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SOME REMARKS ON COTORSION ENVELOPES OF MODULES

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Abstract. In this paper we prove that the extension of pure injective module is pure injective if and only if the cotorsion envelope and the pure injective envelope of any R-module M are isomorphic over M. And we prove that if the product of pure injective envelopes of flat modules is a pure injective envelope and the product of flat covers is a flat cover, then the product of cotorsion envelopes is a cotorsion envelope.

1. Introduction

Throughout this paper, R denotes any ring with identity and all modules are unitary. A left R-module P is called *pure injective* if every diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & \swarrow{\scriptstyle g} & & & & & \\ & & P & & & & & & \end{array}$$

with the upper row pure exact can be completed to a commutative diagram. Equivalently, $\mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(N, P) \to 0$ is exact whenever N is a pure submodule of M. We let $\mathcal{PE}$ be the class of all left pure injective modules. Then $\mathcal{PE}$ is contains all injective modules and closed under isomorphisms, finite direct sums, direct summands, and direct products.

We say that a left R-module C is *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F. We let $\mathcal{C}$ be the class of all cotorsion modules. Then $\mathcal{C}$ is closed under isomorphisms, extensions, finite direct sums, direct summands, and direct products. And every pure injective module is cotorsion (see [5, pg. 52]).

We first recall the definitions of envelopes and covers.

Definition 1.1 ([5]). Let $\mathcal{X}$ be a class of R-modules that is closed under isomorphisms, direct summands, and finite direct sums. An *$\mathcal{X}$-envelope* of a

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module M is a linear map $\phi : M \to X$ with $X \in \mathcal{X}$ (or briefly X) such that the following two conditions hold :

(1) $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to 0$ is exact for any $X' \in \mathcal{X}$;
(2) Any $f : X \to X$ with $f \circ \phi = \phi$ is an automorphism of X.

If $\phi : M \to X$ satisfies (1), and perhaps not (2), ϕ is called an *$\mathcal{X}$-preenvelope* of M.

In particular, if $\mathcal{X} = \mathcal{C}$, $\mathcal{X}$-(pre)envelopes are called *cotorsion (pre)envelopes.*

Dually, an *$\mathcal{X}$-precover* of M is a linear map $\psi : X \to M$ with $X \in \mathcal{X}$ such that $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$ is exact, and if an $\mathcal{X}$-precover $\psi : X \to M$ of M satisfies that any $f : X \to X$ with $\psi \circ f = \psi$ is an automorphism of X, then $\psi : X \to M$ is called an *$\mathcal{X}$-cover* of M.

Note that an $\mathcal{X}$-envelope, if it exists, is unique up to isomorphism and any cotorsion (pre)envelopes and pure injective (pre)envelopes are injections.

Proposition 1.2 ([5, Theorem 3.4.6]). *For any ring R every left R-module has a cotorsion envelope if and only if every left R-module has a flat cover.*

Since it is known that every module has a flat cover([1]), every module M has a cotorsion envelope $C(M)$.

We recall that for a class $\mathcal{X}$ of modules, $\mathcal{X}^\perp$ consists of all modules N such that $\mathrm{Ext}^1_R(X, N) = 0$ for all $X \in \mathcal{X}$ and ${}^\perp\mathcal{X}$ consists of all M such that $\mathrm{Ext}^1_R(M, X) = 0$ for all $X \in \mathcal{X}$.

Lemma 1.3 (Wakamatsu, [4] or [5, Lemma 2.1.2]). *Let $\mathcal{X}$ be closed under extensions. If $\phi : M \to X$ is an $\mathcal{X}$-envelope of M, then* $\mathrm{coker}\phi \in {}^\perp\mathcal{X}$.

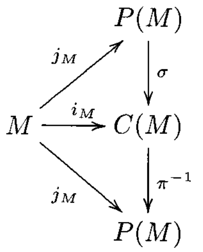
Conversely, if $\phi : M \to X$ is an injection with $X \in \mathcal{X}$ and $E = \mathrm{coker}\phi \in {}^\perp\mathcal{X}$, then for any $X' \in \mathcal{X}$, $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to \mathrm{Ext}^1_R(E, X') = 0$ is exact. So ϕ is an $\mathcal{X}$-preenvelope of M. Such a preenvelope is called a *special $\mathcal{X}$-preenvelope* of M.

