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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO DIFFERENTIAL MONOMIALS SHARE ONE VALUE

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Abstract. We prove four theorems on the uniqueness of non linear differential polynomials sharing one value which improve a result of Yang and Hua, and supplements some results of Lahiri, Xu and Qiu and Banerjee.

1. Introduction definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities). Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a; f)$ the set of distinct a-points of f with multiplicities not greater than m. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f)$=$E_{\infty)}(a; g)$ we say that f, g share the value a CM.

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Yang and Hua [12] studied the problem of non linear differential polynomials when they share the same value a CM. They proved the following result.

Theorem A ([12]). *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Corresponding to entire functions Xu and Qiu proved the following result.

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Theorem B ([10]). *Let f and g be two nonconstant entire functions, $n \geq 12$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

To state the next result we require the following definition.

Definition 1.1 ([4, 5]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values improving *Theorem A* the following result is proved in [5].

Theorem C ([5]). *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $(a, 2)$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

In the same direction recently the present author improved *Theorem B* and supplement *Theorem C.* The present author proved the following two theorems.

Theorem D ([1]). *Let f and g be two nonconstant meromorphic functions such that $n > 22 - [5\Theta(\infty; f) + 5\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$, where n is an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 0)$, then conclusion of the Theorem C holds.*

Theorem E ([1]). *Let f and g be two nonconstant meromorphic functions and $n > \max\{8, 12 - \{3\Theta(\infty; f) + 3\Theta(\infty; g)\}\}$, an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 1)$, then conclusion of the Theorem C holds.*

Now one may ask the following questions which are the motivations of the paper.

(i) In *Theorem A* can the nature of sharing the value a be further relaxed other than the concept of weighted sharing ?

(ii) Can one obtain the same result as in *Theorem D* and *Theorem E* with a different nature of sharing the value a ?

In this paper we shall investigate the possible solutions of the above questions. We now state the following theorems which are the main results of the paper.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions and $n > 22 - [5\Theta(\infty; f) + 5\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$, is an integer. If for $a \in \mathbb{C} - \{0\}$, $\overline{E}_{2)}(a; f^n f') = \overline{E}_{2)}(a; g^n g')$ then conclusion of the Theorem C holds.*

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions and $n > \max[8, \frac{38}{3} - \{3\Theta(\infty; f) + 3\Theta(\infty; g) + \frac{1}{3}\min\{\Theta(\infty; f), \Theta(\infty; g)\}\}]$, is an integer. If for $a \in \mathbb{C} - \{0\}$, $\overline{E}_{3)}(a; f^n f') = \overline{E}_{3)}(a; g^n g')$ and $E_{1)}(a; f^n f') = E_{1)}(a; g^n g')$ then conclusion of the Theorem C holds.*

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions and $n > \max[8, 12 - \{3\Theta(\infty; f) + 3\Theta(\infty; g)\}]$, is an integer. If for $a \in \mathbb{C} - \{0\}$, $\overline{E}_{4)}(a; f^n f') = \overline{E}_{4)}(a; g^n g')$ and $E_{1)}(a; f^n f') = E_{1)}(a; g^n g')$ then conclusion of the Theorem C holds.*

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions and $n \geq 11$, is an integer. If for $a \in \mathbb{C} - \{0\}$, $\overline{E}_{4)}(a; f^n f') = \overline{E}_{4)}(a; g^n g')$ and $E_{2)}(a; f^n f') = E_{2)}(a; g^n g')$ then conclusion of the Theorem C holds.*

Remark 1.1. *Theorem 1.4* improves *Theorem A*.

Though the standard definitions and notations of the value distribution theory are available in [2], we explain some definitions and notations which are used in the paper.

Definition 1.2 ([3]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m, where each a point is counted according to its multiplicity. $\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.3 ([5], [14]). We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.

Definition 1.4. Let m and r be two positive integers such that $1 \leq r < m - 1$ and for $a \in \mathbb{C}$, $\overline{E}_{m)}(a; f) = \overline{E}_{m)}(a; g)$, $E_{r)}(a; f) = E_{r)}(a; g)$. Let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g for which $p > q \geq r + 1$ ($q > p \geq r + 1$), by $\overline{N}_E^{(r+1}(r, a; f)$ the reduced counting function of those a-points of f and g for which $p = q \geq r + 1$, by $\overline{N}_{f \geq m+1}(r, a; f \mid g \neq a)$ $(\overline{N}_{g \geq m+1}(r, a; g \mid f \neq a))$ the reduced counting functions of those a-points of f and g for which $p \geq m+1$ and $q = 0$ ($q \geq m+1$ and $p = 0$).

Definition 1.5. If $r = 0$ in definition 1.4 then we use the same notations as in definition 1.4 except by $N_E^{1)}(r, a; f)$ we mean the common simple a-points of f and g and by $\overline{N}_E^{(2}(r, a; f)$ we mean the reduced counting functions of those a-points of f and g for which $p = q \geq 2$.

