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# THE QUASI-HADAMARD PRODUCTS OF CERTAIN $p$-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The object of the present paper is to show quasi-Hadamard products of certain $p$-valent functions with negative coefficients in the open unit disc. Our results are the generalizations of the corresponding results due to Yaguchi et al. [10], Aouf and Darwish [3], Lee et al. [5] and Sekine and Owa [9].

## 1. Introduction

Let $A_p(n)$ be the class of functions of the form :

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in N = \{1, 2, \ldots\}) \tag{1.1}$$

which are analytic and $p$-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A_p(n)$ is said to be a member of the class $P_p^*(n, \alpha, \beta)$ if it satisfies

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} + p - 2\alpha} \right| < \beta \quad (z \in U) \tag{1.2}$$

for some $\alpha (0 \leq \alpha < p)$ and $\beta (0 < \beta \leq 1)$. The class $P_p^*(n, \alpha, \beta)$ was studied by Aouf [1, 2].

We note that :

(i) For $\beta = 1$, the class $P_p^*(n, \alpha, 1) = P_p^*(n, \alpha) = \{f(z) \in A_p(n) : \mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha (z \in U), 0 \leq \alpha < p\}$ was studied by Yaguchi et al. [10] and Owa and Aouf [7];

(ii) For $\beta = n = 1$, the class $P_p^*(1, \alpha, 1) = P_p^*(\alpha) = \{f(z) \in A_p : \mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha (z \in U), 0 \leq \alpha < p\}$ was studied by Lee et al. [5], Aouf and Darwish [4] and Aouf [3];

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(iii) For $\beta = p = 1$, the class $P_1^*(n,\alpha,1) = \widetilde{C}(\alpha,n) = \{f(z) \in A_1(n) : \mathrm{Re}\left\{f'(z)\right\} > \alpha (z \in U), 0 \leq \alpha < 1\}$ was studied by Sekine and Owa [9];

(iv) For $\beta = p = n = 1$, the class $P_1^*(1,\alpha,1) = \widetilde{C}(\alpha,1)$ was studied by Sarangi and Urelagaddi [8] and Owa [6].

For functions $f_j(z) \in A_p(n)$ defined by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \ \ (a_{k,j} \geq 0; j \in N)\ , \tag{1.3}$$

we denote by $(f_1 * f_2)(z)$ the quasi-Hadamard product of functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k\ . \tag{1.4}$$

For $\beta = 1$ Yaguchi et al. [10] proved the following results :

**Theorem A.** *If* $f_j(z) \in P_p^*(n,\alpha_j,1) = P_p^*(n,\alpha_j) (j = 1,2)$, *then* $(f_1 * f_2)(z) \in P_p^*(n,\gamma)$, *where*

$$\gamma = p - \frac{\prod_{j=1}^{2} (p - \alpha_j)}{p+n}\ . \tag{1.5}$$

The result is sharp.

**Theorem B.** *If* $f_j(z) \in P_p^*(n,\alpha_j) (j = 1,2)$, *then the function*

$$h(z) = z^p - \sum_{k=p+n}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{1.6}$$

*is in the class* $P_p^*(n,\gamma)$, *where*

$$\gamma = p - \frac{2(p-\alpha_0)^2}{p+n} \ \ (\alpha_0 = \min\{\alpha_1,\alpha_2\})\ . \tag{1.7}$$

The result is sharp.

For $\beta = n = 1$, Lee et al. [5] have shown that :

**Theorem C.** *If* $f_j(z) \in P_p^*(1,\alpha,1) = P_p^*(\alpha)\, (j = 1,2)$, *then* $(f_1 * f_2)(z) \in P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{(p-\alpha)^2}{p+1}\ . \tag{1.8}$$

The result is sharp.

Also for $\beta = n = 1$, Aouf and Darwish [3] have proved the following results:

**Theorem D.** *If* $f_j(z) \in P_p^*(\alpha_j)(j = 1, 2)$, *then* $(f_1 * f_2)(z) \in P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{\prod_{j=1}^{2}(p - \alpha_j)}{p+1}. \tag{1.9}$$

The result is sharp.

