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A DIFFERENCE EQUATION FOR MULTIPLE KRAVCHUK POLYNOMIALS

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ABSTRACT. Let $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ be a multiple Kravchuk polynomial with respect to r discrete Kravchuk weights. We first find a lowering operator for multiple Kravchuk polynomials $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ in which the orthogonalizing weights are not involved. Combining the lowering operator and the raising operator by Rodrigues' formula, we find a $(r+1)$-th order difference equation which has the multiple Kravchuk polynomials $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ as solutions. Lastly we give an explicit difference equation for $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ for the case of $r=2$.

1. Introduction

The Kravchuk polynomial $\{K_n^{(p;N)}(x)\}$ $(0<p<1,\ N\in\mathbb{N})$ is a sequence of polynomials such that $\deg(K_n^{(p;N)})=n$ and

$$\sum_{k=0}^{N}\binom{N}{k}K_n^{(p;N)}(k)(-k)_j p^k(1-p)^{N-k}=\begin{cases}0 & \text{if}\quad j=0,1,\dots,n-1\\ \neq 0 & \text{otherwise,}\end{cases} \tag{1.1}$$

where $(a)_j=(a)(a+1)\cdots(a+j-1)$ is the Pochhammer symbol with $(a)_0=1$. With the discrete measure μ given by

$$\mu(x)=\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}\delta(x-k),$$

where $\delta(x)$ is the Dirac delta function, we may express the summation in (1.1) as an integration

$$\int_{-\infty}^{\infty}K_n^{(p,N)}(x)(-x)_j d\mu(x)=\sum_{k=0}^{N}\binom{N}{k}K_n^{(p,N)}(k)(-k)_j p^k(1-p)^{N-k}$$

so that (1.1) actually means the orthogonality of the Kravchuk polynomials.

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The Kravchuk polynomial was first introduced in [11] and after that it has been treated by many authors. We refer, in particular, to [13, 16] for explicit formula and many properties of Kravchuk polynomials. One of the most important properties is that each $K_n^{(p;N)}$ satisfies a second order difference equation

$$x\Delta\nabla y + \left(\frac{Np}{1-p} - \frac{x}{1-p}\right)\Delta y = -\frac{n}{1-p}y,$$

where Δ and ∇ are the forward and the backward difference operators defined by

$$\Delta f(x) = f(x+1) - f(x)$$

and

$$\nabla f(x) = f(x) - f(x-1).$$

The multiple orthogonal polynomials were developed in Padé approximation for a long time ago. See [1, 7, 8, 14, 19] and references therein. Recently many papers have been published in multiple orthogonal polynomials as a generalization of ordinary orthogonal polynomials ([2]-[6], [9]-[10], [12], [15], [17]-[18]). In [5], the authors investigated the properties of several multiple orthogonal polynomials of discrete variables by extending the classical orthogonal polynomials of discrete variables. In particular, they defined the multiple Kravchuk polynomials as follows:

Let $r \in \mathbb{N}$, $\vec{p} = (p_1, p_2, \ldots, p_r)$, and $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index with $|n| = n_1 + n_2 + \cdots + n_r$. Consider r discrete Kravchuk measures μ_i given by

$$\mu_i(x) = \sum_{k=0}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} \delta(x-k), \qquad i = 1, 2, \ldots, r.$$

It is well known that if $0 < p_i < 1$ and $p_i \neq p_j$ for $i \neq j$, then there exists a multi-indexed sequence $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ of polynomials, called a multiple Kravchuk polynomial, such that

$$\deg(K_{\vec{n}}^{(\vec{p};N)}) = |n|$$

and

$$\int K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_k d\mu_i(x) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, 2, \ldots, r.$$

In this case, $K_{\vec{n}}^{(\vec{p};N)}$ can be written by

$$K_{\vec{n}}^{(\vec{p};N)}(x) = \prod_{k=1}^{r} p_k^{n_k} (-N)_{|\vec{n}|} \frac{\Gamma(x+1)\Gamma(N-x+1)}{N!}$$

$$\times \prod_{i=1}^{r} \left[\left(\frac{1-p_i}{p_i}\right)^x \nabla^{n_i} \left(\frac{p_i}{1-p_i}\right)^x\right] \frac{(N-|\vec{n}|)!}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)},$$

where $\Gamma(x)$ is the gamma function. In particular, if $r = 2$, then

$$K_{(n_1,n_2)}^{(p_1,p_2;N)}(x)$$
$$= p_1^{n_1} p_2^{n_2} (-N)_{n_1+n_2} \sum_{j=0}^{n_1+n_2} \sum_{k=0}^{j} \frac{(-n_1)_k}{k!} \frac{(-n_2)_{j-k}}{(j-k)!} \left(\frac{1}{p_1}\right)^k \left(\frac{1}{p_2}\right)^{j-k} \frac{(-x)_j}{(-N)_j}.$$