Definition 1.6. Let $\overline{E}_{m)}(a; f) = \overline{E}_{m)}(a; g)$ for $a \in \mathbb{C}$. Also let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q. We denote by $\overline{N}(r, a; f \mid= p; g \mid= q)$ the reduced counting functions of common a-points of f and g with multiplicity p and q respectively. Also we denote by $\overline{N}_{f>s}(r, a; g)$ ($\overline{N}_{f>s}(r, a; g)$) the reduced counting functions of those a-points of f and g for which $p > q = s$ ($q > p = s$)

Definition 1.7 ([6]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.8 ([6]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right)$$

and

$$H = \left(\frac{F'''}{F''} - \frac{2F''}{F'-1}\right) - \left(\frac{G'''}{G''} - \frac{2G''}{G'-1}\right).$$

Lemma 2.1. *If f, g be two nonconstant meromorphic functions such that $E_{1)}(1; f) = E_{1)}(1; g)$ and $h \not\equiv 0$ then*

$$N(r, 1; f \mid\leq 1) = N(r, 1; g \mid\leq 1) \leq N(r, 0; h) \leq N(r, \infty; h) + S(r, f) + S(r, g).$$

Proof. Since the functions f and g have the same simple one points it can be easily verified by direct computation that the function h is zero whenever $f - 1$ has a simple zero. This proves the lemma. ☐

Lemma 2.2. *Let $\overline{E}_{m)}(1;f) = \overline{E}_{m)}(1;g)$, $E_{1)}(1;f) = E_{1)}(1;g)$ and $h \not\equiv 0$, where $m \geq 3$. Then*

$$\begin{aligned} & N(r,\infty;h) \\ \leq\ & \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,0;g \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2) + \overline{N}(r,\infty;g \mid\geq 2) \\ & +\overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_{f\geq m+1}(r,1;f \mid g \neq 1) \\ & +\overline{N}_{g\geq m+1}(r,1;g \mid f \neq 1) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) the common zeros of $f-1$ and $g-1$ whose multiplicities are different, (iv) those 1-points of f (g) which are not the 1-points of g (f), (v) zeros of f' which are not the zeros of $f(f-1)$, (vi) zeros of g' which are not zeros of $g(g-1)$.

Since all the poles of h are simple, the lemma follows from above. This proves the lemma. □

Lemma 2.3. *Let $\overline{E}_{2)}(1;f) = \overline{E}_{2)}(1;g)$ and $h \not\equiv 0$. Then*

$$\begin{aligned} & N(r,\infty;h) \\ \leq\ & \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,0;g \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2) + \overline{N}(r,\infty;g \mid\geq 2) \\ & +\overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_{f\geq 3}(r,1;f \mid g \neq 1) \\ & +\overline{N}_{g\geq 3}(r,1;g \mid f \neq 1) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'). \end{aligned}$$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.2. This proves the lemma. □

Lemma 2.4 ([7]). *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 2.5. *Let $\overline{E}_{2)}(1;f) = \overline{E}_{2)}(1;g)$. Then*

$$\begin{aligned} & \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) \\ & +2\overline{N}_{g\geq 3}(r,1;g \mid f \neq 1) - \overline{N}_{f>1}(r,1;g) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Proof. Let z_0 be a 1-point of f with multiplicity p and a a-point of g with multiplicity q. If $q = 3$ the possible values of p are as follows (i) $p = 3$ (ii) $p \geq 4$ (iii) $p = 0$. Similarly when $q = 4$ the possible values of p are (i) $p = 3$ (ii) $p = 4$ (iii) $p \geq 5$ (iv) $p = 0$. If $q \geq 5$ we can similarly find the possible values of p. Since $\overline{E}_{2)}(1;f) = \overline{E}_{2)}(1;g)$, we note that the simple 1-points of f are either simple or double 1 points of g and the double 1 points of f are either simple or double 1-points of g. Now the lemma follows from above. This completes the proof of the lemma. □

Lemma 2.6. *Let* $\overline{E}_{m)}(1;f) = \overline{E}_{m)}(1;g)$, $E_{1)}(1;f) = E_{1)}(1;g)$ *and* $h \not\equiv 0$, *where* $m \geq 3$. *Then*

$$\begin{aligned} & 2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) \\ & +m\overline{N}_{g\geq m+1}(r,1;g \mid f \neq 1) - \overline{N}_{f>2}(r,1;g) \\ \leq \;& N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Proof. Since the given condition implies that the simple 1-points of f and g are same the proof of the lemma can be carried out in the line of proof of Lemma 2.5. This proves the lemma. □

Lemma 2.7. *Let* $\overline{E}_{2)}(1;f) = \overline{E}_{2)}(1;g)$. *Then*

$$\begin{aligned} & \overline{N}_{f>1}(r,1;g) + 2\overline{N}_{f\geq 3}(r,1;f \mid g \neq 1) \\ \leq \;& \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f), \end{aligned}$$

where $N_0(r,0;f')$ *is the counting function of those zeros of* f' *which are not the zeros of* $f(f-1)$ *counting according to multiplicity.*

Proof. We note that a 1-point of f with multiplicity 2 is counted only once in the counting function $\overline{N}_{f>1}(r,1;g)$. Also since a 1 point of f with multiplicity $\geq$ 3 may or may not be a 1-point of g, the 1-points of f are counted either once or twice according as those points are counted in $\overline{N}_{f>1}(r,1;g)$ or in $\overline{N}_{f\geq 3}(r,1;f \mid g \neq 1)$ respectively. In other words any 1 point of f with multiplicity ≥ 3 is counted at most twice.