**Theorem E.** *If* $f_j(z) \in P_p^*(\alpha)(j = 1, 2, 3)$, *then* $(f_1 * f_2 * f_3)(z) \in P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{(p - \alpha)^3}{(p+1)^2}. \tag{1.10}$$

The result is sharp.

**Theorem F.** *If* $f_j(z) \in P_p^*(\alpha)(j = 1, 2)$, *then the function*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{1.11}$$

*is in the class* $P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{2(p - \alpha)^2}{p+1}. \tag{1.12}$$

The result is sharp.

Further for $\beta = p = 1$, Sekine and Owa [9] proved the following results :

**Theorem G.** *If* $f_j(z) \in P_1^*(n, \alpha, 1) = \widetilde{C}(\alpha, n)(j = 1, 2)$, *then* $(f_1 * f_2)(z) \in \widetilde{C}(n, \gamma)$, *where*

$$\gamma = 1 - \frac{(1 - \alpha)^2}{n+1}. \tag{1.13}$$

The result is sharp.

**Theorem H.** *If* $f_j(z) \in \widetilde{C}(\alpha, n)(j = 1, 2)$, *then the function*

$$g(z) = z - \sum_{k=n+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{1.14}$$

*is in the class* $\widetilde{C}(\gamma, n)$, *where*

$$\gamma = 1 - \frac{2(1 - \alpha)^2}{n+1}. \tag{1.15}$$

The result is sharp.

In the present paper, we prove some interesting generalizations of the theorems given by Yaguchi et al. [10], Aouf and Darwish [3], Lee et al. [5] and Sekine and Owa [9].

## 2. Quasi-Hadamard products

To prove our main results of quasi-Hadamard products, we need the following lemma given by Aouf ([1] and [2]).

**Lemma 1.** *A function $f(z) \in A_n(p)$ is in the class $P_p^*(n,\alpha,\beta)$ if and only if*

$$\sum_{k=p+n}^{\infty} (1+\beta)ka_k \le 2\beta(p-\alpha) \ . \tag{2.1}$$

Applying the above lemma, we derive :

**Theorem 1.** *If $f_j(z) \in P_p^*(n,\alpha_j,\beta)(j = 1,2,\ldots,m)$, then $(f_1 * f_2 * \cdots * f_m)(z) \in P_p^*(n,\gamma,\beta)$, where*

$$\gamma = p - \frac{\prod_{j=1}^{m} 2\beta(p-\alpha_j)}{2\beta[(1+\beta)(p+n)]^{m-1}} \ . \tag{2.2}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(p-\alpha_j)}{(1+\beta)(p+n)} z^{p+n} \quad (j = 1,2,\ldots,m) \ . \tag{2.3}$$

*Proof.* For $m = 1$, we see that $\gamma = \alpha_1$. For $m = 2$, Lemma 1 gives

$$\sum_{k=p+n}^{\infty} \frac{k(1+\beta)}{2\beta(p-\alpha_j)} a_{k,j} \le 1 \quad (j = 1,2) \ . \tag{2.4}$$

This gives that

$$\sum_{k=p+n}^{\infty} \frac{(1+\beta)k}{\sqrt{\prod_{j=1}^{2} 2\beta(p-\alpha_j)}} \sqrt{a_{k,1}a_{k,2}} \le 1 \ . \tag{2.5}$$

To prove the case when $m = 2$, we have to find the largest $\gamma$ such that

$$\sum_{k=p+n}^{\infty} \frac{(1+\beta)k}{2\beta(p-\gamma)} a_{k,1}a_{k,2} \le 1 \, , \tag{2.6}$$

or such that

$$\frac{\sqrt{a_{k,1}a_{k,2}}}{2\beta(p-\gamma)} \le \frac{1}{\sqrt{\prod_{j=1}^{2} 2\beta(p-\alpha_j)}} \quad (k \ge p+n) \ . \tag{2.7}$$

