Application of the Implicit Restarted Arnoldi Method to the Small-Signal Stability of Power Systems

Dong-Joon Kim† and Young-Hwan Moon*

Abstract – This paper describes the new eigenvalue algorithm exploiting the Implicit Restarted Arnoldi Method (IRAM) and its application to power systems. IRAM is a technique for combining the implicitly shifted mechanism with a k-step Arnoldi factorization to obtain a truncated form of the implicitly shifted QR iteration. The numerical difficulties and storage problems normally associated with the Arnoldi process are avoided. Two power systems, one of which has 36 state variables and the other 150 state variables, have been tested using the ARPACK program, which uses IRAM, and the eigenvalue results are compared with the results obtained from the conventional QR method.

Keywords : Implicit Arnoldi, QR method, Small-signal stability

1. Introduction

The conventional [1, 2] method for small signal stability analysis is not applicable to large-scale power systems because of limitations due to memory capacity, computing time, and computation accuracy. To evaluate the small signal stability of power systems, it is usually required to only calculate a specific set of eigenvalues related to certain features of interest, for example, local mechanical modes, interarea modes, etc. Therefore, significant effort has been expended in developing new methods with basic properties, such as sparsity-based techniques, finding a small specific set of eigenvalues and mathematical robustness with good convergence characteristics and numerical stability [3, 4].

Because the eigenanalysis of modern power systems deals with matrices of very large dimension, sparsity techniques play a key role in the analysis. Two of the more popular sparsity-based eigenvalue techniques for general unsymmetrical matrices are the S-method [3], which is based on the Lanczos method with Cayley transformation, and the modified Arnoldi method [4]. The Lanczos-type method is a very successful method for the symmetrical eigenvalue problem, but has serious flaws in the case of unsymmetrical eigenvalue problems, known as the phenomenon of 'breakdown'. The modified Arnoldi method uses complete reorthogonalization and an iterative process with shift-invert transformation [5, 6]. However, reorthogonalization requires considerable storage and repeatedly finding the eigensystem will become prohibitive due to the cost of flops. To overcome such difficulties, an alternative has been proposed by Saad [8, 9] to restart the iteration with a vector that has been preconditioned so that it is more nearly in a dimensional invariant subspace of interest. This preconditioning takes the form of a polynomial applied to the starting vector. The polynomial is constructed to damp unwanted components from the eigenvector expansion. This technique is referred to as explicit (polynomial restarting). One of the more popular methods is the Arnoldi–Chebyshev method.

This paper describes another restarting approach, which is applicable to very large power systems. This approach is called the implicitly restarted Arnoldi method (IRAM) [5, 6]. IRAM is a technique for combining the implicitly shifted mechanism with a k-step Arnoldi factorization to obtain a truncated form of the implicitly shifted QR iteration. The numerical difficulties and storage problems normally associated with the Arnoldi process are avoided. The algorithm is capable of computing a few (k) eigenvalues with user-specified features, such as largest real part or largest magnitude. Implicit restarting provides a means of extracting interesting information from very large Krylov subspaces while avoiding the storage and numerical difficulties associated with the standard method. It does this by continually compressing the interesting information into a fixed size-dimensional subspace. This is accomplished through the implicit-shift mechanism.

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Received 9 March 2007 ; Accepted 18 June 2007

In this paper, the basic algorithms of IRAM, which are applicable to large-scale power systems, are described as an initial research phase for developing a sparsity-based eigenvalue program and the algorithms using IRAM are applied to two power systems in order to investigate its features.

2. IRAM Algorithm

2.1 Implicit Q Theorem

The Hessenberg decomposition is not unique, but it becomes so if the first column of Q is specified [10]. This is essentially the case provided that H has no zero subdiagonal entries. Hessenberg matrices with this property are said to be unreduced. A very important theorem that clarifies the uniqueness of the Hessenberg reduction is the implicit Q theorem.

Theorem 1. (Implicit Q Theorem) Assume $Q=[q_1,\cdots,q_n]$ and $V=[v_1,\cdots,v_n]$ are orthogonal matrices with the property that both $Q^TAQ=H$ and $V^TAV=G$ are upper Hessenberg where $A\in R^{n\times n}$. Let k denote the smallest positive integer for which $h_{k+1,k}=0$, with the convention that $k=n$ if H is unreduced. If $q_1=v_1$, then $q_i=\pm v_i$ and $|h_{i,i-1}|=|g_{i,i-1}|$ for i = 2:k. Moreover, if $k<n$, then $g_{k+1,k}=0$.

