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OSCILLATION AND NONOSCILLATION THEOREMS FOR NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. By means of a Riccati transform some oscillation or nonoscillation criteria are established for nonlinear differential equations of second order

$(E_1)\qquad [p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)]' + q(t)|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t)) = 0.$

(E_2), (E_3) and (E_4) where $0 < \alpha$ and

$$\tau(t) \le t, \quad \tau'(t) > 0, \quad \tau(t) \to \infty \text{ as } t \to \infty.$$

In this paper we improve some previous results.

1. Introduction

Let $p(t)$, $q(t)$ be positive functions for $t \ge a > 0$. In this paper we are concerned with oscillatory and nonoscillatory properties of differential equations of the type

$$(E_1)\qquad \Big(p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + q(t)|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t)) = 0,$$

where $\alpha > 0$. This equation is also written as follows :

$$\Big(p(t)|x'(t)|^{\alpha-1}\, x'(t)\Big)' + q(t)|x(\tau(t))|^{\alpha-1}\, x(\tau(t)) = 0.$$

For our purpose we denote a function $\Phi(t)$ by

$$(H_1)\qquad \Phi(t) = \int_a^t \frac{1}{p(u)^{1/\alpha}}\, du \quad \text{as} \quad t \ge a.$$

Throughout this paper we always assume that $\Phi(t) \to \infty$ as $t \to \infty$.

By a solution of differential equation (E_1) on some interval $[T_0, \infty)$ we mean a continuously differentiable function $x : [T_0, \infty) \to R$, for some T_0, such that $x(t)$ satisfies the differential equation (E_1) for all $t \ge T_0$. A solution $x(t)$ is oscillatory if it is defined on some interval of the type $[t_x, \infty)$, $t_x \ge T_0$ and

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has unbounded zeros. Otherwise it is said to be nonoscillatory. The equation (E_1) is called oscillatory if all solutions of the equation are oscillatory.

The differential equations of second order are very important in application. For the last twenty years many authors dealt with the linear or nonlinear differential equations of second order. So numerous results were obtained [1-15] and the methods of investigation for differential equations of second order is frequently used for investigation of differential equations of higher order.

By means of Picone's formula Jaros and Kusano [6] investigated the oscillatory properties of (E_1) with $\tau(t) = t$. The various forms, according as $p(t)$, $\tau(t)$, and α, of the differential equation (E_1) have been investigated by Dzurina [3], Kusano and Yosida [7], Li and Yeh [9, 10] in view of a Riccati transform. Kusano and Yosida [7] obtained the following :

Theorem. *The differential equation* $\Big(|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + q(t)|x(t)|^\alpha \operatorname{sgn} x(t) = 0$ *is nonoscillatory if and only if there exists a continuous solution of the integral equation*

$$u(t) = \int_t^\infty q(s)\,ds + \alpha \int_t^\infty |u(s)|^{1+1/\alpha}\,ds \tag{1.1}$$

defined in some neighborhood of ∞.

The following is a well known property : if the inequality

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4} \quad \left(\limsup_{t\to\infty} t^2 q(t) < \frac{1}{4}\right) \tag{1.2}$$

is valid where $q(t)$ is real valued and continuous for large $t > 0$, then the equation $x''(t) + q(t)x(t) = 0$ is oscillatory (nonoscillatory) (see Hartman [3]).

In this paper we obtain a sufficient and necessary condition for (E_1) to be nonoscillatory and get more general conditions than (1.2) for the equation (E_1) either to be oscillatory or not to be oscillatory.

2. Main results

We use a notation as follows : for a nonnegative integer n and a function $u > 0$

$$\ln^{[0]} u = u,\ \ln^{[1]} u = \ln u, \ldots, \ln^{[n+1]} u = \ln\big(\ln^{[n]} u\big).$$

In this paper we assume that

(A_1) $0 < \tau(t) \le t$, $\tau'(t) > 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

(A_2) The constant a is chosen so that $\ln^{[n]} \Phi(\tau(a)) = e$ where $n \ge 1$.

Lemma 1. *Let* X, Y *be positive constants and* $a \ge 1$. *Then*

$$(1-a)Y^a \le X^a - aXY^{a-1}, \tag{2.1}$$

where the equality is valid if and only if $X = Y$.

Theorem 1. *Let the assumptions* (A_1), (A_2) *be satisfied.*

(i) *if the equality*

$$\int_a^\infty \left[q(s)\left[\Phi(\tau(s))\right]^\alpha - \frac{\Big(\alpha/(\alpha+1)\Big)^{\alpha+1}\left[\Phi(\tau(s))\right]'}{\Phi(\tau(s))} \right] ds = \infty \tag{2.2}$$

holds, the equation (E_1) *is then oscillatory.*

(ii) *if for each fixed* $n \geq 1$

$$\int_a^\infty q(s)\Big[\ln^{[n]}\Phi(\tau(s))\Big]^\alpha ds = \infty \tag{2.3}$$

holds, the equation (E_1) *is oscillatory.*

Proof. Assume that equation (E_1) is nonoscillatory. Let $x(t)$ be a nonoscillatory solution. We may assume that $x(t) > 0$ eventually. If $x(t) < 0$ eventually, we consider (E_1) with $x(t) = -y(t)$. There exists a $T_0 > a$ such that $x(t) > 0$, $x(\tau(t)) > 0$ for $T_0 \leq t$. It follows that $p(t)x'(t)^\alpha$ is decreasing for $T_0 \leq t$. It is not difficult to show that $x'(t) > 0$ for $T_0 \leq t$. So we have $p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t) > 0$ for $t \geq T_0$. Consider a Riccati transform

$$W(t) = \Big(\ln^{[n]}\Phi(\tau(t))\Big)^\alpha \frac{p(t)\big|x'(t)\big|^\alpha \operatorname{sgn} x'(t)}{\big|x(\tau(t))\big|^\alpha \operatorname{sgn} x(\tau(t))}. \tag{2.4}$$