When $r = 2$, Van Assche [18] found difference equations having the multiple Charlier and the multiple Meixner polynomials as solutions. But until now there is no difference equation to have multiple Kravchuk's polynomials and multiple Hahn's polynomials as solutions even though they are in the same family of discrete classical multiple orthogonal polynomials.

In this paper, we first find a lowering operator for $K_{\vec{n}}^{(\vec{p};N)}$ in which the orthogonalizing weights are not involved. Combining the lowering operator and the raising operator by Rodrigues' formula, we find a $(r+1)$-th order difference equation which has the multiple Kravchuk polynomial $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ as solutions. Lastly we give an explicit difference equation for $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ for the case of $r = 2$.

2. Main results

For multiple Kravchuk polynomials $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$, we define weights by

$$W_{p_i}^{(N)}(x) = \begin{cases} \frac{N! p_i^x (1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N-x+1)} & \text{if} \quad x = 0, 1, \ldots, N \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for $i = 1, 2, \ldots, r$. Throughout the paper we assume that $0 < p_i < 1$ and $p_i \neq p_j$ for $i \neq j$. Then for any polynomial ϕ and ψ,

$$\int \phi(x)\psi(x) d\mu_i(x) = \sum_{x=0}^{N} \phi(x)\psi(x) W_{p_i}^{(N)}(x).$$

By the summation by parts with the properties $W_{p_i}^{(N)}(-1) = W_{p_i}^{(N)}(N+1) = 0$, it is easy to see that

$$\sum_{x=0}^{N} \Delta\phi(x)\psi(x) W_{p_i}^{(N)}(x) = -\sum_{x=0}^{N} \phi(x) \nabla[\psi(x) W_{p_i}^{(N)}(x)].$$

Let $L_p^{(N)}[\cdot]$ be a difference operator defined by

$$L_p^{(N)}[y] = (1-p)x\nabla y + [(N+1)p - x]y. \tag{2.2}$$

From the Rodrigues formula (see [5])

$$\frac{(N+1)p_i(1-p_i)}{W_{p_i}^{(N+1)}(x)}\nabla\left(W_{p_i}^{(N)}(x)K_{\vec{n}}^{(\vec{p};N)}(x)\right)$$

$$= -K_{\vec{n}+e_i}^{(\vec{p};N+1)}(x), \quad i=1,2,\dots,r,$$

where $e_i = (0,0,\dots,1,\dots,0)$ is the i-th standard unit vector, we obtain a raising operator

$$L_{p_i}^{(N)}[K_{\vec{n}}^{(\vec{p};N)}] = -K_{\vec{n}+e_i}^{(\vec{p};N+1)}, \quad i=1,2,\dots,r.$$

Then the formal adjoint $L^*_{(p,N)}[\cdot]$ of $L_p^{(N)}[\cdot]$ is given by

$$L^*_{(p,N)}[y] = -(1-p)\Delta xy + [(N+1)p - x]y$$

and it can be easily seen that for any polynomial ϕ and ψ,

$$\sum_{x=0}^{N} L_p^{(N)}[\phi](x)\psi(x)W_p^{(N)}(x) = \sum_{x=0}^{N}\phi(x)L^*_{(p,N)}[\psi W_p^{(N)}](x).$$

Now we need a lemma which can be proved by simple calculations.