Considering the above and using Lemma 2.4 we see that

$$\begin{aligned} & \overline{N}_{f>1}(r,1;g) + 2\overline{N}_{f\geq 3}(r,1;f \mid g \neq 1) \\ \leq \;& N(r,0;f' \mid f = 1) \\ \leq \;& N(r,0;f' \mid f \neq 0) - N_0(r,0;f') \\ \leq \;& \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f). \end{aligned}$$

This proves the lemma. □

Lemma 2.8. *Let* $\overline{E}_{2)}(1;f) = \overline{E}_{2)}(1;g)$. *Then*

(i) $\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f)$.

(ii) $\overline{N}_L(r,1;g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,g)$.

Proof. We prove (i) because (ii) can be proved in a similar manner. Using Lemma 2.4 we obtain

$$\begin{aligned}\overline{N}_L(r,1;f) &\le \overline{N}(r,1;f\mid\ge 2)\\ &\le N(r,0;f'\mid f=1)\\ &\le N(r,0;f'\mid f\ne 0)-N_0(r,0;f')\\ &\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f')+S(r,f).\end{aligned}$$

This proves the lemma. □

Lemma 2.9. *Let $\overline{E}_{3)}(1;f)=\overline{E}_{3)}(1;g)$ and $E_{1)}(1;f)=E_{1)}(1;g)$. Then*

$$\begin{aligned}&\overline{N}_{f>2}(r,1;g)+2\,\overline{N}_{f\ge 4}(r,1;f\mid g\ne 1)\\ \le\ &\frac{2}{3}\,\overline{N}(r,0;f)+\frac{2}{3}\,\overline{N}(r,\infty;f)-\frac{2}{3}\,N_0(r,0;f')+S(r,f).\end{aligned}$$

Proof. Using Lemma 2.4 we get

$$\begin{aligned}&\overline{N}_{f>2}(r,1;g)+2\overline{N}_{f\ge 4}(r,1;f\mid g\ne 1)\\ \le\ &\overline{N}(r,1;f\mid\ge 3;g\mid=2)+2\,\overline{N}(r,1;f\mid g\ne 1)\\ \le\ &\frac{2}{3}\,N(r,0;f'\mid f=1)\\ \le\ &\frac{2}{3}\,\overline{N}(r,0;f)+\frac{2}{3}\,\overline{N}(r,\infty;f)-\frac{2}{3}\,N_0(r,0;f')+S(r,f),\end{aligned}$$

where by $\overline{N}(r,1;f\mid\ge 3;g\mid=2)$ we mean the reduced counting function of 1 points of f with multiplicity not less than 3 which are the 1-points of g with multiplicity 2. This completes the proof of the lemma. □

Lemma 2.10. *Let $\overline{E}_{4)}(1;f)=\overline{E}_{4)}(1;g)$ and $E_{1)}(1;f)=E_{1)}(1;g)$. Then*

$$\begin{aligned}&\overline{N}_{f>2}(r,1;g)+2\,\overline{N}_{f\ge 5}(r,1;f\mid g\ne 1)\\ \le\ &\frac{1}{2}\,\overline{N}(r,0;f)+\frac{1}{2}\,\overline{N}(r,\infty;f)-\frac{1}{2}\,N_0(r,0;f')+S(r,f).\end{aligned}$$

Proof. We omit the proof since the lemma can be proved in the line of proof of Lemma 2.9. This proves the lemma. □

Lemma 2.11 ([16]). *If $h\equiv 0$, then f and g share 1 CM.*

Lemma 2.12 ([9, 12]). *If f, g share 1-CM, then one of the following cases holds*

(i) $T(r,f)+T(r,g)\le 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\} + S(r,f)+S(r,g)$
(ii) $f\equiv g$
(iii) $fg\equiv 1$.