It follows that $W(t) > 0$ for $T_0 \leq t$. Since $\dfrac{d\Phi(\tau(t))}{dt} = p(\tau(t))^{-1/\alpha}\tau'(t) > 0$ by (A_1), it is obvious that $p(t)x'(t)^\alpha \leq p(\tau(t))x'(\tau(t))^\alpha$ and that

$$\frac{\left[\Phi(\tau(t))\right]' W(t)^{1/\alpha}}{\ln^{[n]}\Phi(\tau(t))} \leq \frac{x'(\tau(t))\tau'(t)}{x(\tau(t))}, \qquad n \geq 0,$$

where $T_0 < t$ and $' = d/dt$. We note that

$$\frac{d}{dt}\ln^{[n]}\Phi(\tau(t)) = \frac{\left[\Phi(\tau(t))\right]'}{\Phi(\tau(t))\cdot\ln^{[1]}\Phi(\tau(t))\cdots\ln^{[n-1]}\Phi(\tau(t))}, \quad n \geq 1. \tag{2.5}$$

Put $F_{\Phi,\tau,n}(t) = \Phi(\tau(t))\cdot\ln^{[1]}\Phi(\tau(t))\cdots\ln^{[n]}\Phi(\tau(t))$. Since

$$\begin{aligned} W'(t) &= \frac{\alpha\left[\Phi(\tau(t))\right]'}{F_{\Phi,\tau,n}(t)}W(t) - q(t)\Big[\ln^{[n]}\Phi(\tau(t))\Big]^\alpha - \alpha W(t)\,\frac{x'(\tau(t))\tau'(t)}{x(\tau(t))} \\ &\leq \frac{\alpha\left[\Phi(\tau(t))\right]'}{F_{\Phi,\tau,n}(t)}W(t) - q(t)\Big[\ln^{[n]}\Phi(\tau(t))\Big]^\alpha - \frac{\alpha\left[\Phi(\tau(t))\right]'}{\ln^{[n]}\Phi(\tau(t))}W(t)^{1+1/\alpha} \end{aligned}$$

is valid for $n \geq 0$, we obtain $W'(t)$ is not greater than

$$\begin{cases} -\dfrac{\alpha\left[\Phi(\tau(t))\right]'}{\Phi(\tau(t)}\left\{W(t)^{1+1/\alpha} - W(t\right\} - q(t)\Phi(\tau(t))^\alpha, & \text{if } n = 0, \\ -\dfrac{\alpha\left[\Phi(\tau(t))\right]'}{\ln^{[n]}\Phi(\tau(t))}\left\{W(t)^{1+1/\alpha} - \dfrac{W(t)}{F_{\Phi,\tau,n-1}(t)}\right\} - q(t)\Big[\ln^{[n]}\Phi(\tau(t))\Big]^\alpha, & \text{if } n \geq 1 \end{cases}$$

By the inequality (2.1) with

$$\begin{cases} a = \dfrac{\alpha+1}{\alpha} > 1, \quad X = W(t), \quad Y = \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha}, & \text{if } n = 0, \\ a = \dfrac{\alpha+1}{\alpha} > 1, \quad X = W(t), \quad Y = \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha} F_{\Phi,\tau,n-1}(t)^{-\alpha}, & \text{if } n \geq 1, \end{cases}$$

we reach

$$\begin{cases} -\dfrac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \leq W(t)^{1+1/\alpha} - W(t), & \text{if } n = 0, \\ -\dfrac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} F_{\Phi,\tau,n-1}(t)^{-(\alpha+1)} \leq W(t)^{1+1/\alpha} - \dfrac{W(t)}{F_{\Phi,\tau,n-1}(t)}, & \text{if } n \geq 1. \end{cases}$$

Therefore it is obvious that

$$W'(t) \leq \begin{cases} \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1} \dfrac{[\Phi(\tau(t))]'}{\Phi(\tau(t))} - q(t)\Phi(\tau(t))^{\alpha}, & \text{if } n = 0, \\ \dfrac{\Big(\alpha/(\alpha+1)\Big)^{\alpha+1} [\Phi(\tau(t))]'}{\ln^{[n]} \Phi(\tau(t)) \big[F_{\Phi,\tau,n-1}(t)\big]^{\alpha+1}} - q(t)\Big[\ln^{[n]} \Phi(\tau(t))\Big]^{\alpha}, & \text{if } n \geq 1. \end{cases}$$

Integrating from a to t, we have then

$$W(t) \leq W(a) - \int_a^t \left[q(s)\Phi(\tau(s))^{\alpha} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{[\Phi(\tau(s))]'}{\Phi(\tau(s)}\right] ds, \tag{2.6}$$

and for $n \geq 1$
(2.7)

$$W(t) \leq W(a) - \int_a^t \left[q(s)\Big[\ln^{[n]} \Phi(\tau(s))\Big]^{\alpha} - \frac{\Big(\alpha/(\alpha+1)\Big)^{\alpha+1} [\Phi(\tau(s))]'}{\ln^{[n]} \Phi(\tau(s)) \big[F_{\Phi,\tau,n-1}(s)\big]^{\alpha+1}}\right] ds.$$