Lemma 2.1. *Let $W_{p_i}^{(N)}$ be the weights for Kravchuk polynomials as in* (2.1). *Then we have*

(a) $\Delta(-x)_k = -k(-x)_{k-1}$;
(b) $\Delta[x(-x)_{k-1}] = k(-x)_{k-1} - (k-1)^2(-x)_{k-2}$;
(c) $(-x-1)_{k-1} = (-x)_{k-1} - (k-1)(-x)_{k-2}$;
(d) $W_{p_i}^{(N)}(x-1) = \frac{x}{(N+1)p_i}W_{p_i}^{(N+1)}(x)$;
(e) $\nabla W_{p_i}^{(N)}(x) = \left(\frac{1}{1-p_i} - \frac{x}{(N+1)p_i(1-p_i)}\right)W_{p_i}^{(N+1)}(x)$;
(f) $W_{p_i}^{(N-1)}(x) = \frac{N-x}{N(1-p_i)}W_{p_i}^{(N)}(x)$;
(g) $\nabla[(-x)_{k-1}W_{p_i}^{(N-1)}(x)]$

$$= \left(\frac{Np_i - x}{Np_i(1-p_i)}(-x)_{k-1} - \frac{(k-1)x}{Np_i}(-x+1)_{k-2}\right)W_{p_i}^{(N)}(x);$$

(h) $\Delta[x(-x)_{k-1}W_{p_i}^{(N-1)}(x)] = \frac{(N-k)p_i - x}{1-p_i}(-x)_{k-1}W_{p_i}^{(N-1)}(x)$

$$-\frac{(k-1)(N-k+1)p_i}{1-p_i}(-x)_{k-2}W_{p_i}^{(N-1)}(x).$$

Proof. We prove only (a), (g) and (h) because all the others can be proved similarly. From the definition of the forward difference operator, we have

$$\begin{aligned}\Delta(-x)_k &= (-x-1)_k - (-x)_k \\ &= (-x-1)(-x)(-x+1)\cdots(-x+k-2) \\ &\quad - (-x)(-x+1)\cdots(-x+k-1) \\ &= [(-x-1)-(-x+k-1)](-x)(-x+1)\cdots(-x+k-2) \\ &= -k(-x)_{k-1}\end{aligned}$$

which implies (a).

The assertion (g) follows from

$$\begin{aligned}&\nabla[(-x)_{k-1}W_{p_i}^{(N-1)}(x)] \\ &=(-x+1)_{k-1}\nabla W_{p_i}^{(N-1)}(x) + \nabla(-x)_{k-1}W_{p_i}^{(N-1)}(x) \\ &=\left\{\left((-x)_{k-1}+(k-1)(-x+1)_{k-2}\right)\frac{Np_i-x}{Np_i(1-p_i)}\right. \\ &\quad \left.-(k-1)(-x+1)_{k-2}\frac{N-x}{N(1-p_i)}\right\}W_{p_i}^{(N)}(x) \\ &=\left(\frac{Np_i-x}{Np_i(1-p_i)}(-x)_{k-1}-\frac{(k-1)x}{Np_i}(-x+1)_{k-2}\right)W_{p_i}^{(N)}(x).\end{aligned}$$

From the definition of the forward difference operator, we have

$$\begin{aligned}\Delta[x(-x)_{k-1}W_{p_i}^{(N-1)}(x)] &= \Delta[x(-x)_{k-1}]W_{p_i}^{(N-1)}(x) \\ &\quad + (x+1)(-x-1)_{k-1}\Delta W_{p_i}^{(N-1)}(x).\end{aligned}$$

Since

$$\begin{aligned}(x+1)\Delta W_{p_i}^{(N-1)}(x) &= (x+1)W_{p_i}^{(N-1)}(x+1) - (x+1)W_{p_i}^{(N-1)}(x) \\ &= \left(\frac{p_i(N-1-x)}{1-p_i} - (1+x)\right)W_{p_i}^{(N-1)}(x) \\ &= \frac{Np_i-1-x}{1-p_i}W_{p_i}^{(N-1)}(x),\end{aligned}$$

we have by Lemma 2.1 (b) and (c),

$$\begin{aligned}
&\Delta[x(-x)_{k-1}W_{p_i}^{(N-1)}(x)]\\
&= [k(-x)_{k-1} - (k-1)^2(-x)_{k-2}]W_{p_i}^{(N-1)}(x)\\
&\quad + \frac{Np_i - 1 - x}{1-p_i}(-x-1)_{k-1}W_{p_i}^{(N-1)}(x)\\
&= \left(k + \frac{Np_i - 1 - x}{1-p_i}\right)(-x)_{k-1}W_{p_i}^{(N-1)}(x)\\
&\quad - (k-1)\left(k - 1 + \frac{Np_i - 1 - x}{1-p_i}\right)(-x)_{k-2}W_{p_i}^{(N-1)}(x).
\end{aligned} \tag{2.3}$$