2.2 The Double Implicit-shift QR [10,11]

The single-shift QR iteration uses h_{nn} as the best approximate eigenvalue along the diagonal during each iteration:

for $k = 1, 2 ,...$
  $u = H(n,n)$
  $H-\mu I=UR$ (QR factorization)
  $H=RU+\mu I$
end

However, if the eigenvalues a_1 and a_2 of:

$$G=\begin{pmatrix} h_{mm} & h_{mn} \\ h_{nm} & h_{nn} \end{pmatrix} \quad m=n\text{-}1 \tag{1}$$

are complex, then h_{nn} would usually be a poor approximate eigenvalue. To avoid this difficulty it is possible to perform two single-shift QR steps in succession using a_1 and a_2 as shifts:

$$\begin{aligned} H-a_1I&=U_1R_1 \\ H_1&=R_1U_1+a_1I \end{aligned} \tag{2}$$

$$\begin{aligned} H_1-a_2I&=U_2R_2 \\ H_2&=R_2U_2+a_2I\,. \end{aligned} \tag{3}$$

These equations can be manipulated to show that:

$$(U_1U_2)(R_2R_1)=M \tag{4}$$

where M is defined by:

$$M=(H-a_1I)(H-a_2I)\,. \tag{5}$$

Note that M is a real matrix because:

$$M=H^2-sH+tI \tag{6}$$

where:

$$s=a_1+a_2=h_{mn}+h_{nn}=trace(G)\in R$$

and:

$$t=a_1a_2=h_{mm}h_{nn}-h_{mn}h_{nm}=\det(G)\in R$$

Because this step requires $O(n^3)$ flops to compute H_2 from H it is not a practical course of action to compute $H_2=Z^THZ$, where Z is computed from real QR factorization. However, by applying the implicit Q theorem, the double-shift step with $O(n^2)$ flops can be implemented. In particular, we can effect the transition from H to H_2 in flops if we compute Me_1, the first column of M. The first column of M is $Me_1=[x,y,z,0,\cdots,0]^T$ where:

$$x=h_{11}^2+h_{12}h_{21}-sh_{11}+t \tag{7}$$

$$y=h_{21}(h_{11}+h_{22}-s) \tag{8}$$

$$z=h_{21}h_{32}\,. \tag{9}$$

Then we can determine a Householder matrix P_0 such that $P_0(Me_1)$ is a multiple of e_1, and compute Householder matrices $P_1,\cdots,P_{n-2}$ such that if Z_1 is the product $Z_1=P_0P_1\cdots P_{n-2}$, then $Z_1^THZ_1$ is upper

Hessenberg and the first columns of Z and Z_1 are identical.

2.3 k-step Arnoldi Factorization

If $A \in C^{m \times n}$ then a relation of the form:

$$AQ_k = Q_k H_k + r_k e_k^T \tag{10}$$

where $Q_k \in C^{m \times k}$ has orthonormal columns, $Q_k^H r_k = 0$ and $H_k \in C^{k \times k}$ is upper Hessenberg with nonnegative subdiagonal elements and is called a *k*-step Arnoldi factorization of A. These equations are obtained from an Arnoldi process. In particular, if $Q = [q_1, \cdots, q_n]$ and we compare columns in $AQ = QH$, then:

$$Aq_k = \sum_{i=1}^{k+1} h_{ik} q_i, \quad 1 \le k \le n-1. \tag{11}$$

Isolating the last term in the summation gives:

$$h_{k+1} q_{k+1} = Aq_k - \sum_{i=1}^{k} h_{ik} q_i \coloneqq r_k \tag{12}$$

where $h_{ik} = q_i^T A q_k$ for I = 1:*k*. It follows that if $r_k \ne 0$, then q_{k+1} is specified by:

$$q_{k+1} = r_k / h_{k+1,k} \tag{13}$$

where $h_{k+1,k} = |r_k|_2$.

The q_k are called the Arnoldi vectors, and they define an orthonormal basis for the Krylov subspace $\kappa(A, q_1, k)$: $span\{q_1, \cdots, q_k\} = span\{q_1, Aq_1, \cdots, A^{k-1} q_1\}$.

2.4 Implicit Restarted Arnoldi Method (IRAM)

The IRAM determines the restart vector implicitly using the *QR* iteration with shifts. The restart occurs after every *m* step and we assume that $m > j$ where *j* is the number of sought-after eigenvalues. The choice of the Arnoldi length parameter *m* depends on the problem dimension, the effects of orthogonality loss, and system storage constraints. After *m* steps we have the Arnoldi factorization:

$$AQ_C = Q_C H_C + r_C e_m^T \tag{14}$$

The subscript C represents current. The *QR* iteration with shifts is then applied to H_C. Here $p = m - j$, and we have $H_+ = V^T H_C V$ because $V_i^T H^{(i)} V_i = H^{(i+1)}$. The orthogonal matrix $V = V_1 \cdots V_p$, with V_i, the orthogonal matrix associated with the shift μ_i, has two crucial properties:

(a) $[V]_{mi}$ for $i = 1 : j-1$. This is because each V_i is upper Hessenberg and so $V \in R^{m \times m}$ has lower bandwidth $p = m - j$.

