Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 623–629

UNIQUENESS OF ENTIRE FUNCTIONS AND DIFFERENTIAL POLYNOMIALS

Junfeng Xu and Hongxun Yi

Abstract. In this paper, we study the uniqueness of entire functions and prove the following result: Let f and g be two nonconstant entire functions, n, m be positive integers. If $f^n(f^m-1)f'$ and $g^n(g^m-1)g'$ share 1 IM and $n > 4m+11$, then $f \equiv g$. The result improves the result of Fang-Fang.

1. Introduction

Let f be nonconstant meromorphic in the complex plane, we assumed that the reader is familiar with the notations of Nevanlinna theory ([3, 7]).

Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a IM. Let k be a positive integer. Set $E_k(a,f) = \{z : f(z) - a = 0, \exists\ \mathrm{i}, 1 \le i \le k,\ \text{st. } f^{(i)}(z) \neq 0\}$, where a zero point with multiplicity $m(\le k)$ is counted m times in the set.

In addition, we also use the following notations.

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_{k)}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\le k$, by $N_{(k}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\ge k$; and denote the reduced counting function by $\overline{N}_{k)}(r, 1/(f-a))$, $\overline{N}_{(k}(r, 1/(f-a))$, respectively. Set $N_k(r, 1/(f-a)) = \overline{N}(r, 1/(f-a)) + \overline{N}_{(2}(r, 1/(f-a)) + \cdots + \overline{N}_{(k}(r, 1/(f-a))$.

In 1976, Gross [2] proposed the following question.

Question 1. *Whether there exists a finite set S such that $E(S,f) = E(S,g)$ can imply $f \equiv g$ for any pair of nonconstant entire functions f and g ?*

Received April 11, 2006.

2000 *Mathematics Subject Classification.* 30D35.

Key words and phrases. uniqueness, entire function, sharing values.

Supported by Specialized Research Fund for the Doctoral Program of Higher Education (20060422049) and NSF of China (10771121).

Yi [8] gave a positive answer to Question 1. He proved there exists a polynomial set S with 7 elements such that $E(S,f) = E(S,g)$ can imply $f \equiv g$ for any pair of nonconstant entire functions f and g.

In 2002, Fang and Fang [1] proved there exists a differential polynomial d such that for any pair of nonconstant entire functions f and g we can get $f \equiv g$ if $d(f)$ and $d(g)$ share one value CM.

Theorem A ([1]). *Let f and g be two nonconstant entire functions, $n \geq 8$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1* CM, *then $f \equiv g$.*

In 2004, Lin-Yi [4] and Qiu-Fang [5] proved that Theorem A remains valid for $n \geq 7$.

Theorem B. *Let f and g be two nonconstant entire functions, $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1* CM, *then $f \equiv g$.*

Naturally, one can pose the following question: what can be stated if CM is replaced with IM in Theorem B. In ([1]), Fang-Fang obtained the following theorem.

Theorem C. *Let f and g be two nonconstant entire functions, n be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share* IM *and $n \geq 17$, then $f \equiv g$.*

In this paper, we improve Theorem C and prove Theorem C holds for $n \geq 16$ by the different method. In fact, we get

Theorem 1.1. *Let f and g be two nonconstant entire functions, n, m be positive integers. If $f^n(f^m-1)f'$ and $g^n(g^m-1)g'$ share the value 1* IM *and $n > 4m+11$, then $f \equiv g$.*

Remark 1.2. Let $m = 1$ in theorem 1.1, then $n \geq 16$. Obviously, Theorem 1.1 improved Theorem C.

In fact, for transcendental entire functions, we also can extend Theorem 1.1 in view of the fixed-point.

Theorem 1.3. *Let f and g be two transcendental entire functions, n, m be positive integers. If $f^n(f^m-1)f'$ and $g^n(g^m-1)g'$ share z* IM *and $n > 4m+11$, then $f \equiv g$.*

2. Lemmas

Lemma 2.1 ([6]). *Let f be a nonconstant meromorphic function, n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$ where a_i is a meromorphic function satisfying $T(r, a_i) = S(r,f) (i = 1, 2, \ldots, n)$. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

Lemma 2.2 ([6]). *Let f and g be two meromorphic functions, and let k be a positive integer. Then*

$$N(r, \frac{1}{f^{(k)}}) < N(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f).$$

Using the method of [10], we get the following.

