{k}{k+1}\overline{N}_{(f,a),\le k}(r) + \frac{1}{k+1}\overline{N}_{(f,a),\le k}(r) \\ &\quad + \frac{1}{k+1}\overline{N}_{(f,a),\ge k+1}(r) \\ &\le \frac{k}{k+1}\overline{N}_{(f,a),\le k}(r) + \frac{1}{k+1}N_{(f,a)}(r).\end{aligned}$$

This completes the proof of the first inequality of the lemma. The second inequality of the lemma follows immediately because of

$$N_{(f,a)}(r) \le T_f(r) + o\left(T_f(r)\right).$$

□

3.1. Proof of Theorem 1

We suppose that $f_1(z) \not\equiv f_2(z)$. Without loss of generality, we may assume that there exist infinitely many small functions b with respect to f_1 such that $\Theta(0, f_1 - b) > 0$ and $b \not\equiv a_j$ $(j = 1, 2, \ldots, q)$. We denote them by b_k $(k = 1, 2, \ldots, \infty)$. Obviously, $\Theta_{f_1} = \sum_{k=1}^{\infty} \Theta(0, f_1 - b_k)$. Thus there exits a p such that $\sum_{k=1}^{p} \Theta(0, f_1 - b_k) > \Theta_{f_1} - \varepsilon$ holds for ε (> 0). From Theorem B we have

$$(5)\quad \|(p+q-2-\varepsilon)T_{f_1}(r) \le \Sigma_{k=1}^{p}\overline{N}_{(f_1,b_k)}(r) + \Sigma_{j=1}^{q}\overline{N}_{(f_1,a_j)}(r) + o\left(T_{f_1}(r)\right).$$

It is easy to see that

$$(6)\qquad \overline{N}_{(f_1,b_k)}(r) < \left(1 - \Theta(0, f_1 - b_k)\right) T_{f_1}(r) + o\left(T_{f_1}(r)\right).$$

From Lemma 2 we get

$$\begin{aligned}\overline{N}_{(f_1,a_j)}(r) &\le \frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j),\le k_j}(r) + \frac{1}{k_j+1}N_{(f_1,a_j)}(r) \\ &< \frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j),\le k_j}(r) + \frac{1}{k_j+1}\left(1 - \delta(0, f_1 - a_j)\right) T_{f_1}(r) \\ &\quad + o\left(T_{f_1}(r)\right).\end{aligned}$$

Submitting the above inequalities and (6) into (5), we get

$$\begin{aligned}\|(p+q-2-\varepsilon)T_{f_1}(r) &\le \left\{\Sigma_{k=1}^{p}\left(1 - \Theta(0, f_1 - b_k)\right)\right\} T_{f_1}(r) \\ &\quad + \Sigma_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{(f_1,a_j),\le k_j}(r) \\ &\quad + \left\{\Sigma_{j=1}^{q}\frac{1}{k_j+1}\left(1 - \delta(0, f_1 - a_j)\right)\right\} T_{f_1}(r) \\ &\quad + o\left(T_{f_1}(r)\right).\end{aligned}$$

From (1) we have

$$1 \ge \frac{k_1}{k_1+1} \ge \frac{k_2}{k_2+1} \ge \cdots \ge \frac{k_q}{k_q+1} \ge \frac{1}{2}.$$

Hence we can deduce that

$$\begin{aligned}
&\|(p+q-2-\varepsilon)T_{f_1}(r)\\
\leq\ & (p-\Theta_{f_1}+\varepsilon)\,T_{f_1}(r)\\
&+\Sigma_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{(f_1,a_j),\leq k_j}(r)\\
&+\left\{\Sigma_{j=1}^{m-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)(1-\delta(0,f_1-a_j))\right\}T_{f_1}(r)\\
&+\left\{\Sigma_{j=1}^{q}\frac{1-\delta(0,f_1-a_j)}{k_j+1}\right\}T_{f_1}(r)\\
&+o\left(T_{f_1}(r)\right),
\end{aligned}$$

namely,

$$\begin{aligned}
&\left\|\left(\frac{(m-1)k_m}{k_m+1}+B_1-2\varepsilon\right)T_{f_1}(r)\right.\\
&\leq \Sigma_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{(f_1,a_j),\leq k_j}(r)+o\left(T_{f_1}(r)\right),
\end{aligned}$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\delta(0,f_1-a_j)}{k_m+1}+\sum_{j=m}^{q}\frac{k_j+\delta(0,f_1-a_j)}{k_{j+1}}+\Theta_{f_1}-2.$$

Similarly,

$$\begin{aligned}
&\left\|\left(\frac{(n-1)k_n}{k_n+1}+B_2-2\varepsilon\right)T_{f_2}(r)\right.\\
&\leq \Sigma_{j=1}^{q}\frac{k_n}{k_n+1}\overline{N}_{(f_2,a_j),\leq k_j}(r)+o\left(T_{f_2}(r)\right),
\end{aligned}$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\delta(0,f_2-a_1)}{k_n+1}+\sum_{j=n}^{q}\frac{k_j+\delta(0,f_2-a_j)}{k_{j+1}}+\Theta_{f_2}-2.$$