(b) $Ve_1 = \alpha (H_C - \mu_p I)(H_C - \mu_{p-1} I) \cdots (H_c - \mu_1 I) e_1$, where α is a scalar.

We obtain the following transformation:

$$AQ_+ = Q_+ H_+ + r_c e_m^T V \tag{15}$$

where $Q_+ = Q_c V$. Given property (a),

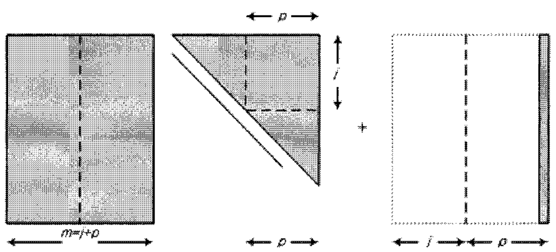

Fig. 1. Step 1: Arnoldi factorization, $Q_{j+p} H_{j+p} + r_m e_m^T$

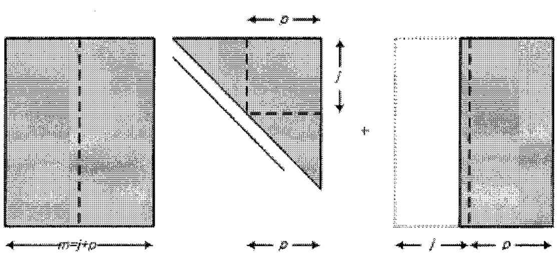

Fig. 2. Step 2: Applying the implicitly shifted step, . $Q_{j+p} V V^T H_{j+p} V + r_m e_m^T V$

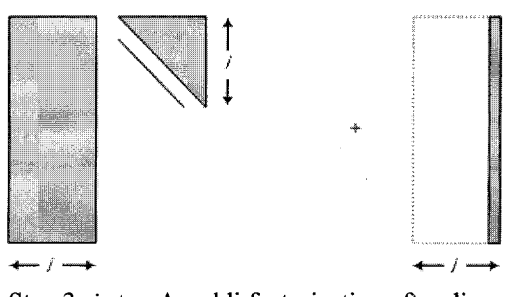

Fig. 3. Step 3: *j*-step Arnoldi factorization after discarding the last columns, $Q_j H_j + v_{mj} r_m e_j^T$.

$AQ_+(:,1:j) = Q_+(:,1:j)H_+(1:j,1:j) + v_{mj}r_c e_m^T$ is a length j Arnoldi factorization. Back to the basic Arnold iteration at step $j+1$ and performing p steps, we can have a new length m Arnoldi factorization.

Fig.1~3 illustrate one cycle of the iteration, to clarify each step of IRAM.

2.5 Shift - Invert Spectral Transformation

The shift-invert spectral transformation with IRAM enhances convergence to a desired portion of the spectrum. If λ is an eigenpair for A and $\sigma \neq \lambda$, then:

$$(A-\sigma)^{-1}x = xv \text{, where } v = 1/(\lambda-\sigma). \quad (16)$$

These transformed eigenvalues of largest magnitude are precisely the eigenvalues that are easily computed by a Krylov method. Once they are determined, it is simple to transform back to the original problem.

$$\lambda_j = \sigma + \frac{1}{v_j} \quad (17)$$

In addition, the complex shift-invert method requires twice the storage requirements of the real shift-invert method.

3. Small-Signal Stability of Multimachine Systems

3.1 Formulation of the State Equations [12]

The linearized model of all machines and its control devices can be expressed in the following form:

$$px_g = A_g x_g + B_g v_{gg} \quad (18)$$

$$i_{gg} = C_g x_g - D_g v_{gg} \quad (19)$$

where:

$$x_g = [x_{g1}^t,\cdots,x_{gn}^t]^T,\quad v_{gg} = [v_{gg1}^t,\cdots,v_{ggn}^t]^T$$
$$v_c = [v_{c1}^t,\cdots,v_{cn}^t]^T,\quad i_{gg} = [i_{gg1}^t,\cdots,i_{ggn}^t]^T.$$