Lemma 2.13. *Let $\overline{E}_{4)}(1;f)=\overline{E}_{4)}(1;g)$ $E_{2)}(1;f)=E_{2)}(1;g)$ then the conclusion of Lemma 2.12 holds.*

Proof. Let $h \equiv 0$. Then the result follows from Lemma 2.11 and Lemma 2.12. So we suppose that $h \not\equiv 0$. Then by the second fundamental theorem, Lemma 2.1 and 2.2 we get

$$\begin{aligned} T(r,f)+T(r,g) &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g) \\ &\quad +N(r,1;f\mid=1)+\overline{N}(r,1;f\mid\geq 2)+\overline{N}(r,1;g) \\ &\quad -N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g) \qquad (2.1)\\ &\leq N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g) \\ &\quad +\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_{f\geq 5}(r,1;f\mid g\neq 1) \\ &\quad +\overline{N}_{g\geq 5}(r,1;g\mid f\neq 1)+\overline{N}(r,1;f\mid\geq 2)+\overline{N}(r,1;g) \\ &\quad +S(r,f)+S(r,g). \end{aligned}$$

Since

$$\overline{N}(r,1;f\mid=4;\ g\mid=3)+\overline{N}(r,1;f\mid=4)\leq 2\,\overline{N}(r,1;f\mid=4)$$

and

$$\overline{N}(r,1;f\mid=3;\ g\mid=4)+\overline{N}(r,1;g\mid=4)\leq 2\,\overline{N}(r,1;g\mid=4),$$

we see that

$$\begin{aligned} &\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_{f\geq 5}(r,1;f\mid g\neq 1)+\overline{N}_{g\geq 5}(r,1;g\mid f\neq 1) \\ &+\overline{N}(r,1;f\mid\geq 2)+\overline{N}(r,1;g) \\ \leq\ & \overline{N}(r,1;f\mid=4;\ g\mid=3)+\overline{N}(r,1;f\mid\geq 6)+\overline{N}(r,1;g\mid=4;f\mid=3) \\ &+\overline{N}(r,1;g\mid\geq 6)+\overline{N}(r,1;f\mid\geq 5)+\overline{N}(r,1;g\mid\geq 5)+\overline{N}(r,1;f\mid=2) \\ &+\overline{N}(r,1;f\mid=3)+\overline{N}(r,1;f\mid=4)+\overline{N}(r,1;f\mid\geq 5)+N(r,1;g\mid=1) \\ &+\overline{N}(r,1;g\mid=2)+\overline{N}(r,1;g\mid=3)+\overline{N}(r,1;g\mid=4)+\overline{N}(r,1;g\mid\geq 5) \\ \leq\ & \frac{1}{2}\,N(r,1;f\mid=1)+\frac{1}{2}\,N(r,1;g\mid=1)+\overline{N}(r,1;f\mid=2)+\overline{N}(r,1;g\mid=2) \\ &+\overline{N}(r,1;f\mid=3)+\overline{N}(r,1;g\mid=3)+2\,\overline{N}(r,1;f\mid=4)+2\,\overline{N}(r,1;g\mid=4) \\ &+2\,\overline{N}(r,1;f\mid\geq 5)+2\,\overline{N}(r,1;g\mid\geq 5)+\overline{N}(r,1;f\mid\geq 6)+\overline{N}(r,1;g\mid\geq 6) \\ \leq\ & \frac{1}{2}\,[N(r,1;f)+N(r,1;g)] \\ \leq\ & \frac{1}{2}[T(r,f)+T(r,g)]. \end{aligned}$$

Now the lemma follows from (2.1). This completes the proof of the lemma. □

Lemma 2.14 ([1], [15]). *If* $h \equiv 0$ *and*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)}{T(r)} < 1,$$

then $f \equiv g$ *or* $f.g \equiv 1$.

Lemma 2.15 ([8], [11]). *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 2.16 ([1]). *Let f be a nonconstant meromorphic function and $F = \frac{f^{n+1}}{a(n+1)}$, n being a positive integer. Then $T(r, F) \leq T(r, F') + N(r, 0; f) - N(r, 0; f') + S(r, f)$.*

Lemma 2.17 ([1]). *Let f, g be two nonconstant meromorphic functions and $F = \frac{f^{n+1}}{a(n+1)}$, $G = \frac{g^{n+1}}{a(n+1)}$, where $n(> 2)$ is an integer. Then $F' \equiv G'$ implies $F \equiv G$*

Lemma 2.18 ([12]). *Let f, g be two nonconstant meromorphic functions and $n > 6$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1, c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

Lemma 2.19. *Let $\overline{E}_{2)}(1; f) = \overline{E}_{2)}(1; g)$ and $h \not\equiv 0$. Then*

$$\begin{aligned} & T(r, f) \\ \leq\ & N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) \\ & + \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g) + 2\, \overline{N}_{f \geq 3}(r, 1; f \mid g \neq 1) \\ & + \overline{N}_{f>1}(r, 1; g) - \overline{N}_{g \geq 3}(r, 1; g \mid f \neq 1) - m(r, 1; g) \\ & + S(r, f) + S(r, g). \end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} & T(r, f) + T(r, g) \\ \leq\ & \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) \\ & + \overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N_0(r, 0; f') - N_0(r, 0; g') \\ & + S(r, f) + S(r, g). \end{aligned} \tag{2.2}$$