Further, by using (2.5), we need to find the largest $\gamma$ such that

$$\frac{1}{2\beta(p-\gamma)} \leq \frac{(1+\beta)k}{\prod_{j=1}^{2} 2\beta(p-\alpha_j)} \quad (k \geq p+n) . \tag{2.8}$$

It follows from (2.8) that

$$\gamma \leq p - \frac{\prod_{j=1}^{2} 2\beta(p-\alpha_j)}{2\beta(1+\beta)k} \quad (k \geq p+n) . \tag{2.9}$$

Defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{\prod_{j=1}^{2} 2\beta(p-\alpha_j)}{2\beta(1+\beta)k} , \tag{2.10}$$

we see that $\varphi'(k) \geq 0$ for $k \geq p+n$. This implies that

$$\gamma \leq \varphi(p+n) = p - \frac{\prod_{j=1}^{2} 2\beta(p-\alpha_j)}{2\beta(1+\beta)(p+n)} . \tag{2.11}$$

Therefore, the result is true for $m = 2$.

Suppose that the result is true for any positive integer $m$. Then we have $(f_1 * f_2 * \cdots * f_m * f_{m+1})(z) \in P_p^*(n, \lambda, \beta)$, where

$$\lambda = p - \frac{2\beta(p-\gamma)2\beta(p-\alpha_{m+1})}{2\beta(1+\beta)(p+n)} , \tag{2.12}$$

where $\gamma$ is given by (2.2). After a simple calculation, we have

$$\lambda = p - \frac{\prod_{j=1}^{m+1} 2\beta(p-\alpha_j)}{2\beta[(1+\beta)(p+n)]^m} . \tag{2.13}$$

Thus, the result is true for $m+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer $m$.

Finally, taking the functions $f_j(z)$ defined by (2.3), we have

(2.14)

$$(f_1 * f_2 * \cdots * f_m)(z) = z^p - \left\{ \prod_{j=1}^{m} \frac{2\beta(p-\alpha_j)}{(1+\beta)(p+n)} \right\} z^{p+n} = z^p - A_{p+n} z^{p+n} ,$$

which shows that

$$\sum_{k=p+n}^{\infty} \left[ \frac{(1+\beta)k}{2\beta(p-\gamma)} \right] A_k$$

$$= \frac{(1+\beta)(p+n)}{2\beta(p-\gamma)} \left\{ \prod_{j=1}^{m} \frac{2\beta(p-\alpha_j)}{(1+\beta)(p+n)} \right\} = 1 \,. \tag{2.15}$$

Consequently, the result is sharp. □

Putting $\alpha_j = \alpha$ $(j = 1, 2, \ldots, m)$ in Theorem 1, we have :

**Corollary 1.** *If* $f_j(z) \in P_p^*(n, \alpha, \beta)(j = 1, 2, \ldots, m)$, *then* $(f_1 * f_2 * \cdots * f_m)(z) \in P_p^*(n, \gamma, \beta)$, *where*

$$\gamma = p - \frac{[2\beta(p-\alpha)]^m}{2\beta[(1+\beta)(p+n)]^{m-1}} \,. \tag{2.16}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(p-\alpha)}{(1+\beta)(p+n)} z^{p+n} \;\; (j = 1, 2, \ldots, m) \,. \tag{2.17}$$

Putting $\beta = 1$ in Theorem 1, we have :

**Corollary 2.** *If* $f_j(z) \in P_n^*(n, \alpha_j, 1) = P_n^*(n, \alpha_j)(j = 1, 2, \ldots, m)$, *then* $(f_1 * f_2 * \cdots * f_m)(z) \in P_p^*(n, \gamma)$, *where*

$$\gamma = p - \frac{\prod_{j=1}^{m} (p-\alpha_j)}{(p+n)^{m-1}} \,. \tag{2.18}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{p-\alpha_j}{p+n} z^{p+n} \;\; (j = 1, 2, \ldots, m) \,. \tag{2.19}$$

Putting $n = 1$ in Corollary 1, we have :