$A_g \sim D_c$ are block diagonal matrices composed of the corresponding device matrices. The interconnecting transmission network is represented by the node equations:

$$\begin{bmatrix} i_{ng} \\ i_{nl} \end{bmatrix} = \begin{bmatrix} Y_{gg} & Y_{gl} \\ Y_{\lg} & Y_{ll} \end{bmatrix} \begin{bmatrix} v_{ng} \\ v_{nl} \end{bmatrix} \quad (20)$$

$$i_{nl} = J_l v_{nl} \quad (21)$$

where:

J_l: nonlinear load bus linearized coefficient

i_{ng}, v_{ng} : voltage and current of generator bus, $i_{ng} = [i_{ng1}^t,\cdots,i_{ngn}^t]^T$, $v_{ng} = [v_{ng1}^t,\cdots,v_{ngn}^t]^T$

i_{nl}, v_{nl} : voltage and current of load bus , $i_{nl} = [i_{nl1}^t,\cdots,i_{nln}^t]^T$, $v_{nl} = [v_{nl1}^t,\cdots,v_{nln}^t]^T$

$Y_{gg} \sim Y_{ll}$: admittance matrix of network

Equating Equation (20), associated with the admittances of the load and generator buses, and Equation (21) we obtain:

$$i_{ng} = [Y_{gg} - Y_{gl}(Y_{ll} - J_l)^{-1}Y_{\lg}]v_{ng} \quad (22)$$

Network equations are written in a synchronously rotating R-I reference frame. For synchronous machines, Park's equations are expressed in local d-q coordinates fixed on the generator rotor. It is necessary to transform the network input variables, such as terminal voltages, into the local d-q coordinate fixed on the generator rotor. The following transformation matrices are used to change the reference frame:

$$i_{dq} = i_{gg} = T_1 i_{ng} + T_2\delta \quad (23)$$

$$v_{ng} = T_3 v_{\theta v} = T_3 v_{dq} = T_3 v_{gg} \quad (24)$$

$$T_1 = \begin{bmatrix} \sin\delta_0 & -\cos\delta_0 \\ \cos\delta_0 & \sin\delta_0 \end{bmatrix} \quad (25)$$

$$T_2 = \begin{bmatrix} I_{R0}\cos\delta_0 + I_{I0}\sin\delta_0 \\ -I_{R0}\sin\delta_0 + I_{I0}\cos\delta_0 \end{bmatrix} \quad (26)$$

$$T_3 = \begin{bmatrix} -V_0\sin\theta_0 & \cos\theta_0 \\ V_0\cos\theta_0 & \sin\theta_0 \end{bmatrix} \quad (27)$$

$$\delta = T_4 x_g$$

The elements of T_4, which are related to generator angles, are equal to unity and the others are zero. Using the transformation matrices above, we obtain the complete system state matrix [12]:

$$px_g = Ax_g, \quad (28)$$

where:

$$A = A_g + B_g(T_1 Y_g T_3 + D_g)^{-1}(C_g - T_2 T_4) \quad (29)$$

$$Y_g = [Y_{gg} - Y_{gl}(Y_{ll} - J_l)^{-1}Y_{\lg}]. \quad (30)$$

3.2 Implementation of IRAM

To apply the shift-invert transformation around a specified point λ_s, the following transformation $A_t = (A - \lambda_s I)^{-1}$ can be used to magnify the eigenvalues of A close to λ_s.

In IRAM, the only operation involving A_t is the matrix-vector multiplication $A_t v_i$, and the solution of the equations is:

$$(A - \lambda_s I) q_i = v_i . \tag{31}$$

By substituting the expression for the state matrix A of a power system, given by Equation (29), in Equation (31), the extended system matrix can be rewritten as:

$$\begin{bmatrix} A_D - \lambda_s I & B_D \\ C_D & -(Y_N + Y_{De}) \end{bmatrix} \begin{bmatrix} q_i \\ u_i \end{bmatrix} = \begin{bmatrix} v_i \\ 0 \end{bmatrix}, \tag{32}$$

where A_g, B_g, and D_g in Equation (31) are replaced by A_D, B_D, and D_D, respectively. In addition, C_D and Y_N are:

$$C_D = \begin{bmatrix} C_g - T_2 T_4 \\ 0 \end{bmatrix} \tag{33}$$

$$Y_N = \begin{bmatrix} T_1 Y_{gg} T_3 & T_1 Y_{gl} \\ Y_{lg} T_3 & Y_{ll} \end{bmatrix}. \tag{34}$$

The solution of Equation (31), to compute q_i, involves three steps:

1. calculate $Y_{De}(\lambda_s) = D_D - C_D(\lambda_s I - A_D)^{-1} B_D$
2. solve for $u_i : (Y_N + Y_{De}(\lambda_s)) u_i = -C_D(\lambda_s I - A_D)^{-1} v_i$
3. calculate: $q_i = (\lambda_s I - A_D)^{-1}(B_D u_i - v_i)$.