If z_0 be a common simple zero of $f - 1$ and $g - 1$ then it is easy to verify that $h(z_0) = 0$. Since $h \not\equiv 0$,

$$\begin{aligned} N_E^{1)}(r, 1; f) \leq N(r, 0; h) & \leq T(r, h) + O(1) \\ & \leq N(r, \infty; h) + S(r, f) + S(r, g). \end{aligned}$$

Hence by Lemmas 2.3 and 2.5 we get

$$\begin{aligned} & \overline{N}(r, 1; f) + \overline{N}(r, 1; g) \\ \leq\ & N_E^{1)}(r, 1; f) + \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) \\ & + \overline{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \overline{N}(r, 1; g) \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
\leq\ & \overline{N}(r,0;f \mid\geq 2)+\overline{N}(r,\infty;f \mid\geq 2)+\overline{N}(r,0;g \mid\geq 2)\\
&+\overline{N}(r,\infty;g \mid\geq 2)+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\\
&+\overline{N}_{f\geq 3}(r,1;f \mid\ g\neq 1)+\overline{N}_{g\geq 3}(r,1;g \mid\ f\neq 1)\\
&+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)\\
&+\overline{N}_{f\geq 3}(r,1;f \mid\ g\neq 1)+T(r,g)-m(r,1;g)\\
&+O(1)-\overline{N}_L(r,1;f)-\overline{N}_L(r,1;g)-\overline{N}_E^{(2}(r,1;f)\\
&-2\overline{N}_{g\geq 3}(r,1;g \mid\ f\neq 1)+\overline{N}_{f>1}(r,1;g)+\overline{N}_0(r,0;f')\\
&+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,0;f \mid\geq 2)+\overline{N}(r,\infty;f \mid\geq 2)+\overline{N}(r,0;g \mid\geq 2)\\
&+\overline{N}(r,\infty;g \mid\geq 2)+T(r,g)-m(r,1;g)+\overline{N}_L(r,1;f)\\
&+\overline{N}_L(r,1;g)+2\overline{N}_{f\geq 3}(r,1;f \mid\ g\neq 1)+\overline{N}_{f>1}(r,1;g)\\
&-\overline{N}_{g\geq 3}(r,1;g \mid\ f\neq 1)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\
&+S(r,f)+S(r,g).
\end{aligned}
$$

From (2.2) and (2.3) the lemma follows. This proves the lemma. □

Lemma 2.20. *Let* $\overline{E}_{3)}(1;f)=\overline{E}_{3)}(1;g)$, $E_{1)}(1;f)=E_{1)}(1;g)$ *and* $h\not\equiv 0$. *Then*

$$
\begin{aligned}
T(r,f)\ \leq\ & N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\
&+\overline{N}_{f>2}(r,1;g)+2\,\overline{N}_{f\geq 4}(r,1;f \mid g\neq 1)\\
&-2\,\overline{N}_{g\geq 4}(r,1;g \mid\ f\neq 1)-m(r,1;g)\\
&+S(r,f)+S(r,g).
\end{aligned}
$$

Proof. Using Lemmas 2.2 and 2.6 we note that

$$
\begin{aligned}
(2.4)\qquad & \overline{N}(r,1;f)+\overline{N}(r,1;g)\\
\leq\ & N(r,1;f \mid= 1)+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)\\
&+\overline{N}_{f\geq 4}(r,1;f \mid\ g\neq 1)+\overline{N}(r,1;g)\\
\leq\ & \overline{N}(r,0;f \mid\geq 2)+\overline{N}(r,\infty;f \mid\geq 2)+\overline{N}(r,0;g \mid\geq 2)\\
&+\overline{N}(r,\infty;g \mid\geq 2)+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\\
&+\overline{N}_{f\geq 4}(r,1;f \mid\ g\neq 1)+\overline{N}_{g\geq 4}(r,1;g \mid\ f\neq 1)\\
&+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)\\
&+\overline{N}_{f\geq 4}(r,1;f \mid\ g\neq 1)+T(r,g)-m(r,1;g)\\
&+O(1)-2\,\overline{N}_L(r,1;f)-2\,\overline{N}_L(r,1;g)-\overline{N}_E^{(2}(r,1;f)
\end{aligned}
$$

$$\begin{aligned}
&-3\,\overline{N}_{g\geq 4}(r,1;g\mid\ f\not=1)+\overline{N}_{f>2}(r,1;g)+\overline{N}_0(r,0;f')\\
&+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,0;g\mid\geq 2)\\
&+\overline{N}(r,\infty;g\mid\geq 2)+T(r,g)-m(r,1;g)\\
&+2\,\overline{N}_{f\geq 4}(r,1;f\mid\ g\not=1)+\overline{N}_{f>2}(r,1;g)\\
&-2\,\overline{N}_{g\geq 4}(r,1;g\mid\ f\not=1)+\overline{N}_0(r,0;f')\\
&+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).
\end{aligned}$$

