Design of Time-varying Stochastic Process with Dynamic Bayesian Networks

Hyun-Cheol Cho†, M. Sami Fadali** and Kwon-Soon Lee *

Abstract – We present a dynamic Bayesian network (DBN) model of a generalized class of nonstationary birth-death processes. The model includes birth and death rate parameters that are randomly selected from a known discrete set of values. We present an on-line algorithm to obtain optimal estimates of the parameters. We provide a simulation of real-time characterization of load traffic estimation using our DBN approach.

Keywords : Adaptive estimation, Birth-Death process, Convergence property, Dynamic Bayesian networks

1. Introduction

The birth-death process is a well known Markov stochastic process. The fundamental characteristic of the process is that its state is only dependent on neighboring states. Birth-death processes are still widely applied in science and engineering. A general birth-death process was used for modeling of stochastic epidemics in [1]. The author also investigated the behavior of state probabilities with an almost infinity time interval for the case of a periodic transition intensity matrix of the process. In [1], the authors provided a review of Markov-driven fluid queues with special attention given to the heterogeneous on/off model. In addition, an approximation procedure in which the original heterogeneous arrival process is replaced by a homogeneous birth-death arrival process was presented. In [2], the authors studied stochastic modeling of the powder coating process based on a birth-death population balance including theoretically-derived one-step transition probabilities. The population balance equation was obtained under steady-state conditions and its dynamics were shown to have a Bernoulli distribution at equilibrium. In [3], the author used the high volatility of share prices in stock markets to build a model that leads to a particular cross-sectional size distribution. The model focuses on both transient and steady-state behavior of the market capitalization of the stock, which in turn is modeled as a birth-death process. In [4], the authors investigated the fluid queue models with infinite buffer capacity in which the fluid flow is governed by a birth-death process with quadratic arrival and service rates on a finite state space. Recently, in [5] the author used generalized birth-death processes to model the ageing process.

Most applications of birth-death modeling concern stationary or time-invariant characteristics in dynamic systems in which the model parameters are assumed constant. In the design step, the parameters are selected from observation data to optimally reflect system dynamics. However, parameter selection becomes suboptimal in real-time implementation if the statistics of the system change appreciably. The results of the fixed model become progressively less accurate unless the model parameters are updated. The additional computationally task of parameter correction becomes necessary to maintain model fidelity.

We develop an adaptive estimation for parameters of the birth-death process using DBN technique. DBN is a graphical modeling approach for temporal states of dynamic systems. We refer the reader to [6] for more information on DBNs. We represent the parameters of the birth-death process with a DBN and sequentially estimate their values from given observations. We use the time-moving average of a Bernoulli random variable as an update rule for the parameters. The convergence of the proposed estimation is analytically investigated based on a stochastic convergence theorem and the stability of its dynamics is demonstrated. We apply our estimation algorithm to a time-varying M/M/1 birth-death process for modeling road traffic.

This paper is organized as follows: Section 2 provides a brief review of the birth-death process. In Section 3, we propose an adaptive parameter estimation algorithm for the birth-death process using a DBN model. In Section 4, the convergence and stability of the estimation algorithm are studied. A simulation example and conclusions are respectively provided in Sections 5 and 6.

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2. Preliminaries

We start with a brief review of a generic continuous birth-death process. Usually, a birth-death process is a special modification of a continuous-time Markov chain (See *Definition* 1) which is homogeneous, aperiodic, and irreducible [7].

***Definition* 1** *Given a stochastic process* $X(t)$, $t \geq 0$ *where* $X(t)$ *is the state of the dynamic system at time t, a continuous-time Markov chain is a continuous time, discrete-valued random process such that for an infinitesimal time step* δ,

$$p(X(t+\delta)=j \mid X(t)=i)=q_{ij}\delta \quad (1)$$

$$p(X(t+\delta)=i \mid X(t)=i)=1-\sum_{j\neq i} q_{ij}\delta \quad (2)$$

The main feature of a birth-death process is that its state changes are only between adjacent states. This characteristic leads to the restriction that the state transition parameter at each step q_{ij} in (1) and (2) satisfies $q_{ij}=0$ for $|i-j|>1$. We state the motion characteristics of the generic birth-death process as follows:

1) A birth occurs between time t and $t+\delta$ with probability $\lambda_i\delta + o(\delta)$ where $o(\delta)$ represents an infinitesimal change of higher order than δ. A birth increases the system state by one to i+1, where the parameter λ_i is the birth (arrival) rate in state i.
2) A death occurs between t and $t+\delta$ with probability $\mu_i\delta + o(\delta)$, which decreases the system state by one to i–1, where μ_i is the death (departure) rate in state i. Note $\mu_0 = 0$.
3) It is assumed that the birth and the death are statistically independent of each other.