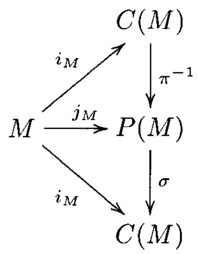
2. Main results

Let $i_M : M \to C(M)$ and $j_M : M \to P(M)$ be a cotorsion envelope and a pure injective envelope of an R-module M, respectively. Since $P(M)$ is cotorsion, there exists a morphism $\pi : C(M) \to P(M)$ with $\pi \circ i_M = j_M$.

Proposition 2.1. *If a morphism $\pi : C(M) \to P(M)$ with $\pi \circ i_M = j_M$ is an isomorphism, then any morphism $\sigma : C(M) \to P(M)$ with $\sigma \circ i_M = j_M$ is an isomorphism.*

Proof. Consider the following two diagrams ;

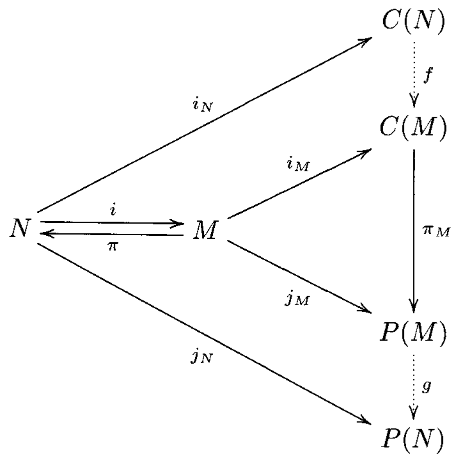




Since $\pi^{-1}\circ\sigma\circ j_M = j_M$ and j_M is an envelope, $\pi^{-1}\circ\sigma$ is an isomorphism. Similarly, since $\sigma\circ\pi^{-1}\circ i_M = i_M$ and i_M is an envelope, $\sigma\circ\pi^{-1}$ is an isomorphism. So σ is an isomorphism. □

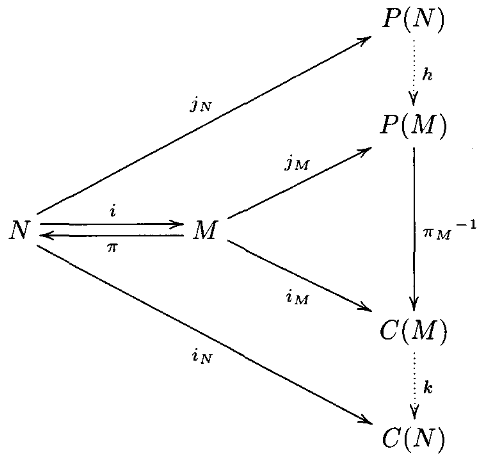
Proposition 2.2. *Let $\mathcal{CP}$ be the class of all R-modules M such that any linear map $\pi_M : C(M) \to P(M)$ with $\pi_M \circ i_M = j_M$ is an isomorphism. Then $\mathcal{CP}$ is closed under direct summands and finite direct sums.*

Proof. Let N be a direct summand of an R-module $M \in \mathcal{CP}$ and let $M = N\oplus K$ for a submodule K of M. For any map $\pi_N : C(N) \to P(N)$ with $\pi_N\circ i_N = j_N$, consider the diagram ;



where $\pi : M \to N$ and $i : N \to M$ are the natural maps. Since $C(N)$ is a cotorsion envelope of N, for $i_M\circ i : N \to C(M)$, there exists $f : C(N) \to C(M)$ such that $f\circ i_N = i_M \circ i$. Similarly, for $j_N \circ \pi : M \to C(N)$, there exists $g : P(M) \to P(N)$ such that $g\circ j_M = j_N \circ \pi$, since $P(M)$ is a pure injective envelope of M. So $g\circ\pi_M\circ f\circ i_M = g\circ\pi_M\circ i_M\circ i = g\circ j_M\circ i = j_N\circ\pi\circ i = j_N$. Let $\phi = g\circ\pi_M\circ f$. Then $\phi : C(M) \to P(M)$ satisfies $\phi\circ i_N = j_N$.

On the other hand, for ${\pi_M}^{-1}$, there exist $h : P(N) \to P(M)$ and $k : C(M) \to C(N)$ such that $h \circ j_N = j_M \circ i$ and $k \circ i_M = i_N \circ \pi$.



So $k \circ {\pi_M}^{-1} \circ h \circ j_N = k \circ {\pi_M}^{-1} \circ j_M \circ i = k \circ i_M \circ i = i_N \circ \pi \circ i = i_N$. Let $\psi = k \circ {\pi_M}^{-1} \circ h$. Then ψ maps $P(N)$ to $C(N)$ and we have $\psi \circ \phi \circ i_N = i_N$ and $\phi \circ \psi \circ j_N = j_N$. Since i_N and j_N are envelopes, $\psi \circ \phi$ and $\phi \circ \psi$ are isomorphisms. So ψ is an isomorphism, and thus $N \in \mathcal{CP}$.