Lemma 2.3. *Let F and G be two nonconstant meromorphic functions sharing the value 1* IM. *Let*

$$H = (\frac{F''}{F'} - 2\frac{F'}{F-1}) - (\frac{G''}{G'} - 2\frac{G'}{G-1}).$$

If $H \not\equiv 0$, then

$$\begin{aligned} &T(r,F) + T(r,G) \\ &\leq 2\Big(N_2(r,F) + N_2(r,G) + N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G})\Big) \\ &+ 3\Big(\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G})\Big) \\ &+ S(r,F) + S(r,G). \end{aligned}$$

Lemma 2.4. *Let f and g be two nonconstant entire functions. Let $F = f^n(f^m-1)f'$ and $G = g^n(g^m-1)g'$, where m, n are two positive integers. If F and G share the value 1* IM *and $n > m+2$, then $S(r,f) = S(r,g)$.*

Proof. By Lemma 2.1, we get

$$\begin{aligned} (n+m)T(r,f) &= T(r,f^n(f^m-1)) + S(r,f) \\ &\leq T(r,F) + T(r,f') + S(r,f). \end{aligned}$$

Hence

$$T(r,F) \geq (n+m-1)T(r,f) + S(r,f).$$

By the second fundamental theorem and Lemma 2, then

$$\begin{aligned} T(r,F) &= \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) + S(r,f) \\ &= \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f^m-1}) + \overline{N}(r,\frac{1}{f'}) \\ &\quad + \overline{N}(r,\frac{1}{G-1}) + S(r,f) \\ &= (2+m)T(r,f) + T(r,G) + S(r,f). \end{aligned}$$

Noticing that

$$T(r,G) \leq T(r,g^n(g^m-1)) + T(r,g') \leq (n+m+1)T(r,g) + S(r,g).$$

Hence

$$(n-3)T(r,f) \leq (n+m+1)T(r,g) + S(r,g) + S(r,f).$$

Similarly, we have

$$(n-3)T(r,g) \leq (n+m+1)T(r,f) + S(r,g) + S(r,f).$$

This proves the lemma. □

3. Proofs of the theorems

In this section we present the proofs of the main results.

Proof of Theorem 1.1. Let

$$F = f^n(f^m-1)f', \qquad G = g^n(g^m-1)g', \tag{1}$$

$$\begin{aligned} F_1 &= \frac{1}{n+m+1}f^{n+m+1} - \frac{1}{n+1}f^{n+1}, \\ G_1 &= \frac{1}{n+m+1}g^{n+m+1} - \frac{1}{n+1}g^{n+1}, \end{aligned} \tag{2}$$

where F and G share the value 1 IM. Then by Lemma 2.1, we have

$$T(r, F_1) = (n+m+1)T(r, f) + S(r, f) \tag{3}$$

$$T(r, G_1) = (n+m+1)T(r, g) + S(r, f). \tag{4}$$

Noting that $F_1' = F$, we have

$$m(r, \frac{1}{F_1}) \leq m(r, \frac{1}{F}) + S(r, f).$$

Then by the first fundamental theorem, we have

$$T(r, F_1) = T(r, F) + N(r, \frac{1}{F_1}) - N(r, \frac{1}{F}) + S(r, f).$$

Obviously, we have

$$N(r, \frac{1}{F_1}) = (n+1)N(r, \frac{1}{f}) + N(r, \frac{1}{f^m - \frac{n+m+1}{n+1}}), \tag{5}$$

$$N(r, \frac{1}{F}) = nN(r, \frac{1}{f}) + N(r, \frac{1}{f^m-1}) + N(r, \frac{1}{f'}). \tag{6}$$