An illustration of a birth-death model with N+1 states is shown in Fig. 1. Here, the parameters λ_i, $i = 0, \cdots, N-1$ and μ_i, $i = 1, \cdots, N$ are the birth and death rates. Note that birth never occurs at state N and death never occurs at 0.

Let $X(t)$ be a state representing the number of customers in the queuing system at time t, which is simply a difference between total births and total deaths at the current time. Moreover, $p_i(t) \equiv p(X(t)=i)$ is the probability of finding the system in state i at time t. The Chapman-Kolmogorov dynamic equations for an N+1 state generic birth-death model are given by [7]:

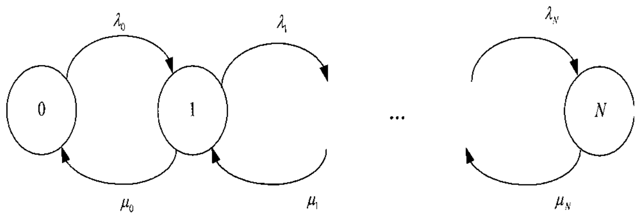

Fig. 1. A finite birth-death process model

$$\frac{d}{dt}p_0(t)=-\lambda_0 p_0(t)+\mu_1 p_1(t) \quad (3)$$

$$\frac{d}{dt}p_i(t)=-(\lambda_i+\mu_i)p_i(t)+\lambda_{i-1}p_{i-1}(t)+\mu_{i+1}p_{i+1}(t) \quad (4)$$

where $i = 1, \cdots, N$, and

$$\sum_{i=0}^{N} p_i(t)=1 \quad (5)$$

The matrix form of (3) and (4) is given by

$$\frac{d}{dt}P(t)=Q^T P(t) \quad (6)$$

where the state probability vector is

$$P(t)=[P_0(t) \quad P_1(t) \quad \cdots \quad P_N(t)]^T \quad (7)$$

and the infinitesimal generator matrix is

$$Q=\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1} & -(\lambda_{N-1}+\mu_{N-1}) & \lambda_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_N & -\mu_N \end{bmatrix} \quad (8)$$

A general solution of the differential equation of (6) is easily obtained as

$$P(t)=\exp(Q^T t)P(0) \quad (9)$$

where $P(0)$ is an initial stochastic vector. The steady-state probabilities are obtained by setting the left sides of (3) and (4) to zero and are given by

$$p_i^* = \lim_{t\to\infty} p_i(t)=\prod_{j=1}^{i}\left(\frac{\lambda_{j-1}}{\mu_j}p_0^*\right), \quad i=1,\cdots,N \quad (10)$$

where

$$p_0^* = \lim_{t\to\infty} p_0(t)=\left(1+\sum_{i=1}^{N}\prod_{j=1}^{i}\left(\frac{\lambda_{j-1}}{\mu_j}\right)\right)^{-1} \quad (11)$$

This formula expresses the stationary probabilities in terms of the birth and death rates in the queuing systems. Thus, the birth-death process is completely specified by these rates. For the homogeneous case $\lambda_i = \lambda$, $i = 0, \cdots, N-1$ and μ_i, $i = 1, \cdots, N$, Equation (10) becomes

$$p_i^* = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i, \qquad i = 1, \cdots, N \tag{12}$$

The parameters of the general birth-death process are usually assumed fixed, i.e. the matrix in (8) of the stationary process does not change with time. However, for a nonstationary birth-death process, the model parameters and the rate matrix in (8) are time-varying. By simply introducing time as an argument of the matrix in (6), we have the time-varying dynamic equation of the birth-death process as

$$\frac{d}{dt}P(t) = Q^T(t)P(t) \tag{13}$$

3. Adaptive Estimation Algorithm

In this Section, we propose an adaptive parameter estimation algorithm of the birth-death process. This algorithm is based on observation sequence from systems such that the parameter update is sequentially accomplished using current and preceding observation data. The observation is periodically obtained during a given sampling time. This framework is constructed with a simple discrete-type DBN model, depicted in Fig. 2.