**Corollary 3.** *If* $f_j(z) \in P_p^*(1, \alpha, \beta) = P_p^*(\alpha, \beta)(j = 1, 2, \ldots, m)$, *then* $(f_1 * f_2 * \cdots * f_m)(z) \in P_p^*(\gamma, \beta)$, *where*

$$\gamma = p - \frac{[2\beta(p-\alpha)]^m}{2\beta[(1+\beta)(p+1)]^{m-1}} \,. \tag{2.20}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(p-\alpha)}{(1+\beta)(p+1)} z^{p+1} \;\; (j = 1, 2, \ldots, m) \,. \tag{2.21}$$

Putting $\beta = n = 1$ in Corollary 1, we have :

**Corollary 4.** *If* $f_j(z) \in P_p^*(1,\alpha,1) = P_p^*(\alpha)(j = 1,2,\ldots,m)$, *then* $(f_1 * f_2 * \cdots * f_m)(z) \in P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{(p-\alpha)^m}{(p+1)^{m-1}} \quad . \tag{2.22}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{p-\alpha}{p+1} z^{p+1} \quad (j = 1,2,\ldots,m) . \tag{2.23}$$

Putting $\beta = p = 1$ in Corollary 1, we have :

**Corollary 5.** *If* $f_j(z) \in P_1^*(n,\alpha,1) = \widetilde{C}(n,\alpha)(j = 1,2,\ldots,m)$, *then* $(f_1 * f_2 * \cdots * f_m)(z) \in \widetilde{C}(n,\gamma)$, *where*

$$\gamma = 1 - \frac{(1-\alpha)^m}{(1+n)^{m-1}} \quad . \tag{2.24}$$

The result is sharp for the functions

$$f_j(z) = z - \frac{1-\alpha}{1+n} z^{1+n} \quad (j = 1,2,\ldots,m) . \tag{2.25}$$

*Remark* 1. (i) Corollary 4 (when $\beta = n = 1$) is the generalization of Theorem E given by Aouf and Darwish [3];
(ii) Corollary 2 is the generalization of Theorem A given by Yaguchi et al. [10]. Also Corollary 2 (when $n = 1$) is the generalization of Theorem D given by Aouf and Darwish [3];
(iii) Corollary 4 is the generalization of Theorem C given by Lee et al. [5];
(iv) Corollary 5 is the generalization of Theorem G given by Sekine and Owa [9].

**Theorem 2.** *If* $f_j(z) \in P_p^*(n,\alpha_j,\beta)(j = 1,2,\ldots,m)$ *and*

$$h(z) = z^p - \sum_{k=p+n}^{\infty} (\sum_{j=1}^{m} a_{k,j}^2) z^k \ , \tag{2.26}$$

*then* $h(z) \in P_p^*(n,\gamma,\beta)$, *where*

$$\gamma = p - \frac{m[2\beta(p-\alpha_0)]^2}{2\beta(1+\beta)(p+n)} \ (\alpha_0 = \min\{\alpha_1,\alpha_2,\ldots,\alpha_m\}) \ . \tag{2.27}$$

The result is sharp for the functions $f_j(z)$ given by (2.3).

*Proof.* Since Lemma 1 gives

$$\sum_{k=p+n}^{\infty} \left\{ \frac{(1+\beta)k}{2\beta(p-\alpha_j)} \right\}^2 a_{k,j}^2 \le \left\{ \sum_{k=p+n}^{\infty} \frac{(1+\beta)k}{2\beta(p-\alpha_j)} a_{k,j} \right\}^2 \le 1 \tag{2.28}$$

for $j = 1, 2, \dots, m$, we have

$$\sum_{k=p+n}^{\infty} \frac{1}{m} \left\{ \frac{(1+\beta)k}{2\beta(p-\alpha_j)} \right\}^2 (\sum_{j=1}^{m} a_{k,j}^2) \le 1 . \tag{2.29}$$

Note that we have to find the largest $\gamma$ such that

$$\sum_{k=p+n}^{\infty} \left\{ \frac{(1+\beta)k}{2\beta(p-\gamma)} \right\} (\sum_{j=1}^{m} a_{k,j}^2) \le 1 . \tag{2.30}$$

This implies that

$$\gamma \le p - \frac{m[2\beta(p-\alpha_0)]^2}{2\beta(1+\beta)k} \quad (k \ge p+n) , \tag{2.31}$$

that is, that

$$\gamma \le p - \frac{m[2\beta(p-\alpha_0)]^2}{2\beta(1+\beta)(p+n)} , \tag{2.32}$$

which completes the proof of Theorem 2.