Substituting the relation $x(-x)_{k-2} = -(-x)_{k-1} + (k-2)(-x)_{k-2}$ into the equation (2.3), we prove (h). □

Theorem 2.2. *For the multiple Kravchuk polynomial* $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ *we have*

$$\Delta K_{\vec{n}}^{(\vec{p};N)}(x) = \sum_{i=1}^{r} n_i K_{\vec{n}-e_i}^{(\vec{p};N-1)}(x). \tag{2.4}$$

Proof. By Lemma 2.1 (g), we have for $i = 1, 2, \ldots, r$,

$$\begin{aligned}
\sum_{x=0}^{N} \Delta K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_k W_{p_i}^{(N-1)}(x) &= -\sum_{x=0}^{N} K_{n}^{(\vec{p};N)}(x)\nabla[(-x)_k W_{p_i}^{(N-1)}(x)]\\
&= -\sum_{x=0}^{N} K_{n}^{(\vec{p};N)}(x)\psi(x)W_{p_i}^{(N)}(x),
\end{aligned}$$

where

$$\psi(x) = \frac{Np_i - x}{Np_i(1-p_i)}(-x)_k - \frac{kx}{Np_i}(-x+1)_{k-1}$$

is a polynomial of degree $\le k+1$. Hence, we obtain by the orthogonality

$$\sum_{x=0}^{N} \Delta K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_k W_{p_i}^{(N-1)}(x) = 0, \quad 0 \le k \le n_i - 2.$$

Let V be a space of polynomials defined by

$$V = \Big\{\phi \mid \deg(\phi) = |n| - 1 \quad \text{and} \quad \sum_{x=0}^{N} \phi(x)(-x)_k W_{p_i}^{(N-1)}(x) = 0,$$
$$0 \le k \le n_i - 2,\ i = 1, 2, \ldots, r\Big\}.$$

Then clearly $\Delta K_{\vec{n}}^{(\vec{p};N)} \in V$ and the dimension of V is r. Since

$$\sum_{x=0}^{N} K_{\vec{n}-e_1}^{(\vec{p};N-1)}(x)(-x)_k W_{p_i}^{(N-1)}(x) = 0, \quad 0 \le k \le n_i - 2, \quad i = 1, 2, \ldots, r,$$

we prove $K_{\vec{n}-e_1}^{(p;N-1)} \in V$. By the same process we obtain that $K_{\vec{n}-e_i}^{(\vec{p};N-1)} \in V$ for $i = 1, 2, \ldots, r$. Assume now that

$$d_1 K_{n-e_1}^{(\vec{p};N-1)}(x) + d_2 K_{n-e_2}^{(\vec{p};N-1)}(x) + \cdots + d_r K_{n-e_r}^{(\vec{p};N-1)}(x) = 0, \tag{2.5}$$

where d_j's ($i = 1, 2, \ldots, r$) are constants. Multiplying (2.5) by $(-x)_{n_k-1} W_{p_k}^{(N-1)}(x)$ and then taking summations, we obtain

$$\sum_{i=1}^{r} d_i \sum_{x=0}^{N} K_{n-e_i}^{(\vec{p};N-1)}(x)(-x)_{n_k-1} W_{p_k}^{(N-1)}(x) = 0.$$

Since the multi-index is normal (see [5] for the definition),

$$\sum_{x=0}^{N} K_{n-e_i}^{(\vec{p};N-1)}(x)(-x)_{n_k-1} W_{p_k}^{(N-1)}(x) = \begin{cases} 0 & \text{if} \quad i \ne k \\ \ne 0 & \text{if} \quad i = k \end{cases}$$

so that $d_k = 0$ for $k = 1, 2, \ldots, r$. Hence, $\{K_{\vec{n}-e_i}^{(\vec{p};N-1)}(x)\}_{i=1}^{r}$ is linearly independent so that $\Delta K_{\vec{n}}^{(\vec{p};N)}$ can be represented as a linear combination of $\{K_{\vec{n}-e_i}^{(\vec{p};N-1)}(x)\}$. Let

$$\Delta K_{\vec{n}}^{(\vec{p};N)}(x) = \sum_{i=1}^{r} d_i K_{\vec{n}-e_i}^{(\vec{p};N-1)}(x), \tag{2.6}$$