As indicated in Fig. 2, the current state $X(k)$ is caused by the preceding state $X(k\text{-}1)$ and the state observation is temporal. According to the characteristic of the birth-death

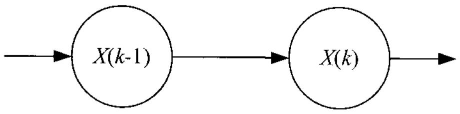

Fig. 2. A simple DBN for the parameter estimation

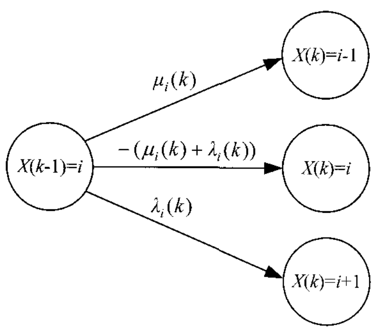

Fig. 3. The parameters of the birth-death process

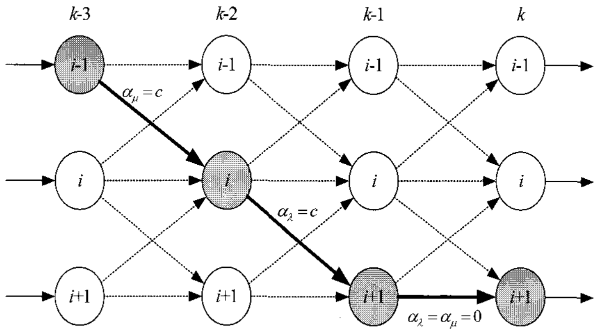

Fig. 4. Observation sequence for the birth-death process

process, a certain state at $k-1$, $x_i(k-1)$, $i = 1, \cdots, N$ causes three neighboring states (See Fig. 3).

First, we define update rules for the parameters in Fig. 3 as

$$\lambda_i(k) = \left(\frac{k-1}{k}\right)\lambda_i(k-1) + \left(\frac{1}{k}\right)\alpha_{\lambda_i}(k),\ i = 0, \cdots, N-1 \tag{14}$$

$$\mu_i(k) = \left(\frac{k-1}{k}\right)\mu_i(k-1) + \left(\frac{1}{k}\right)\alpha_{\mu_i}(k),\ i = 1, \cdots, N \tag{15}$$

where α_μ, $\alpha_\lambda > 0$. These are time-averaging expressions including the previous average and random variables α_μ and α_λ. The random variables are equal to 0 or a positive constant c depending on the observed state transition, i.e.

$$\begin{cases} \alpha_{\lambda_i}(k) = c\,, & \textit{if } X(k) = i+1 \mid X(k-1) = i \\ \alpha_{\lambda_i}(k) = 0\,, & \textit{otherwise} \end{cases} \tag{16}$$

$$\begin{cases} \alpha_{\mu_i}(k) = c\,, & \textit{if } X(k) = i-1 \mid X(k-1) = i \\ \alpha_{\mu_i}(k) = 0\,, & \textit{otherwise} \end{cases} \tag{17}$$

Note that random variables in (16) and (17) are both zero if there is no change in the current state, i.e. $\alpha_\lambda = \alpha_\mu = 0$, when $X(k)=i$ given $X(k-1)=i$.

Example: Given an observation sequence, e.g. $X(k-3)=i-1$, $X(k-2)=i$, $X(k-1)=i+1$, and $X(k)=i+1$, according to (16) and (17), we have $\alpha_\mu(k-2)=c$, $\alpha_\lambda(k-1)=c$, $\alpha_\lambda(k)=\alpha_\mu(k)=0$ for the observations shown in Fig. 4. This changes the parameter values using (14) and (15) so that $\lambda_i(k-2)$ is decreased, but $\mu_i(k-2)$ is relatively increased. At $k-1$, $\lambda_i(k-1)$ is increased, $\mu_i(k-1)$ is decreased, and both $\lambda_i(k)$ and $\mu_i(k)$ at k are decreased such that $-(\lambda_i(k)+\mu_i(k))$ is increased. This procedure is sequentially conducted by comparing two observations at k and $k-1$. The example demonstrates that the algorithm is simple and provides efficient online parameter estimation for a large sample.

4. Convergence Analysis of the Estimation

In this Section, we examine the asymptotic behavior of the parameter estimation algorithm of Section 3. By using a stochastic convergence theorem, we analytically prove the convergence and stability of the algorithm for a large sample, i.e. an infinite number of observations.