By (4), (5) and (6), we have

$$\begin{aligned} T(r, F_1) \leq\ & T(r, F) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^m - \frac{n+m+1}{n+1}}) \\ & -N(r, \tfrac{1}{f^m-1}) - N(r, \tfrac{1}{f'}) + S(r, f). \end{aligned} \tag{7}$$

Likewise, we have

$$\begin{aligned} T(r, G_1) \leq\ & T(r, G) + N(r, \frac{1}{g}) + N(r, \frac{1}{g^m - \frac{n+m+1}{n+1}}) \\ & -N(r, \tfrac{1}{g^m-1}) - N(r, \tfrac{1}{g'}) + S(r, g). \end{aligned} \tag{8}$$

Let H be the same as Lemma 2.3, and $H \not\equiv 0$, then by (1), we have

$$N_2(r, F) + N_2(r, \frac{1}{F}) \leq 2N(r, \frac{1}{f}) + N(r, \frac{1}{f^m-1}) + N(r, \frac{1}{f'}), \tag{9}$$

$$N_2(r, G) + N_2(r, \frac{1}{G}) \leq 2N(r, \frac{1}{g}) + N(r, \frac{1}{g^m-1}) + N(r, \frac{1}{g'}). \tag{10}$$

Utilizing Lemma 2.3 and (7)-(10), we have

$$\begin{aligned}
&T(r,F_1)+T(r,G_1)\\
&\le 5N(r,\frac{1}{f})+\ N\big(r,\frac{1}{f^m-\frac{n+m+1}{n+1}}\big)\\
&\quad+N(r,\frac{1}{f^m-1})+\ N\big(r,\frac{1}{f'}\big)+S(r,f)\\
&\quad+5N(r,\frac{1}{g})+\ N\big(r,\frac{1}{g^m-\frac{n+m+1}{n+1}}\big)\\
&\quad+N(r,\frac{1}{g^m-1})+\ N\big(r,\frac{1}{g'}\big)+S(r,g)\\
&\quad+3\big(\overline{N}\big(r,\frac{1}{F}\big)+\overline{N}\big(r,\frac{1}{G}\big)\big)+S(r,F)+S(r,G).
\end{aligned} \tag{11}$$

Noting that

$$\overline{N}(r,\tfrac{1}{F})=\overline{N}(r,\tfrac{1}{f})+\overline{N}(r,\tfrac{1}{f^m-1})+\overline{N}(r,\tfrac{1}{f'}). \tag{12}$$

Similarly,

$$\overline{N}(r,\frac{1}{G})\le\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g^m-1})+\overline{N}(r,\frac{1}{g'}). \tag{13}$$

Combining the equation (11),(12) and (13), we have

$$\begin{aligned}
T(r,F_1)+T(r,G_1)\le\ &8N(r,\frac{1}{f})+N(r,\frac{1}{f^m-\frac{n+m+1}{n+1}})\\
&+4N(r,\frac{1}{f^m-1})+4N(r,\frac{1}{f'})+S(r,f)\\
&+8N(r,\frac{1}{g})+N(r,\frac{1}{g^m-\frac{n+m+1}{n+1}})\\
&+4N(r,\frac{1}{g^m-1})+4N(r,\frac{1}{g'})+S(r,g).
\end{aligned} \tag{14}$$

By Lemma 2.4, we get

$$N(r,\frac{1}{f'})\le N(r,\frac{1}{f})+S(r,f),\quad N(r,\frac{1}{g'})\le N(r,\frac{1}{g})+S(r,g).$$

From this and with (3), (4), (14), we have

$$(n-4m-11)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g). \tag{15}$$

By Lemma 2.4, $S(r,f)=S(r,g)$. Hence we can get a contradiction from $n>4m+11$. Therefore, $H\equiv 0$.