Consequently, $W(t) \leq 0$ for t large enough, which contradicts that $W(t) > 0$. In order to prove (ii) it is enough to show that the integral

$$\int_a^{\infty} \frac{[\Phi(\tau(s))]'}{\ln^{[n]} \Phi(\tau(s)) \Big[\Phi(\tau(s)) \cdot \ln^{[1]} \Phi(\tau(s)) \cdots \ln^{[n-1]} \Phi(\tau(s))\Big]^{\alpha+1}} ds$$

is finite for $1 \leq n$. Because this integral is transformed into

$$\int_{\Phi(\tau(a))}^{\infty} \frac{du}{\ln^{[n]} u \Big[u \cdot \ln^{[1]} u \cdots \ln^{[n-1]} u\Big]^{\alpha+1}}$$

and because the value of the integral is finite, the proof is complete. □

Remark 1. Let the assumption (A_1) be satisfied. If the equation (E_1) is nonoscillatory, there exists a nonoscillatory solution $x(t)$. We may assume that $x(t) > 0$ eventually. As seen in the proof of Theorem 1 there exists a $T_0 > a$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x'(t) > 0$ for $t \geq T_0$. Consider a Riccati transform $U(t) = p(t)x'(t)^\alpha / x(\tau(t))^\alpha$. We obtain then $U(t) > 0$ for $t \geq T_0$ and

$$U'(t) \leq -q(t) - \alpha p(\tau(t))^{-1/\alpha}\tau'(t)U(t)^{1+1/\alpha} < -q(t). \tag{2.8}$$

Thus $U(t) < 0$ eventually if $\int^\infty q(s)\,ds = \infty$. Therefore if the equality

$$\int_a^\infty q(s)\,ds = \infty. \tag{H_2}$$

holds, the equation (E_1) is oscillatory.

Remark 2. Dzurina proved that $\big(p(t)y'(t)\big)' + q(t)y(\tau(t)) = 0$ is oscillatory if $\liminf_{t\to\infty} \dfrac{\Phi(\tau(t))^2 q(t)}{\big[\Phi(\tau(t))\big]'} > 1/4$. The following theorem gives a sufficient condition for (2.2) to be valid.

Theorem 2. *Let the assumption (A_1) be valid. Then if the inequality*

$$\liminf_{t\to\infty} \frac{q(t)\big[\Phi(\tau(t))\big]^{\alpha+1}}{\big[\Phi(\tau(t))\big]'} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{2.9}$$

holds, the equation (E_1) is oscillatory.

Theorem 3. *Let ε be a positive number with $0 < \varepsilon < \alpha$. if the formula*

$$\int_a^\infty \Phi(\tau(s))^{\alpha-\varepsilon}\, q(s)\,ds = \infty \tag{2.10}$$

holds for $0 < \varepsilon < \alpha$, the equation (E_1) is then oscillatory.

Proof. Let $\alpha > \varepsilon > 0$ be valid. From (2.9) there exists a $T_* > a$ such that

$$\frac{\Phi(\tau(t))^{\alpha+1}q(t)}{\big[\Phi(\tau(t))\big]'} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \epsilon, \quad \text{for} \quad t \geq T_*.$$

Put $\delta = \big(\alpha/(\alpha+1)\big)^{\alpha+1} + \epsilon$. Clearly for $t \geq T_*$ we have

$$\Phi(\tau(t))^{\alpha-\varepsilon}q(t)/\big[\Phi(\tau(t))\big]' > \delta\big[\Phi(\tau(t))\big]'/\Phi(\tau(t))^{1+\varepsilon}.$$

It is clear that $\big[\Phi(\tau(t))\big]'/\Phi(\tau(t))^{1+\varepsilon}$ is integrable on $[a, \infty)$. Moreover, we have

$$q(t)\big[\Phi(\tau(t))\big]^\alpha - \frac{\delta\big[\Phi(\tau(t))\big]'}{\Phi(\tau(t))} \geq q(t)\big[\Phi(\tau(t))\big]^{\alpha-\varepsilon} - \frac{\delta\big[\Phi(\tau(t))\big]'}{\Phi(\tau(t))^{1+\varepsilon}}$$

for $0 < \varepsilon < \alpha$ and for sufficiently large t. It is clear that the equality (2.2) is valid if (2.10) is valid . Therefore the equation (E_1) is oscillatory. □

Remark 3. Theorem 3 is a general result of a known sufficient condition

$$\int^\infty s^{1-\varepsilon}q(s)\,ds = \infty, \qquad 0 < \varepsilon < 1 \tag{2.11}$$

for the differential equation (1.3) to be oscillatory. In fact this equality, (2.3) and (2.10) are concerned with (H_2).

Remark 4. Immediately we obtain an extension of Theorem 3. For any $\rho > 0$ and for any integer $n \geq 1$ the inequality $t^\rho > (\ln t)^n$ is valid for sufficiently large t. The differential equation (E_1) is therefore oscillatory if the equality

$$\int_a^\infty \Phi(\tau(s))^{\alpha-\varepsilon}\left[\ln \Phi(\tau(s))\right]^n q(s)\, ds = \infty, \qquad n \geq 1 \tag{2.12}$$

holds for $0 < \varepsilon < \alpha$.