4.1 Stochastic Convergence

Since the parameter estimation rules of (14) and (15) have the exact same structure, we only analyze the stability of one but the results will apply to both. First, we rewrite (14) as

$$\begin{aligned}\lambda_i(k) &= a(k)\lambda_i(k-1)+b(k)\alpha_\lambda(k) \\ &= \frac{1}{k}\sum_{n=1}^{k}\alpha_\lambda(n)\end{aligned} \tag{18}$$

where $a(k)=k^{-1}(k-1)$ and $b(k)=k^{-1}$. For simplicity, we let c = 1 in (16), thus note α_λ = 1 or 0 such that this variable is governed by the Bernoulli distribution

$$p(\alpha=\alpha_\lambda)=q^{\alpha_\lambda}(1-q)^{1-\alpha_\lambda}, \qquad \alpha_\lambda=0,1 \tag{19}$$

where $q=p(\alpha_\lambda=1)\in(0,1)$. The mean and mean square for the Bernoulli distribution are given by

$$E(\alpha)=E(\alpha^2)=q \tag{20}$$

***Definition* 2** [8] *For a sequence of random variables X on some probability space, X(k) converges in probability to X^*, if, for* $\varepsilon>0$, $\lim_{k\to\infty}P\{|X(k)-X^*|>\varepsilon\}=0$.

***Definition* 3** [8] *We say that a sequence of random variables X(k) converges to X^* in a mean-squared sense, if* $\lim_{k\to\infty}E\{(X(k)-X^*)^2\}=0$.

***Theorem* 1** [8] *If a sequence of random variables X(k) converges to X^* in mean square, then it converges to X^* in probability.*

***Lemma* 1** The sequence of the random variable λ_i in (18) asymptotically converges in mean-square to q.

Proof We seek to prove that

$$\lim_{k\to\infty}E\left\{(\lambda_i(k)-q)^2\right\}=0 \tag{21}$$

where $q=E\{\alpha\}$. By substituting the second term of (18) in (21), we expand the limit as

$$\begin{aligned}&\lim_{k\to\infty}E\left\{\left(\frac{1}{k^2}\left[\sum_{n=1}^{k}\alpha_\lambda(n)\right]^2-\frac{2q}{k}\sum_{n=1}^{k}\alpha_\lambda(n)+q^2\right)\right\} \\ &=\lim_{k\to\infty}E\left(\frac{1}{k^2}\left[\sum_{n=1}^{k}\alpha_\lambda(n)^2+\sum_{n=1}^{k}\sum_{\substack{l=1\\l\neq n}}^{k}\alpha_\lambda(n)\alpha_\lambda(l)\right]-\frac{2q}{k}\sum_{n=1}^{k}\alpha_\lambda(n)+q^2\right)\end{aligned} \tag{22}$$

For i.i.d. Bernoulli trials, the expression becomes

$$\lim_{k\to\infty}\left(\frac{q}{k^2}+\frac{k(k-1)}{k^2}q^2-2q^2+q^2\right)=0 \tag{23}$$

This result shows that the estimation asymptotically converges to a constant value. Similarly, the estimation of (15) has the identical convergence property.

4.2 Stability Analysis

We discuss the stability of the time-varying dynamic system corresponding to our learning algorithm. The estimation rule of (14) is rewritten in vector form as

$$m(k+1)=F(k)m(k)+G(k)u(k) \tag{24}$$

where the state vector $m(k)=[\lambda_0(k)\ \cdots\ \lambda_{N-1}(k)]^T$, the input vector $u(k)=[\alpha_{\lambda_0}\ \cdots\ \alpha_{\lambda_{N-1}}]^T$, and the corresponding matrices $F(k)=(k-1/k)I_N$ and $G(k)=(1/k)I_N$.

***Theorem* 2** [9] *Consider an unforced linear discrete time-varying system as $x(k+1)=F(k)x(k)$. Its solution vector is $x(k)=\phi(k,k_0)x(k_0)$, $k_0<k$, where the state-transition matrix $\phi(k,k_0)=F(k)F(k-1)\cdots F(k_0)$. If a norm of the solution $\|x(k)\|\to 0$ as $k\to\infty$ for any initial state $x(k_0)$, this system is asymptotically stable. This is equivalent to the condition $\|\phi(k,k_0)\|\to 0$ as $k\to\infty$.*

***Lemma* 2** The dynamic equation (5.13) is asymptotically stable for any initial state $m(k_0)$ at initial time k_0.

Proof The state-transition matrix from k_0 to k for (24) is

$$\phi(k,k_0)=\left[\prod_{i=k_0}^{k}\left(\frac{i-1}{i}\right)\right]I_N=\left(\frac{k_0-1}{k}\right)I_N \tag{25}$$

where $k_0\geq 1$ and $k_0 \ll k$. Applying a limit to (25), we have

$$\lim_{k\to\infty}\phi(k,k_0)=\left[\lim_{k\to\infty}\left(\frac{k_0-1}{k}\right)\right]I_N=\mathbf{0} \tag{26}$$

From *Theorem* 5.4, we conclude that the recursion (24) is asymptotically stable.