Because matrix Y_N has the sparsity structure of a nodal admittance matrix, the method can be applied to very large systems by using sparsity-based techniques for the solution of algebraic equations. The matrix product $C_D(\lambda_s I - A_D)^{-1} B_D$ is block diagonal and each (2 × 2) diagonal block can be obtained from the product $C_D{}^i(\lambda_s I^i - A_D{}^i)^{-1} B_D{}^i$, where $C_D{}^i$, $(\lambda_s I^i - A_D{}^i)$, and $B_D{}^i$ are blocks associated with the i-th system component, such as a generator.

4. Case Study

This section describes the testing of two systems, an 11-bus system and a 39-bus system, with the ARPACK program [6], in which the IRAM algorithm was implemented with Fortran 77, using BLAS and LAPACK in part. To apply IRAM with single precision to the power system for small-signal stability, the PSS tuning program, PWRSTAB [12], and ARPACK program were integrated into one program. However, the eigenvalues of IRAM were calculated using ARPACK and those from the QR method were calculated by using the PWRSTAB program. The two results are compared.

4.1 Two-Area System

An 11-bus system, as shown in Fig. 4, which has four machines equipped with static exciters, is of order 36 with one unstable mode, which is caused by rapid-response exciter systems. The eigenvalues were calculated by ARPACK (slightly modified to handle complex matrices) with every shift point from (0.0, j15.0) to (0.0, j1.0) with a decrement of −1.0. Table 1 shows the comparison of the two results, the QR method, and IRAM. They produced almost identical results.

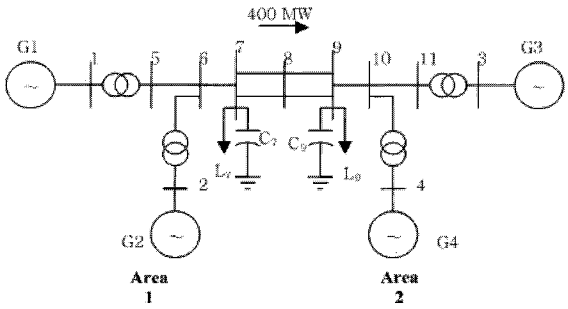

Fig. 4. Two-area system

Table 1. Comparison of the eigenvalue results from the QR method and IRAM

No.	*QR*		*IRAM*	
	Real	***Imag***	***Real***	***Imag***
1	−18.659	±16.458	−18.659	±16.458
2	−19.171	±10.152	−19.171	±10.152
3	−0.466	±7.332	−0.466	±7.332
4	−0.665	±7.162	−0.665	±7.162
5	0.049	±3.867	0.049	±3.867

4.2 England 39-bus system

The England 39-bus system has 150 state variables including 10 machine models, nine exciter models, and nine governor models. Table 2 presents the identical eigenvalue results from the two methods. This indicates that IRAM provides reliable results irrespective of system size. In this paper, the calculation speed of IRAM for a large-scale power system could not be investigated because we did not use a large-scale eigenvalue program that exploits sparsity and, therefore, could not study large-scale power systems; these will be studied in the next research phase.

Table 2. Comparison of the eigenvalue results of QR method and IRAM (150 order system)

No.	*QR*		*IRAM*	
	Real	***Imag***	***Real***	***Imag***
1	−0.283	±7.715	−0.283	±7.715
2	−0.145	±7.612	−0.145	±7.612
3	−0.091	±7.105	−0.091	±7.105
4	−0.196	±6.261	−0.196	±6.261
5	−0.127	±3.987	−0.127	±3.987
6	−0.069	±1.338	−0.069	±1.338

5. Conclusion

This paper describes the implicit restated Arnoldi method algorithm, which is applicable to large-scale power systems, and its application to small-size power systems to observe the salient features of the IRAM algorithm. The ARPACK program was used to apply IRAM with shift and invert spectral transformation to power systems for small-signal stability. The two area 11-bus system with 36 state variables and the England 39-bus system with 150 state variables were tested using IRAM and the eigenvalue results compared with results obtained from the QR method. They show identical eigenvalue outcomes for both systems. Therefore, the research results of this paper indicate that IRAM provides reliable calculation results for the eigenvalues of concern, regardless of system size.

In the ongoing research phase, an efficient sparsity-based eigenvalue algorithm applicable to very large power systems will be developed using the IRAM algorithm.

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