Corollary 1. *Let the assumption* (A_1) *be satisfied.*

$$\liminf_{t\to\infty} \frac{\tau(t)^{1+\alpha} q(t)}{\tau'(t)} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{2.13}$$

is valid. Then the equation

$$\Big(|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + q(t)|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t)) = 0 \tag{E_2}$$

is oscillatory.

For example, if $\liminf_{t\to\infty} \dfrac{q(t)\tau^2(t)}{\tau'(t)} > \dfrac{1}{4}$ holds, then the equation $x''(t) + q(t)x(\tau(t)) = 0$ is oscillatory. It is clear that $\Phi(t) = t$ if $a = 0$ in (H_1). The differential equation $x''(t) + q(t)x(\sqrt{t}) = 0$ is oscillatory if $\int_0^\infty \big(\sqrt{s}\, q(s) - 1/(8s\sqrt{s})\big)\, ds = \infty$ by (i) of Theorem 1, or if $\int^\infty s^{1/2-\varepsilon} q(s)\, ds = \infty$ for $1/2 > \varepsilon > 0$ by Theorem 3.

Put $s = \Phi(t)$. By the change of variables $(t, x) \to (s, Y)$ the differential equation (E_1) is reduced to

$$\Big(|Y'|^\alpha \operatorname{sgn} Y'\Big)' + Q(s)|Y|^\alpha \operatorname{sgn} Y = 0, \qquad s \geq 0$$

where $Q(s) = (p \circ \Phi^{-1})(s)^{1/\alpha}(q \circ \Phi^{-1})(s)$ and $Y(s) = (x \circ \Phi^{-1})(s)$. We need the following lemma which can be obtained from the change of variables given above and Lemma 1 of Kusano and Yosida [7].

Lemma 2. *Let* $F(t, x)$ *be a continuous function on* $[t_0, \infty) \times R$ *which are nondecreasing in* x *and satisfies* $\operatorname{sgn} F(t, x) = \operatorname{sgn} x$ *for each fixed* $t \geq t_0 > 0$. *Let* α *be a positive constant. If the differential inequality*

$$\Big(p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + F(t, x(\tau(t))) \leq 0$$

has an eventually positive solution, then the differential equation

$$\Big(p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + F(t, x(\tau(t))) = 0$$

also has an eventually positive solution.

Example 1. Consider a differential equation

$$(2.14)\qquad \begin{aligned}&\Big(p(t)|x'(t)|^{\alpha}\operatorname{sgn} x'(t)\Big)'\\ &+\delta\Big[\Phi(t)\Big]'\Phi(t)^{-2+\frac{1}{\alpha+1}}\times\Phi(\tau(t))^{-\frac{\alpha^2}{\alpha+1}}|x(\tau(t))|^{\alpha}\operatorname{sgn} x(\tau(t))=0.\end{aligned}$$

We seek a positive solution of the type $x(t) = \Phi(t)^c$. Put $f(c) = \alpha(c-1)|c|^{\alpha} + \delta$. Then it is obvious that $c = \alpha/(\alpha+1)$ and $\delta = \big(\alpha/(\alpha+1)\big)^{\alpha+1}$ are the values satisfying $\min_{c>0} f(c) = 0$. Because that $f(c)$ has no real zero if $\delta > \big(\alpha/(\alpha+1)\big)^{\alpha+1}$ holds, (2.14) is then oscillatory. If $\delta \le \big(\alpha/(\alpha+1)\big)^{\alpha+1}$ holds, $f(c)$ has real zeros. So the given equation then is nonoscillatory. Therefore the delay differential equation (2.14) is nonoscillatory if and only if $\delta \le \big(\alpha/(\alpha+1)\big)^{\alpha+1}$.

Remark 5. In Example 1 we note that $\Phi(\tau(t))/\Phi(t) \le 1$. By L'Hospital's rule it follows that

$$(2.15)\qquad \frac{\Phi(t)^{-1+\frac{1}{\alpha+1}}}{\Phi(\tau(t))^{-1+\frac{1}{\alpha+1}}} \approx \frac{\Phi(t)^{-2+\frac{1}{\alpha+1}}\,[\Phi(t)]'}{\Phi(\tau(t))^{-2+\frac{1}{\alpha+1}}\,[\Phi(\tau(t))]'} \le 1$$

for t large enough. Therefore we obtain

$$(2.16)\qquad \begin{aligned}&\frac{\Phi(\tau(t))^{1+\alpha}}{[\Phi(\tau(t))]'}\cdot\delta\Big[\Phi(t)\Big]'\Phi(t)^{-2+\frac{1}{\alpha+1}}\Phi(\tau(t))^{-\frac{\alpha^2}{\alpha+1}}\\ &\equiv\delta\frac{\Phi(t)^{-2+\frac{1}{\alpha+1}}\,[\Phi(t)]'}{\Phi(\tau(t))^{-2+\frac{1}{\alpha+1}}\,[\Phi(\tau(t))]'} \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\end{aligned}$$

for t large enough if $\delta \le \big(\alpha/(\alpha+1)\big)^{\alpha+1}$. Thus if the inequality

$$(2.17)\qquad q(t) \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\Big[\Phi(t)\Big]'\Phi(t)^{-2+\frac{1}{\alpha+1}}\Phi(\tau(t))^{-\frac{\alpha^2}{\alpha+1}}$$

holds, (E_1) is nonoscillatory by Lemma 2.