From (2.2) and (2.4) the lemma follows. This proves the lemma. □

Lemma 2.21. *Let* $\overline{E}_{4)}(1;f)=\overline{E}_{4)}(1;g)$, $E_{1)}(1;f)=E_{1)}(1;g)$ *and* $h\not\equiv 0$. *Then*

$$\begin{aligned}
T(r,f)\ \leq\ & N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\
&+\overline{N}_{f>2}(r,1;g)+2\,\overline{N}_{f\geq 5}(r,1;f\mid g\not=1)\\
&-2\,\overline{N}_{g\geq 5}(r,1;g\mid\ f\not=1)-m(r,1;g)\\
&+S(r,f)+S(r,g).
\end{aligned}$$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.19. This completes the proof of the lemma. □

Lemma 2.22 ([13]). *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)})\leq k\overline{N}(r,\infty;f)+N(r,0;f)+S(r,f).$$

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F=\frac{f^{n+1}}{a(n+1)}$ and $G=\frac{g^{n+1}}{a(n+1)}$. Then $F^{'}=\frac{f^nf'}{a}$ and $G^{'}=\frac{g^ng'}{a}$. Since $\overline{E}_{2)}(a;f^nf')=\overline{E}_{2)}(a;g^ng')$, it follows that $\overline{E}_{2)}(1;F^{'})=\overline{E}_{2)}(1;G^{'})$. If possible, we suppose that $H\not\equiv 0$. Then by Lemmas 2.7, 2.8 and 2.19 we obtain

$$\begin{aligned}
(3.1)\quad T(r,F^{'})\ \leq\ & N_2(r,0;F^{'})+N_2(r,\infty;F^{'})+N_2(r,0;G^{'})\\
&+N_2(r,\infty;G^{'})+2\,\overline{N}(r,0;F^{'})+2\,\overline{N}(r,\infty;F^{'})\\
&+\overline{N}(r,0;G^{'})+\overline{N}(r,\infty;G^{'})+S(r,F^{'})+S(r,G^{'}).
\end{aligned}$$

We see that

$$N_2(r,0;F^{'})+N_2(r,\infty;F^{'})\leq 2\overline{N}(r,0;f)+N(r,0;f')+2\overline{N}(r,\infty;f),$$

$$N_2(r,0;G^{'})+N_2(r,\infty;G^{'})\leq 2\overline{N}(r,0;g)+N(r,0;g')+2\overline{N}(r,\infty;g),$$

$$2\overline{N}(r,0;F^{'})+2\overline{N}(r,\infty;F^{'})\leq 2\overline{N}(r,0;f)+2N(r,0;f')+2\overline{N}(r,\infty;f)$$

and

$$\overline{N}(r,0;G^{'})+\overline{N}(r,\infty;G^{'})\leq \overline{N}(r,0;g)+N(r,0;g')+\overline{N}(r,\infty;g).$$

Also by Lemma 2.15 we get

$$T(r,F') \leq 2T(r,F)+S(r,F)=2(n+1)T(r,f)+S(r,f)$$

and

$$T(r,G') \leq 2T(r,G)+S(r,G)=2(n+1)T(r,g)+S(r,g).$$

So $S(r,F')$ and $S(r,G')$ can be replaced by $S(r,f)$ and $S(r,g)$ respectively. So by Lemmas 2.16 and 2.22 we get from (3.1) for $\varepsilon(>0)$

$$\begin{aligned} T(r,F) &\leq T(r,F')+N(r,0;f)-N(r,0;f')+S(r,f) \\ &\leq 4\overline{N}(r,0;f)+N(r,0;f)+3\overline{N}(r,0;g)+4\overline{N}(r,\infty;f) \\ &\quad +3\overline{N}(r,\infty;g)+2N(r,0;f')+2N(r,0;g')+S(r,f)+S(r,g) \\ &\leq 7T(r,f)+5T(r,g)+(6-6\Theta(\infty;f)+\varepsilon)T(r,f) \\ &\quad +(5-5\Theta(\infty;g)+\varepsilon)T(r,g)+S(r,f)+S(r,g) \\ &\leq \{23-6\Theta(\infty;f)-5\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \end{aligned}$$

So using Lemma 2.15 we get

$$(n+1)T(r,f) \leq \{23-6\Theta(\infty;f)-5\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \tag{3.2}$$

In a similar manner we obtain

$$(n+1)T(r,g) \leq \{23-5\Theta(\infty;f)-6\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$[n-22+5\Theta(\infty;f)+5\Theta(\infty;g)+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon]T(r) \leq S(r). \tag{3.4}$$

Since $\varepsilon(>0)$ is arbitrary, (3.4) implies a contradiction. Hence $H \equiv 0$.