This is,

$$\frac{F''}{F'}-2\frac{F'}{F-1}=\frac{G''}{G'}-2\frac{G'}{G-1}.$$

By integration, we have

$$F=((b+1)G+(a-b-1))/(bG+(a-b)),$$

where $a(\neq 0), b$ are two constants. Then by the same argument of Lemma 6 in [1], we have $f \equiv g$. We complete the proof of Theorem 1.1. □

Proof of Theorem 1.3. We just need to replace F and G in (1) by

$$F = \frac{f^n(f^m - 1)f'}{z}, \quad G = \frac{g^n(g^m - 1)g'}{z}, \tag{16}$$

and thus $F_1 = Fz$. Since the term of $m(r, \frac{1}{z}) = O(\log r)$, it will be occurred during the proof of Theorem 1.1, especially occurred in the right of (15), but we still can get a contradiction for f and g are transcendental. This proves Theorem 1.3. □

4. Final remarks

Remark 4.1. It follows from the proof of Theorem 1.3 that if the condition "$f^n(f^m - a)f'$ and $g^n(g^m - a)g'$ share z IM" is replaced by condition "$f^n(f^m - a)f'$ and $g^n(g^m - a)g'$ share $\alpha(z)$ IM", where $\alpha(z)$ is a meromorphic function such that $\alpha \not\equiv 0, \infty$ and $T(r, \alpha) = o\{T(r, f), T(r, g)\}$, the conclusion of Theorem 1.3 still holds.

Theorem 4.2. *Let f and g be two transcendental entire functions, n, m be positive integers. If $f^n(f^m - a)f'$ and $g^n(g^m - a)g'$ share α IM and $n > 4m + 11$, then $f \equiv g$.*

Similarly, we can get the following theorem by applying the method of Theorem 1.1.

Theorem 4.3. *Let f and g be two nonconstant entire functions, n be a positive integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share the value 1 IM and satisfy one of the following condition,*

(i) *if $E_k(1, f^n(f - 1)f') = E_k(1, g^n(g - 1)g'), k(\geq 3), n(\geq 7)$;*
(ii) *if $E_2(1, f^n(f - 1)f') = E_2(1, g^n(g - 1)g'), n(\geq 8)$;*
(iii) *if $E_1(1, f^n(f - 1)f') = E_1(1, g^n(g - 1)g'), n(\geq 12)$;*
then $f \equiv g$.

As we all known, the condition of n in Theorem 2′, Theorem 3′, Theorem 4′ (see [1]) are $n(\geq 8)$, $n(\geq 9)$ and $n(\geq 14)$, respectively. Obviously, the condition of (i), (ii), (iii) of Theorem 4.2 improved these theorems, respectively.

References

[1] C. Y. Fang and M. L. Fang, *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl. **44** (2002), no. 5-6, 607–617.
[2] F. Gross, *Factorization of meromorphic functions and some open problems*, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), pp. 51–67. Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977.
[3] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, The Clarendon Press, Oxford, 1964.
[4] W. C. Lin and H. X. Yi, *Uniqueness theorems for meromorphic function*, Indian J. Pure Appl. Math. **35** (2004), no. 2, 121–132.

[5] H. L. Qiu and M. L. Fang, *On the uniqueness of entire functions*, Bull Korean Math. Soc. **41** (2004), no. 1, 109–116.
[6] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.
[7] L. Yang, *Value distribution theory*, Translated and revised from the 1982 Chinese original. Springer-Verlag, Berlin; Science Press, Beijing, 1993.
[8] H. X. Yi, *A question of Gross and the uniqueness of entire functions*, Nagoya Math. J. **138** (1995), 169–177.
[9] ———, *Meromorphic functions that share one or two values*, Complex Variables Theory Appl. **28** (1995), no. 1, 1–11.
[10] ———, *Meromorphic functions that share one or two values II*, Kodai Math. J. **22** (1999), no. 2, 264–272.

Junfeng Xu
Department of Mathematics
Shandong University
Jinan, Shandong 250100, P. R. China
E-mail address: jfxu@mail.sdu.edu.cn

Hongxun Yi
Department of Mathematics
Shandong University
Jinan, Shandong 250100, P. R. China
E-mail address: hxyi@sdu.edu.cn