Now let $M, N \in \mathcal{CP}$ and let $M \oplus N = K$. By [5, Theorem 1.2.5], $C(K) = C(M) \oplus C(N)$ and $P(K) = P(M) \oplus P(N)$. Since $\pi_M : C(M) \to P(M)$ and $\pi_N : C(N) \to P(N)$ are isomorphisms, $\pi_M \oplus \pi_N : C(M) \oplus C(N) \to P(M) \oplus P(N)$ is an isomorphism. So $M \oplus N \in \mathcal{CP}$. □

We now extend a result of Xu ([5, Theorem 3.5.1]).

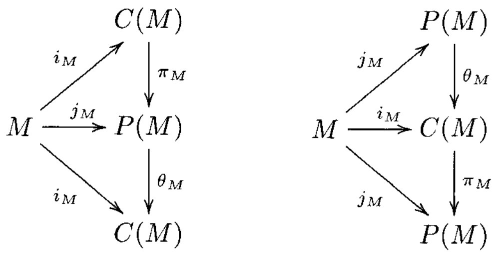
Theorem 2.3. *For any ring R, the followings are equivalent ;*

(1) *For any exact sequence of left R-modules $0 \to P' \to P \to P'' \to 0$ with P' and P'' pure injective, P is also pure injective.*
(2) *For any left R-module M, $P(M)/M$ is flat.*
(3) *Every left cotorsion module is pure injective.*
(4) *For any left R-module M, $\pi_M : C(M) \to P(M)$ is an isomorphism.*

Proof. Since (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by [5, Theorem 3.5.1], it suffices to show that (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2).

For (3) $\Rightarrow$ (4), assume that every left cotorsion module is pure injective. Then $C(M)$ is pure injective. So there exists $\theta_M : P(M) \to C(M)$ such that

$\theta_M(m) = m$ for all $m \in M$. Consider the diagrams ;



Since $(\theta_M \circ \pi_M) \circ i_M = i_M$ and $i_M : M \to C(M)$ is a cotorsion envelope, $\theta_M \circ \pi_M$ is an isomorphism. Similarly, since $j_M : M \to P(M)$ is a pure injective envelope, $\pi_M \circ \theta_M$ is also an isomorphism. So π_M is an isomorphism.

Now we want to prove (4) $\Rightarrow$ (2). Let $i_M : M \to C(M)$ be a cotorsion envelope of M. Then by Lemma 1.3, $P(M)/M \cong C(M)/M \in \mathcal{C}^{\perp}$. So $P(M)/M$ is flat by [5, Lemma 3.4.1]. □

Example 2.4. Every von Neumann regular ring trivially satisfies the conditions of the previous proposition since every module over such a ring is flat.

Proposition 2.5. *If R is a right coherent ring, then for every flat R-module F, any linear map $\pi_F : C(F) \to P(F)$ is an isomorphism.*

Proof. By [5, Theorem 3.4.2] $C = C(F)$ is flat. So by [5, Lemma 3.1.6], $P(C)/C$ is also flat. Then the exact sequence $0 \to C \to P(C) \to P(C)/C \to 0$ is split. Thus C is pure injective. From the proof of the previous Proposition $\pi_F : C(F) \to P(F)$ is an isomorphism. □

It is an important and intriguing question when taking envelopes and covers commutes with taking direct products or sums. For example, taking injective envelopes commutes with direct sums if and only if the ring is left Noetherian. See [3] for consequences of such commutativity and [2] where this problem is considered for torsion free covers over an integral domain and products. We think a basic question is when taking cotorsion envelopes commutes with taking products. The next result is just a small initial step in trying to give a satisfactory answer to this question.