5. Simulation Example

We apply the proposed estimation algorithm to road traffic modeling, which is represented by a general birth-death process. The traffic data pattern was generated using MATLAB based on the road traffic observed in a section of Taiwan freeway No. 1 during the rush hours of. 7:00 to 10:00 a.m. [10]. The traffic data appears to be Poisson distributed and can be realized by using the MATLAB© command, *poissrnd*. The mean value of the Poisson traffic is assumed to be uniformly distributed in the range [8, 12] and is generated using the command, *rand*. Using a sampling period of 0.5 min over a period of 3 hours, the total number of observations is 360. Fig. 5 shows a plot of the traffic history used for this simulation. We set up 21 states (i.e. N=20) to construct our birth-death model and, since the process has M/M/1 structure, the birth and death rates are identical for each state [7], i.e. $\lambda_i = \lambda$ and $\mu_i = \mu$ where $i = 0, \cdots, 20$.

We estimate the two unknown parameters λ and μ from the observations using the algorithm of Section 3. Fig. 6 shows the trajectories of the estimated parameters. The trajectories indicate that the birth rate exceeds the death rate until about 9:00 a.m. and then drops below it. We infer that the traffic first rapidly increases but later decreases.

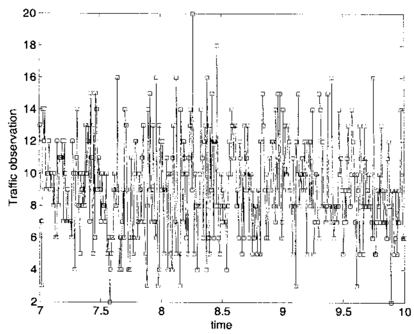

Fig. 5. The traffic observation

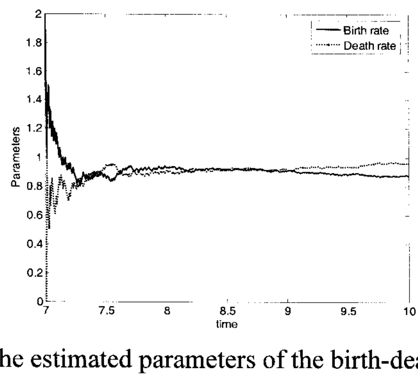

Fig. 6. The estimated parameters of the birth-death process

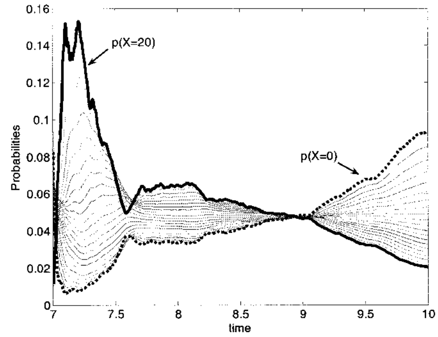

Fig. 7. Trajectory of the state probabilities for the traffic volumes, X=0, ⋯ ,20

The two curves appear to be almost symmetric because the probability that a current state is equal to the preceding one is low. After 8:00 a.m., the curves stabilize but changes persist due to the nonstationary statistics of the traffic.

Based on the estimated birth and death rates, we calculate the 21 state probabilities $p(X(t)=i)$, $i = 0, \cdots, 20$. MATLAB numerically solves the dynamic equation (13) for this process. Its solution trajectory for the probabilities is illustrated in Fig. 7. Following a short transient phase from 7:20 to 7:35 a.m., we note that until 9:00 a.m. $p(X=20)$ is the highest probability. After 9:00 a.m., $p(X=20)$ becomes the lowest probability. In addition, the changes in the trajectory of the probability $p(X=0)$ are in the reverse direction of those in $p(X=20)$. These conductions are analytically justified by the formula in (12). Specifically, as state X is increased, the corresponding probability is decreased if the birth rate is smaller than the death rate and vice versa.

6. Conclusions

We introduce a new parameter estimation algorithm for a generalized birth-death process. We prove the asymptotic stability of the estimation algorithm. We apply the algorithm to road traffic estimation using data from computer simulation and illustrate the algorithm's applicability. Future work includes more realistic applications using highway traffic data. A potential application is to select the arrival (birth) rate based on traffic flow. This framework involves estimation of highway conditions by the DBN approach and decision making to select one of several arrival rates.

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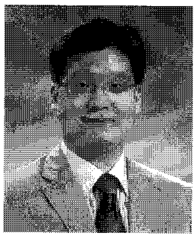

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