Since

$$\overline{N}(r,0;f') \leq T(r,f')-m(r,\frac{1}{f'}) \leq 2T(r,f)-m(r,\frac{1}{f'})+S(r,f),$$

we note that

$$\begin{aligned} &\overline{N}(r,0;F')+\overline{N}(r,\infty;F')+\overline{N}(r,0;G')+\overline{N}(r,\infty;G') \\ \leq\ &\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g) \\ &+\overline{N}(r,0;f')+\overline{N}(r,0;g') \\ \leq\ &4T(r,f)+4T(r,g)-m(r,0;f')-m(r,0;g')+S(r) \\ \leq\ &8T(r)-m(r,0;f')-m(r,0;g')+S(r). \end{aligned} \tag{3.5}$$

Also using Lemma 2.15 we get

$$\begin{aligned} T(r,F')+m(r,\frac{1}{f'}) &= m(r,\frac{f^nf'}{a})+m(r,\frac{1}{f'})+N(r,\infty;\frac{f^nf'}{a}) \\ &\geq m(r,\frac{f^n}{a})+N(r,\infty;f^n) \\ &= T(r,f^n)+O(1) \\ &= nT(r,f)+O(1). \end{aligned} \tag{3.6}$$

Similarly

$$T(r,G') + m(r,\frac{1}{g'}) \geq nT(r,g) + O(1). \tag{3.7}$$

From (3.6) and (3.7) we get

$$\max\{T(r,F'),T(r,G')\} \geq nT(r) - m(r,\frac{1}{f'}) - m(r,\frac{1}{g'}) + O(1). \tag{3.8}$$

By (3.5) and (3.8) applying Lemma 2.14 we get either $F' \equiv G'$ or $F'G' \equiv 1$.

If $F' \equiv G'$, then by Lemma 2.17 we obtain $F \equiv G$ or $f \equiv dg$, where d is some $(n+1)$ th root of unity.

If $F'G' \equiv 1$ then $f^n f' g^n g' = a^2$. Set $f_1 = a^{-\frac{1}{n+1}} f$ and $g_1 = a^{-\frac{1}{n+1}} g$, then $f_1^n f_1' g_1^n g_1' = 1$. So using Lemma 2.18 we get $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2 = -a^2$. This completes the proof of the theorem. □

Proof of Theorem 1.3. Let F and G be defined as in the proof of Theorem 1.1. Since $\overline{E}_{4)}(a; f^n f') = \overline{E}_{4)}(a; g^n g')$ and $E_{1)}(a; f^n f') = E_{1)}(a; g^n g')$ it follows that $\overline{E}_{4)}(1; F') = \overline{E}_{4)}(1; G')$ and $E_{1)}(1; F') = E_{1)}(1; G')$. If possible, we suppose that $H \not\equiv 0$. Then by Lemmas 2.10 and 2.21 we obtain

$$\begin{aligned} T(r,F') \leq\ & N_2(r,0;F') + N_2(r,\infty;F') + N_2(r,0;G') \\ & + N_2(r,\infty;G') + \frac{1}{2}\,\overline{N}(r,0;F') + \frac{1}{2}\,\overline{N}(r,\infty;F') \\ & + S(r,F') + S(r,G'). \end{aligned} \tag{3.9}$$

We see that

$$N_2(r,0;F') + N_2(r,\infty;F') \leq 2\overline{N}(r,0;f) + N(r,0;f') + 2\overline{N}(r,\infty;f),$$

$$N_2(r,0;G') + N_2(r,\infty;G') \leq 2\overline{N}(r,0;g) + N(r,0;g') + 2\overline{N}(r,\infty;g),$$

and

$$\frac{1}{2}\overline{N}(r,0;F') + \frac{1}{2}\overline{N}(r,\infty;F') \leq \frac{1}{2}[\overline{N}(r,0;f) + N(r,0;f') + \overline{N}(r,\infty;f)].$$

Again using Lemma 2.15 and proceeding in the same way as done in the proof of Theorem 1.1 we can show that $S(r,F')$ and $S(r,G')$ can be replaced by $S(r,f)$ and $S(r,g)$ respectively. So by Lemma 2.16 and Lemma 2.22 we obtain from (3.9) for $\varepsilon > 0$

$$\begin{aligned} T(r,F) &\leq T(r,F') + N(r,0;f) - N(r,0;f') + S(r,f) \\ &\leq 2\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,0;f) + \frac{3}{2}N(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g) \\ &\quad + 3\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\ &\leq (7 - 3\Theta(\infty;f) + \varepsilon)T(r,f) + (6 - 3\Theta(\infty;g) + \varepsilon)T(r,g) + S(r) \\ &\leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \end{aligned}$$

So using Lemma 2.15 we get

$$(n+1)T(r,f) \leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \tag{3.10}$$

Similarly we can obtain

$$(n+1)T(r,g) \leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \tag{3.11}$$

From (3.10) and (3.11) we obtain

$$[n - 12 + 3\Theta(\infty;f) + 3\Theta(\infty;g) - 2\varepsilon] \leq S(r). \tag{3.12}$$

Since $\varepsilon(>0)$ is arbitrary, we get a contradiction from (3.12). Hence $H \equiv 0$.