Example 2. Consider a differential equation $x''(t)+\dfrac{\sigma(1-\sigma)}{t^{2-\sigma/2}}\,x(\sqrt{t}) = 0$ where $0 < \sigma \le 1$. It is obvious that $x(t) = t^{\sigma}$ is a positive solution for $t > 0$. Because $\tau(t) = \sqrt{t}$, we have $\dfrac{\tau(t)^2\,q(t)}{\tau'(t)} = 2t\sqrt{t}\,q(t) = \dfrac{2\sigma(1-\sigma)}{t^{1/2-\sigma/2}} \le 1/4$ for sufficiently large t.

Theorem 4. *Consider a differential equation*

$$(E_3)\qquad \Big(p(t)|x'(t)|^{\alpha}\operatorname{sgn} x'(t)\Big)' + q(t)|x(t)|^{\alpha}\operatorname{sgn} x(t) = 0.$$

(i) *The differential equation* (E_3) *is oscillatory if for any* $\varepsilon > 0$, *there exists a* $T_\varepsilon > a$ *such that*

$$p(t)^{1/\alpha}\Phi(t)^{1+\alpha}q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon \tag{2.18}$$

is valid for $t > T_\varepsilon$.

(ii) *The differential equation* (E_3) *is nonoscillatory if*

$$p(t)^{1/\alpha}\Phi(t)^{1+\alpha}q(t) \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{2.19}$$

is valid for sufficiently large t.

Proof. We prove (ii). Note that $\Phi'(t) = p(t)^{-1/\alpha}$. Consider a differential equation

$$\Big(p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + \delta\frac{[\Phi(t)]'}{\Phi(t)^{\alpha+1}}|x(t)|^\alpha \operatorname{sgn} x(t) = 0. \tag{2.20}$$

By direct calculation we can show that this equation has a positive solution $x(t) = \Phi(t)^{\alpha/(1+\alpha)}$ if $\delta = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$ (see [9]). So (2.20) is nonoscillatory. If thus (2.19) is valid, (E_3) is nonoscillatory by Lemma 2. We note that (2.20) is nonoscillatory if and only if $\delta \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. □

Dzurina [3] proved that $\int^\infty \left(sq(s) - \frac{1}{4s}\right) ds = \infty$ is one of the criteria for oscillation of the equation $(p(t)x'(t))' + q(t)x(t) = 0$. This is the case of (2.2) with $n = 0$, $\alpha = 1$, and $\tau(t) = t$.

Theorem 5. *Under the condition* (A_1) *we assume that* $p(t) \ge p(\tau(t))\big(\tau'(t)\big)^{-\alpha}$ *and that* (2.9) *are valid for sufficiently large* t. *Then the equation* (E_3) *is oscillatory.*

Proof. Note that (E_1) is oscillatory if the condition (2.9) is valid. Because

$$\Phi(t) \ge \Phi(\tau(t)), \quad \Phi'(t) = p(t)^{-1/\alpha}, \quad [\Phi(\tau(t))]' = p(\tau(t))^{-1/\alpha}\tau'(t) \tag{2.21}$$

we obtain $\dfrac{\Phi(t)^{\alpha+1}}{\Phi'(t)} \ge \dfrac{\Phi(\tau(t))^{\alpha+1}}{[\Phi(\tau(t))]'}$ from which, combined with (2.18), (E_3) is oscillatory. □

Corollary 2. *Under the assumption* (A_1) *if the equation* (E_3) *is nonoscillatory, the equation* (E_1) *is nonoscillatory.*

Proof. Assume that (E_3) is nonoscillatory. Then a nonoscillatory solution $x(t)$ of (E_3) exists. We may suppose that there is a T, $T \ge a$ such that $x(t) > 0$, $x(\tau(t)) > 0$ are valid for $t \ge T$(if $x(t) < 0$ eventually, is a nonoscillatory solution we put $x(t) = -y(t)$ and proceed the same process). It is not difficult to show that $x'(t) > 0$ for $t \geq T$. So we have $x(\tau(t)) \leq x(t)$ if $T \leq \tau(t)$. On the other hand, it is obvious that $x(t)$ is a positive solution of the differential equation

$$\Big(p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t)\Big)' + q(t)\frac{|x(t)|^\alpha \operatorname{sgn} x(t)}{|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t))}|y(\tau(t))|^\alpha \operatorname{sgn} y(\tau(t)) = 0.$$

It follows that $1 \leq \dfrac{|x(t)|^\alpha \operatorname{sgn} x(t)}{|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t))}$ for t large enough because a function $k(u) = |u|^\alpha \operatorname{sgn} u$ is increasing. By Lemma 2 the differential equation

$$\Big(p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t)\Big)' + q(t)|y(\tau(t))|^\alpha \operatorname{sgn} y(\tau(t)) = 0.$$

is nonoscillatory. □

Corollary 3. *Let (A_2) be satisfied. For an arbitrary nonnegative integer n, we assume that the equality*

$$\int_a^\infty q(s)\Big[\ln^{[n]}\Phi(s)\Big]^\alpha ds = \infty$$

is valid. Then the equation (E_3) is oscillatory.

Remark 6. From Theorem 2 with $n = 2$, $\alpha = 1$ it is clear that the equation $x''(t) + q(t)x(t) = 0$ is oscillatory if

$$\int^\infty q(s)\ln(\ln s)\,ds = \infty$$

is valid.

Corollary 4. *Consider a differential equation*

$$(E_4) \qquad \Big(|x'(t)|^\alpha \operatorname{sgn} x'(t)\Big)' + q(t)|x(t)|^\alpha \operatorname{sgn} x(t) = 0.$$

(i) *The differential equation (E_4) is oscillatory if for any $\varepsilon > 0$, there exists a $T_\varepsilon > a$ such that*

$$t^{1+\alpha}q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon \tag{2.22}$$

is valid for $t > T_\varepsilon$.