Theorem 2.6. *Let R be a right coherent ring. If the product of pure injective envelopes of flat modules F_i is a pure injective envelope of $\prod F_i$ and if the product of flat covers is a flat cover, then the product of cotorsion envelopes is a cotorsion envelope.*

Proof. Consider the pushout diagram of a flat cover $\phi_i : F_i \to M_i$ and a pure injective envelope $\sigma_i : F_i \to G_i$;

$$\begin{array}{ccc} F_i & \xrightarrow{\phi_i} & M_i \\ {\scriptstyle \sigma_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ G_i & \xrightarrow[\psi_i]{} & C_i \end{array}$$

Then it is also a pullback diagram of a cotorsion envelope π_i and a flat cover ψ_i by [5, Theorem 3.4.8]. Let $\pi = \prod \pi_i, \sigma = \prod \sigma_i, \phi = \prod \phi_i$, and $\psi = \prod \psi_i$. Then

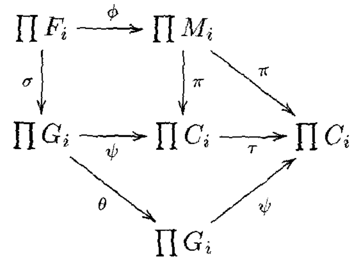
$$\begin{array}{ccc} \prod F_i & \xrightarrow{\phi} & \prod M_i \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \pi} \\ \prod G_i & \xrightarrow[\psi]{} & \prod C_i \end{array}$$

is also a pullback diagram of π and ψ.

Consider the full pullback diagram ;

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod K_i & \xrightarrow{\alpha} & \prod F_i & \xrightarrow{\phi} & \prod M_i & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \prod K_i & \xrightarrow[\beta]{} & \prod G_i & \xrightarrow[\psi]{} & \prod C_i & \longrightarrow & 0 \\ & & & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ & & & & \prod D_i & = & \prod D_i & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $K_i = \ker\psi_i, D_i = \mathrm{coker}\pi_i$ and f, g, α, β are the natural maps. Since R is right coherent, $\prod F_i$ and $\prod G_i$ are flat modules. So $\prod D_i$ is flat, and thus π is a cotorsion preenvelope. Now let τ be an endomorphism of $\prod C_i$ with $\tau \circ \pi = \pi$. Since ψ is a flat cover, there exists $\theta : \prod G_i \to \prod G_i$ with $\tau \circ \psi = \psi \circ \theta$. Consider the pair of linear maps $\theta \circ \sigma : \prod F_i \to \prod G_i$ and $\phi : \prod F_i \to \prod M_i$.



Since $\psi \circ \theta \circ \sigma = \tau \circ \psi \circ \sigma = \pi \circ \phi$, there exists a linear map $k : \prod F_i \to \prod F_i$ such that $\sigma \circ k = \theta \circ \sigma$ and $\phi \circ k = \phi$. Since ϕ is a flat cover, k is an isomorphism. And since $\sigma : \prod F_i \to \prod G_i$ is a pure injective envelope, there exists $\theta' : \prod G_i \to \prod G_i$ with $\sigma \circ k^{-1} = \theta' \circ \sigma$. So $\sigma = \sigma \circ k \circ k^{-1} = \theta \circ \sigma \circ k^{-1} = \theta \circ \theta' \circ \sigma$. Since σ is a pure injective envelope, $\theta \circ \theta'$ is an isomorphism and so θ is surjective. It follows that τ is a surjection. Moreover, if $\tau(x) = 0$ for some $x \in \prod C_i$, there exists $y \in G_i$ with $x = \psi(y)$. So $0 = \tau(x) = \tau \circ \psi(y) = \psi \circ \theta(y) = \psi(y)$, and then there exists $z \in \prod K_i$ with $y = \beta(z) = \sigma(z)$. So $x = \psi(y) = \psi \circ \beta(z) = 0$. Thus τ *is an isomorphism*, and hence π is a cotorsion envelope. □

Example 2.7. If a ring R is left perfect and right coherent, then an R-module F is flat if and only if it is projective. So all modules are cotorsion and $C(M) = M$ for any R-module M. Moreover, for every flat module F, $P(F) = C(F)$ by Proposition 2.5. Thus R is an example of a ring satisfying the condition of Theorem 2.6.

An open question. We would like to prove the advanced Theorem : Let R be a right coherent ring. If the product of pure injective envelopes of flat modules F_i is a pure injective envelope of $\prod F_i$, then the product of flat covers is a flat cover if and only if the product of cotorsion envelopes is a cotorsion envelope.

References

[1] L. Bican, R. El Bashir, and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390.

[2] E. Enochs, *Torsion free covering modules. II*, Arch. Math. (Basel) **22** (1971), 37–52.

[3] E. Enochs, J. R. Garcia Rozas, and L. Oyonarte, *Are covering (enveloping) morphisms minimal?*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 2863–2868.

[4] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), no. 2, 298–325.

[5] J. Xu, *Flat covers of modules*, Lecture Notes in Mathematics, 1634, Springer-Verlag, Berlin, 1996.

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