Hence

$$\begin{aligned}
&\left\|\left(\frac{(m-1)k_m}{k_m+1}+B_1-2\varepsilon\right)T_{f_1}(r)+\left(\frac{(n-1)k_n}{k_n+1}+B_2-2\varepsilon\right)T_{f_2}(r)\right.\\
\leq\ & \Sigma_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{(f_1,a_j),\leq k_j}(r)+\Sigma_{j=1}^{q}\frac{k_n}{k_n+1}\overline{N}_{(f_2,a_j),\leq k_j}(r)\\
&+o\left(T_{f_1}(r)+T_{f_2}(r)\right).
\end{aligned}$$

Let a_0 be a nonzero meromorphic function on $\mathbb{C}$ such that

$$a_0\in(\mathcal{R}(f_1)\cap\mathcal{R}(f_2))\setminus\{a_j\}_{j=1}^{q}.$$

Since $f_1(z) \not\equiv f_2(z)$, there exists $1 \le j \le q$ such that $\frac{(f_1,\tilde{a}_j)}{(f_1,\tilde{a}_0)} \not\equiv \frac{(f_2,\tilde{a}_j)}{(f_2,\tilde{a}_0)}$. Without loss of generality, we may assume that $j = 1$, namely $\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} \not\equiv \frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}$. From (2), we have $f_1 = f_2$ on $\cup_{j=1}^q\{z : \nu_{(f_1,a_j),\le k_j}(z) > 0\}$. It is easy to see that $(a_i^*, a_j) = 0$ on $\{z : (f_1, a_i)(z) = 0 \quad \text{and} \quad (f_2, a_j)(z) = 0\}$ $(0 \le i < j \le q)$. So we deduce by Lemma 1 that

$$
\begin{aligned}
& \Sigma_{j=1}^q \overline{N}_{(f_1,a_j),\le k_j}(r) \\
\le\ & N\left(r, \nu_{\frac{(f_1,a_1)}{(f_1,a_0)} - \frac{(f_2,a_1)}{(f_2,a_0)}}\right) + \sum_{0\le i<j\le q} N(r, \nu_{(a_i^*,a_j)}) \\
=\ & N_{\left(\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} - \frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}\right)\cdot\frac{a_{1k_1}}{a_{0k_0}}}(r) + \sum_{0\le i<j\le q} N_{a_{i1}a_{j0}-a_{i0}a_{j1}}(r) \\
\le\ & N_{\left(\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} - \frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}\right)}(r) + N_{a_{1k_1}}(r) \\
& + \sum_{0\le i<j\le q}\left(N_{\frac{a_{i1}a_{j0}}{a_{i0}a_{j1}}-1}(r) + N_{a_{i0}a_{j1}}(r)\right) + O(1) \\
\le\ & T_{\left(\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)} - \frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}\right)}(r) + T_{a_1}(r) \\
& + \sum_{0\le i<j\le q}\left(T_{\frac{a_j}{a_i}}(r) + T_{a_i}(r) + T_{a_j}(r)\right) + O(1) \\
\le\ & T_{\frac{(f_1,\tilde{a}_1)}{(f_1,\tilde{a}_0)}}(r) + T_{\frac{(f_2,\tilde{a}_1)}{(f_2,\tilde{a}_0)}}(r) + T_{a_1}(r) \\
& + \sum_{0\le i<j\le q}\left(T_{\frac{a_j}{a_i}}(r) + T_{a_i}(r) + T_{a_j}(r)\right) + O(1) \\
\le\ & T_{f_1}(r) + T_{f_2}(r) + o\left(T_{f_1}(r) + T_{f_2}(r)\right).
\end{aligned}
$$

Similarly,

$$\Sigma_{j=1}^q \overline{N}_{(f_2,a_j),\le k_j}(r) \le T_{f_1}(r) + T_{f_2}(r) + o\left(T_{f_1}(r) + T_{f_2}(r)\right).$$

Hence from above discussion, we obtain

$$
\begin{aligned}
& \Big\| \left(\frac{(m-1)k_m}{k_m+1} + B_1 - 2\varepsilon\right) T_{f_1}(r) + \left(\frac{(n-1)k_n}{k_n+1} + B_2 - 2\varepsilon\right) T_{f_2}(r) \\
\le\ & \left(\frac{k_m}{k_m+1} + \frac{k_n}{k_n+1}\right)\left(T_{f_1}(r) + T_{f_2}(r)\right) + o\left(T_{f_1}(r) + T_{f_2}(r)\right),
\end{aligned}
$$

namely,

$$\|(A_1 - 2\varepsilon)\, T_{f_1}(r) + (A_2 - 2\varepsilon)\, T_{f_2}(r) \le o\left(T_{f_1}(r) + T_{f_2}(r)\right).$$

Letting $r \to \infty$ and $\varepsilon \to 0$, we have a contradiction with (3) and (4). Therefore, we complete the proof of Theorem 1.

3.2. Proof of Corollary 1

Since $\Theta_{f_i} \geq 0$ and $\delta(0, f_1 - a_j) \geq 0$ $(j = 1, 2, \ldots, q)$, then it implies from Theorem 1 that Corollary 1 follows.

3.3. Proof of Corollary 2

Letting $n = m$, Corollary 2 follows immediately from Corollary 1.

3.4. Proof of Corollary 3

Letting $m = 3$, Corollary 3 follows immediately from Corollary 2.

3.5. Proof of Corollary 4

From (1) we have

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}.$$

Hence we can get from Corollary 3 that Corollary 4 follows.

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