(ii) *The differential equation (E_4) is nonoscillatory if*

$$t^{1+\alpha}q(t) \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{2.23}$$

is valid for sufficiently large t.

Consider the differential equation $x''(t) + q(t)x(t) = 0$. Then it is clear that $\Phi(t) = t$. If we apply Corollary 4 to this differential equation with $n = 1$ and $\alpha = 1$ we obtain (1.2).

Remark 7. In the case of $\tau(t) = t \pm \lambda$, $\lambda > 0$ Li and Yeh [10] proved that $\left(|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\right)' + q(t)|x(t-\lambda)|^{\alpha} \operatorname{sgn} x(t-\lambda) = 0$ is nonoscillatory if and only if the differential equation (E_4) is nonoscillatory if and only if

$$\left(|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\right)' + q(t)|x(t+\lambda)|^{\alpha} \operatorname{sgn} x(t+\lambda) = 0$$

is nonoscillatory.

Theorem 6. *Assume that (A_1) is satisfied and that $\int_a^\infty q(s)\,ds < \infty$ holds. The differential equation (E_1) is nonoscillatory if and only if there exists a continuously differentiable function of the integral equation*

$$\begin{aligned} U(t) = &\int_t^\infty q(s)\left[\exp - \int_{\tau(s)}^s p(u)^{-1/\alpha}\left|U(u)\right|^{1/\alpha}\operatorname{sgn} U(u)\,du\right]^\alpha ds \\ &+ \alpha\int_t^\infty p(s)^{-1/\alpha}\left|U(s)\right|^{1+1/\alpha} ds. \end{aligned} \tag{2.24}$$

Proof. Consider a Riccati transform

$$V(t) = \frac{p(t)\left|x'(t)\right|^{\alpha}\operatorname{sgn} x'(t)}{\left|x(t)\right|^{\alpha}\operatorname{sgn} x(t)} \tag{2.25}$$

of the differential equation (E_1). Then we note that

$$\frac{x'(t)}{x(t)} = p(t)^{-1/\alpha}\left|V(t)\right|^{1/\alpha}\operatorname{sgn} V(t).$$

It follows that

$$\begin{aligned} V'(t) = &-q(t)\left|\frac{x(\tau(t))}{x(t)}\right|^{\alpha} - \alpha p(t)^{-1/\alpha}\left|V(t)\right|^{1+1/\alpha} \\ = &-q(t) \times \left[\exp - \int_{\tau(t)}^t \alpha p(u)^{-1/\alpha}\left|V(u)\right|^{1/\alpha}\operatorname{sgn} V(u)\,du\right]^\alpha \\ &- \alpha p(t)^{-1/\alpha}\left|V(t)\right|^{1+1/\alpha} \end{aligned} \tag{2.26}$$

is valid for sufficiently large t. Assume that there exists a continuously differentiable function $U(t)$ satisfying (2.24). Differentiating both sides of (2.24) we obtain the fact that $U(t)$ is a solution of (2.26). Then

$$x(t) = x(t_0)\exp\left[\int_{t_0}^t p(u)^{-1/\alpha}\left|U(u)\right|^{1/\alpha}\operatorname{sgn} U(u)\right] du$$

is a nonoscillatory solution of (E_1). In order to prove the converse let $x(t)$ be a nonoscillatory solution of (E_1). Without loss of generality we may assume that $x(t) > 0$ eventually. If $x(t) < 0$ eventually we put $x(t) = -y(t)$. Because then $V(t)$, given by (2.25), is positive eventually it follows that

$$\int_t^\infty q(s)\left[\exp - \int_{\tau(s)}^s p(u)^{-1/\alpha}\left|V(u)\right|^{1/\alpha}\operatorname{sgn} V(u)\,du\right]^\alpha ds < \infty$$

and that $\frac{1}{\alpha}V(t)^{-(1+1/\alpha)}V'(t)+p(t)^{-1/\alpha} \le 0$ for sufficiently large t, from which we obtain $V(t)\Phi(t)^\alpha \le 1$. Consequently we get $V(t) \to 0$ as $t \to \infty$. From this fact we can easily show that $\int_t^\infty p(s)^{-1/\alpha}\big|V(s)\big|^{1+1/\alpha}\,ds < \infty$. □

Remark 8. If $\liminf_{t\to\infty} p(t)^{-1/\alpha}\big|U(t)\big|^{1+1/\alpha} > 0$ (or $\limsup_{t\to\infty} p(t)^{-1/\alpha}\big|U(t)\big|^{1+1/\alpha} < 0$) and $\lim_{t\to\infty}\{t-\tau(t)\} = \infty$, it is clear that

$$\lim_{t\to\infty}\int_{\tau(t)}^t p(s)^{-1/\alpha}\big|U(s)\big|^{1/\alpha}\,ds = \infty \text{ (or } -\infty)$$

is valid.

Remark 9. Theorem 6 implies that the differential equation (E_1) has a positive solution $x(t)$ if and only if there is a positive function $U(t)$ satisfying (2.24). On the other hand, if $\tau(t) = t$ and $p(t) = 1$, we obtain Theorem in Introduction.