Now proceeding in the same way as in the proof of Theorem 1.1 we obtain either $F^{'} \equiv G^{'}$ or $F^{'}G^{'} \equiv 1$. Again proceeding in the same manner as in the proof of Theorem 1.1 we obtain the conclusion of Theorem 1.3. This proves the theorem. □

Proof of Theorem 1.2. Let F and G be defined as in the proof of Theorem 1.1. Then by the given condition of the theorem it follows that $\overline{E}_{3)}(1;F^{'}) = \overline{E}_{3)}(1;G^{'})$ and $E_{1)}(1;F^{'}) = E_{1)}(1;G^{'})$. Suppose that $H \not\equiv 0$. Then using Lemmas 2.9, 2.15 and 2.20 and proceeding in the same way as in the proof of Theorem 1.1 we can obtain

(3.13)

$$(n+1)T(r,f) = T(r,F) \leq [\frac{41}{3} - \frac{10}{3}\Theta(\infty;f) - 3\,\Theta(\infty;f) + 2\,\varepsilon]T(r) + S(r).$$

Similarly we get

(3.14)

$$(n+1)T(r,g) = T(r,G) \leq [\frac{41}{3} - 3\,\Theta(\infty;f) - \frac{10}{3}\,\Theta(\infty;f) + 2\,\varepsilon]T(r) + S(r).$$

From (3.13) and (3.14) we obtain

$$[n - \frac{38}{3} + 3\,\Theta(\infty;f) + 3\,\Theta(\infty;g) + \frac{1}{3}\,\{\Theta(\infty;f), \Theta(\infty;g)\} - 2\varepsilon] \leq S(r). \tag{3.15}$$

Since $\varepsilon(>0)$ is arbitrary, (3.15) leads to a contradiction. Hence $H \equiv 0$.

Now the theorem can be proved in the line of the proof of Theorem 1.1. This proves the theorem. □

Proof of Theorem 1.4. Let F and G be defined as in the proof of Theorem 1.1. Since $\overline{E}_{4)}(a;f^nf') = \overline{E}_{4)}(a;g^ng')$ and $E_{2)}(a;f^nf') = E_{2)}(a;g^ng')$ it follows that $\overline{E}_{4)}(1;F^{'}) = \overline{E}_{4)}(1;G^{'})$ and $E_{2)}(1;F^{'}) = E_{2)}(1;G^{'})$. If possible let us suppose that

$$\begin{aligned} & T(r,F^{'}) + T(r,G^{'}) \\ \leq\; & 2\{N_2(r,0;F^{'}) + N_2(r,0;G^{'}) + N_2(r,\infty;F^{'}) + N_2(r,\infty;G^{'})\} \\ & + S(r,F^{'}) + S(r,G^{'}). \end{aligned} \tag{3.16}$$

Then by Lemmas 2.16 and 2.22 we get from (3.16)

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & T(r,F^{'})+T(r,G^{'})+N(r,0;f)-N(r,0;f') \\
& +N(r,0;g)-N(r,0;g')+S(r,f)+S(r,g) \\
\leq\ & 2N_2(r,0;F^{'})+2N_2(r,0;G^{'})+2N_2(r,\infty;F^{'}) \\
& +2N_2(r,\infty;G^{'})+N(r,0;f)-N(r,0;f') \\
& +N(r,0;g)-N(r,0;g')+S(r,f)+S(r,g) \\
\leq\ & 2\{2\overline{N}(r,0;f)+N(r,0;f')+2\overline{N}(r,\infty;f)\} \\
& +2\{2\overline{N}(r,0;g)+N(r,0;g')+2\overline{N}(r,\infty;g)\} \\
& +N(r,0;f)-N(r,0;f')+N(r,0;g) \\
& -N(r,0;g')+S(r,f)+S(r,g) \\
\leq\ & 4\overline{N}(r,0;f)+2N(r,0;f)+5\overline{N}(r,\infty;f) \\
& +4\overline{N}(r,0;g)+2N(r,0;g)+5\overline{N}(r,\infty;g)+S(r) \\
\leq\ & 6T(r,f)+5\overline{N}(r,\infty;f)+6T(r,g)+5\overline{N}(r,\infty;g)+S(r).
\end{aligned}$$

So by Lemma 2.15 we get

$$\begin{aligned}
(n-5)T(r,f)+(n-5)T(r,g) &\leq 5\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;g)+S(r) \\
&\leq 5T(r,f)+5T(r,g)+S(r).
\end{aligned}$$

That is

$$(n-10)T(r,f)+(n-10)T(r,g) \leq S(r)$$

which is a contradiction.

Therefore the inequality (3.16) does not hold. So from Lemma 2.13 we see that either $F^{'} \equiv G^{'}$ or $F^{'}G^{'} \equiv 1$. Again proceeding in the same manner as in the proof of Theorem 1.1 we obtain the conclusion of Theorem 1.4. This proves the theorem. □

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