Putting $\alpha_j = \alpha$ $(j = 1, 2, \dots, m)$ in Theorem 2, we have :

**Corollary 6.** *If $f_j(z) \in P_p^*(n, \alpha, \beta)(j = 1, 2, \dots, m)$ and $h(z)$ is defined by (2.26), then $h(z) \in P_p^*(n, \gamma, \beta)$, where*

$$\gamma = p - \frac{m[2\beta(p-\alpha)]^2}{2\beta(1+\beta)(p+n)} . \tag{2.33}$$

The result is sharp for the functions $f_j(z)$ defined by (2.17).

Putting $\beta = 1$ in Theorem 2, we have :

**Corollary 7.** *If $f_j(z) \in P_p^*(n, \alpha_j)(j = 1, 2, \dots, m)$ and $h(z)$ is defined by (2.26), then $h(z) \in P_p^*(n, \gamma)$, where*

$$\gamma = p - \frac{m(p-\alpha_0)^2}{p+n} \ (\alpha_0 = \min\{\alpha_1, \alpha_2, \dots, \alpha_m\}) . \tag{2.34}$$

The result is sharp for the functions $f_j(z)$ defined by (2.19).

Putting $\beta = 1$ in Corollary 6, we have :

**Corollary 8.** *If $f_j(z) \in P_p^*(n, \alpha)(j = 1, 2, \dots, m)$ and $h(z)$ is defined by (2.26), then $h(z) \in P_p^*(n, \gamma)$, where*

$$\gamma = p - \frac{m(p-\alpha)^2}{p+n} . \tag{2.35}$$

The result is sharp for the functions $f_j(z)$ defined by

$$f_j(z) = z^p - \frac{p-\alpha}{p+n} z^{p+n} \ (j = 1, 2, \ldots, m)\ . \tag{2.36}$$

Putting $n = 1$ in Corollary 6, we have :

**Corollary 9.** *If* $f_j(z) \in P_p^*(\alpha, \beta)(j = 1, 2, \ldots, m)$ *and* $h(z)$ *is defined by* (2.26) *with* $n = 1$, *then* $h(z) \in P_p^*(\gamma, \beta)$, *where*

$$\gamma = p - \frac{m[2\beta(p-\alpha)]^2}{2\beta(1+\beta)(p+1)}\ . \tag{2.37}$$

The result is sharp for the functions $f_j(z)$ defined by (2.21).

Putting $\beta = n = 1$ in Corollary 6, we have :

**Corollary 10.** *If* $f_j(z) \in P_p^*(\alpha)(j = 1, 2, \ldots, m)$ *and* $h(z)$ *is defined by* (2.26) *with* $n = 1$, *then* $h(z) \in P_p^*(\gamma)$, *where*

$$\gamma = p - \frac{m(p-\alpha)^2}{p+1}\ . \tag{2.38}$$

The result is sharp for the functions $f_j(z)$ defined by (2.23).

Putting $\beta = p = 1$ in Corollary 6, we have :

**Corollary 11.** *If* $f_j(z) \in \widetilde{C}(n, \alpha)(j = 1, 2, \ldots, m)$ *and* $h(z)$ *is defined by* (2.26) *with* $p = 1$, *then* $h(z) \in \widetilde{C}(n, \gamma)$, *where*

$$\gamma = 1 - \frac{m(1-\alpha)^2}{1+n}\ . \tag{2.39}$$

The result is sharp for the functions $f_j(z)$ defined by (2.25).

*Remark* 2. (i) Corollary 7 is the generalization of Theorem B given by Yaguchi et al. [10];
(ii) Corollary 10 is the generalization of Theorem F given by Aouf and Darwish [3];
(iii) Corollary 11 is the generalization of Theorem H given by Sekine and Owa [9].

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