Theorem 7. *Assume that (A_1) is satisfied and that $\int_a^\infty q(s)\,ds < \infty$ holds. If the differential equation (E_3) is nonoscillatory and if*

$$\int_t^\infty Q_1(s)\,ds \le \int_t^\infty q(s)\,ds \tag{2.27}$$

is valid where $Q_1 \in C\big([a,\infty)\big)$ is a positive function , then the differential equation

$$[p(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)]' + Q_1(t)|x(\tau(t))|^\alpha \operatorname{sgn} x(\tau(t)) = 0. \tag{E_1'}$$

is also nonoscillatory.

Proof. Because the differential equation (E_3) is nonoscillatory there is a positive function $W(t)$ satisfying

$$W(t) = \int_t^\infty q(s)\,ds + \alpha\int_t^\infty p(s)^{-1/\alpha}\big|W(s)\big|^{1+1/\alpha}\,ds \tag{2.28}$$

for sufficiently large t by Theorem in Introduction. We denote a set Γ of functions and a mapping F by

$$\Gamma = \{y \in C\big([t_0,\infty)\big) \mid 0 \le y(t) \le W(t)\} \quad \text{for} \quad t \ge t_0, \tag{2.29}$$

where t_0 is chosen so large that (2.28) is valid

$$\begin{aligned}(Fy)(t) = &\int_t^\infty Q_1(s)\left[\exp - \int_{\tau(s)}^s p(u)^{-1/\alpha}\big|y(u)\big|^{1/\alpha} \operatorname{sgn} y(u)\,du\right]^\alpha ds \\ &+ \alpha\int_t^\infty p(s)^{-1/\alpha}\big|y(s)\big|^{1+1/\alpha}\,ds.\end{aligned} \tag{2.30}$$

Clearly the set Γ is a nonempty, closed, bounded, convex subset of B-space $C\big([t_0,\infty)\big)$ with topology of the uniform convergence of functions on every compact subinterval of $[t_0,\infty)$. In view of (2.29) it is obvious that F maps from Γ to Γ and that $F(\Gamma)$ is relatively compact subset of $C\big([t_0,\infty)\big)$. Also if we take account of (2.27), (2.28) and (2.29), F is continuous by Lebesgue dominated convergence theorem. Thus there exists a function $U(t) \in \Gamma$ such that $(FU)(t) = U(t)$ by Schauder fixed point theorem. It is obvious that $U(t)$ is a nonnegative solution satisfying

$$\begin{aligned} U(t) = &\int_t^\infty Q_1(s)\left[\exp - \int_{\tau(s)}^s p(u)^{-1/\alpha}\left|U(u)\right|^{1/\alpha} \operatorname{sgn} U(u)\, du\right]^\alpha ds \\ &+ \alpha \int_t^\infty p(s)^{-1/\alpha}\left|U(s)\right|^{1+1/\alpha} ds. \end{aligned} \tag{2.31}$$

Consequently, the differential equation (E_1') is nonoscillatory by Theorem 6. □

Theorem 8. *Let $\int^\infty q(s)\,ds < \infty$. Suppose that*

$$\liminf_{t\to\infty} \left[\Phi(\tau(t))\right]^\alpha \int_t^\infty q(s)\,ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{2.32}$$

is valid. Then the equation (E_1) is oscillatory.

Proof. It is sufficient to show that (2.32) implies (2.2). Assume that (2.2) is not valid. Then for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that for all $t \geq T_\varepsilon$,

$$\int_t^\infty \left[q(s)\left[\Phi(\tau(s))\right]^\alpha - \frac{\left(\alpha/(\alpha+1)\right)^{\alpha+1}\left[\Phi(\tau(s))\right]'}{\Phi(\tau(s))}\right] ds < \varepsilon.$$

So we have

$$\int_t^\infty \left[\Phi(\tau(s))\right]^\alpha\left[q(s) - \frac{\left(\alpha/(\alpha+1)\right)^{\alpha+1}\left[\Phi(\tau(s))\right]'}{\left[\Phi(\tau(s))\right]^{\alpha+1}}\right] ds < \varepsilon.$$

We note that $\Phi(t)$ is increasing. Thus the left side above is not less than

$$\left[\Phi(\tau(t))\right]^\alpha \int_t^\infty \left[q(s) - \frac{\left(\alpha/(\alpha+1)\right)^{\alpha+1}\left[\Phi(\tau(s))\right]'}{\left[\Phi(\tau(s))\right]^{\alpha+1}}\right] ds.$$

On the other hand, it is obvious that

$$\int_t^\infty \frac{\left[\Phi(\tau(s))\right]'}{\left[\Phi(\tau(s))\right]^{\alpha+1}}\, ds = \frac{1}{\alpha}\left[\Phi(\tau(t))\right]^{-\alpha}.$$

Consequently for any $\varepsilon > 0$, we obtain

$$\left[\Phi(\tau(t))\right]^\alpha \int_t^\infty q(s)\,ds < \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} + \varepsilon$$

which contradicts (2.32). □

Remark 10. Theorem 8 implies Li's result [9] if $\tau(t) = t$.

Remark 11. In Example 1 the differential equation (E_1) with

$$q(t) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left[\Phi(t)\right]' \Phi(t)^{-2+\frac{1}{\alpha+1}} \Phi(\tau(t))^{-\frac{\alpha^2}{\alpha+1}}$$

is nonoscillatory. By (2.16) it is obvious that

$$\left[\Phi(t)\right]' \Phi(t)^{-2+\frac{1}{\alpha+1}} \Phi(\tau(t))^{-\frac{\alpha^2}{\alpha+1}} \le \frac{\left[\Phi(\tau(t))\right]'}{\Phi(\tau(t))^{1+\alpha}}.$$

for t large enough. Thus we obtain

$$\int_t^{\infty} q(s)\, ds \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \Phi(\tau(t))^{-\alpha}.$$

Example 3. We seek a positive solution $x(t) = (\ln t)^c$ of a differential equation $(t\, x'(t))' + \dfrac{\delta}{t(\ln t)^2} x(t) = 0$ We obtain an equation $f(c) = c(c-1) + \delta$. Then it is obvious that $f(c) \ge f(1/2) = -1/4 + \delta$ for all $c > 0$. Therefore the differential equation is nonoscillatory if and only if $\delta \le 1/4$. By Theorem 4 it is obvious that $(t\, x'(t))' + q(t)\, x(t) = 0$ is nonoscillatory if $t(\ln t)^2 q(t) \le 1/4$. Note that this condition implies (2.19) with $n = 0$ and $\Phi(t) = \ln t$

Example 4. Let $v > 0$. Consider a delay differential equation
(2.33)

$$\left(\frac{1}{t^\alpha}|x'(t)|^{\alpha-1} x'(t)\right)' + \delta \cdot 2^{\alpha+1} \frac{t^{-3+2/(\alpha+1)}}{(t-v)^{2\alpha-2+2/(\alpha+1)}} |x(t-v)|^{\alpha-1} x(t-v) = 0,$$

where $t > v$. It follows that $\Phi(t) = t^2/2$ if we put $a = 0$. Because $q(t) = \delta \cdot 2^{\alpha+1} \dfrac{t^{-3+2/(\alpha+1)}}{(t-v)^{2\alpha-2+2/(\alpha+1)}}$, it is easily checked that $x(t) = \left(t^2/2\right)^{\alpha/(\alpha+1)}$ is a nonoscillatory solution of (2.33) if $\delta = \left(\alpha/(\alpha+1)\right)^{\alpha+1}$. Moreover, (2.33) is nonoscillatory if and only if $\delta \le \left(\alpha/(\alpha+1)\right)^{\alpha+1}$

Example 5. Consider a differential equation

$$(2.34) \qquad \left(t^{-\beta}|x'(t)|^\alpha \operatorname{sgn} x'(t)\right)' + \frac{\alpha\delta}{t^{\alpha+1}} \left|x\left(t^{\frac{\alpha}{\alpha+\beta}}\right)\right|^\alpha \operatorname{sgn} x\left(t^{\frac{\alpha}{\alpha+\beta}}\right) = 0,$$

where $\beta > 0$. It is clear that $\Phi(t) = \frac{\alpha}{\alpha+\beta} t^{\frac{\alpha+\beta}{\alpha}}$ if we put $a = 0$. Setting $q(t) = \alpha\delta/t^{\alpha+1}$ we obtain

$$\left[\Phi(\tau(t))\right]^\alpha \int_t^\infty q(s)\, ds = \delta\left(\frac{\alpha}{\alpha+\beta}\right)^\alpha.$$

So the differential equation (2.34) is nonoscillatory if $\delta \ge \dfrac{(\alpha+\beta)^\alpha}{(\alpha+1)^{\alpha+1}}$.

Example 6. Let $a = 0$ and $0 < m \le 1$. Consider for $t \ge 1$ the following delay differential equation

$$(2.35) \qquad \left(|x'(t)|^{\alpha-1} x'(t)\right)' + K t^{m-2-m\alpha+(1-m)/(\alpha+1)} |x(t^m)|^{\alpha-1} x(t^m) = 0.$$

It is not difficult to show that $x(t) = t^{\alpha/(\alpha+1)}$ is a positive solution of (2.35) if $K = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. Thus (2.35) is oscillatory if and only if $K \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. On the other hand, we put $\Phi(t) = t$, $\tau(t) = t^m$ and $n = 0$ in Theorem 2. Then

$$\frac{\Phi(\tau(t))^{\alpha+1} q(t)}{\left[\Phi(\tau(t))\right]'} = \frac{K}{m} t^{\alpha(m-1)/(\alpha+1)} \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

is valid for sufficiently large t.

Example 7. Let $a = e$ and $0 < m \le 1$. Consider for $t \ge 1$ the following delay differential equation

$$\left(t^\alpha |x'(t)|^{\alpha-1} x'(t)\right)' + \frac{\alpha K}{m^\alpha t (\ln t)^{\alpha+1}} |x(t^m)|^{\alpha-1} x(t^m) = 0, \tag{2.36}$$

where K is constant. It is clear that $\Phi(t) = \ln t$. Thus we get

$$\liminf_{t\to\infty} \Phi(\tau(t))^\alpha \int_t^\infty q(s)\, ds = \limsup_{t\to\infty} \Phi(\tau(t))^\alpha \int_t^\infty q(s)\, ds = K.$$

The differential equation (2.36) is oscillatory by Theorem 8 if $K > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$. Note that the differential equation

$$\left(t^\alpha |x'(t)|^{\alpha-1} x'(t)\right)' + Q(t) |x(t^m)|^{\alpha-1} x(t^m) = 0 \tag{2.37}$$

has a positive solution $x(t) = \left(\ln t\right)^{\alpha/(\alpha+1)}$ where

$$Q(t) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{m^{-\alpha^2/(\alpha+1)}}{t\,(\ln t)^{\alpha+1}}.$$

So (2.37) is nonoscillatory. It is not difficult to show that

$$\lim_{t\to\infty} \Phi(\tau(t))^\alpha \int_t^\infty Q(s)\, ds = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} m^{\alpha/(\alpha+1)} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}.$$

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