where d_i's are constants. By the summation by parts and Lemma 2.1(g), we have

$$\begin{aligned} &\sum_{x=0}^{N} \Delta K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_{n_k-1} W_{p_k}^{(N-1)}(x) \\ &= -\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x) \nabla[(-x)_{n_k-1} W_{p_k}^{(N-1)}(x)] \\ &= -\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x) \frac{Np_k - x}{Np_k(1-p_k)}(-x)_{n_k-1} W_{p_k}^{(N)}(x) \\ &\quad + \frac{n_k - 1}{Np_k} \sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x) x(-x+1)_{n_k-2} W_{p_k}^{(N)}(x). \end{aligned} \tag{2.7}$$

By the orthogonality of $\{K_{\vec{n}}^{(\vec{p};N)}(x)\}$ with respect to $W_{p_i}^{(N)}$, the last summation of the right hand side in the equation (2.7) becomes zero. By the orthogonality again the right hand side of (2.7) becomes

$$-\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x)\frac{Np_k - x}{Np_k(1-p_k)}(-x)_{n_k-1}W_{p_k}^{(N)}(x)$$

$$= \frac{1}{Np_k(1-p_k)}\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x)x(-x)_{n_k-1}W_{p_k}^{(N)}(x)$$

$$= -\frac{1}{Np_k(1-p_k)}\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_{n_k-1}(N-x)W_{p_k}^{(N)}(x)$$

$$= -\frac{1}{p_k}\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x).$$

Using the raising operator $L_{p_k}^{(N-1)}[\cdot]$ and the symmetric property with the formal adjoint $L^*_{(p_k,N-1)}[\cdot]$, we have

$$\sum_{x=0}^{N} K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$= -\sum_{x=0}^{N} L_{p_k}^{(N-1)}[K_{\vec{n}-e_k}^{(\vec{p};N-1)}](-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$(2.8)\qquad = -\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)L^*_{(p_k,N-1)}[(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)]$$

$$= (1-p_k)\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)\Delta[x(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)]$$

$$-\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(Np_k - x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x).$$

By a simple calculation with Lemma 2.1(h), we have

$$(1-p_k)\Delta[x(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)] - (Np_k - x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$= \left(-n_k p_k(-x)_{n_k-1} - (n_k-1)(N-n_k+1)p_k(-x)_{n_k-2}\right)W_{p_k}^{(N-1)}(x).$$

Hence, we have by the orthogonality and the equation (2.8),

$$\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)\Big\{(1-p_k)\Delta[x(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)]$$

$$-(Np_k-x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)\Big\}$$

$$=-n_kp_k\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$-(n_k-1)(N-n_k+1)p_k\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-2}W_{p_k}^{(N-1)}(x)$$

$$=-n_kp_k\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

from which we have

$$\sum_{x=0}^{N} \Delta K_{\vec{n}}^{(\vec{p};N)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$=n_k\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x).$$

Note that

$$\sum_{i=1}^{r} d_i\sum_{x=0}^{N} K_{n-e_i}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$=d_k\sum_{x=0}^{N} K_{n-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x).$$

Multiplying (2.6) by $(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$ and then taking summations on both sides, we obtain

$$n_k\sum_{x=0}^{N} K_{\vec{n}-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

$$=d_k\sum_{x=0}^{N} K_{n-e_k}^{(\vec{p};N-1)}(x)(-x)_{n_k-1}W_{p_k}^{(N-1)}(x)$$

which implies $d_k=n_k$ for $k=1,2,\ldots,r$. □

In particular if $r=2$, the lowering equation (2.4) becomes

$$\Delta K^{(p_1,p_2;N)}_{(n_1,n_2)}(x) = n_1 K^{(p_1,p_2;N-1)}_{(n_1-1,n_2)}(x) + n_2 K^{(p_1,p_2;N-1)}_{(n_1,n_2-1)}(x).$$

Theorem 2.3. *The multiple Kravchuk polynomial* $\{K^{(\vec{p};N)}_{\vec{n}}(x)\}$ *satisfies a difference equation*

$$L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N-1)}_{p_r}[\Delta K^{(\vec{p};N)}_{\vec{n}}]$$

$$= -\sum_{i=1}^{r} n_i L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N+r-i)}_{p_{i-1}} L^{(N+r-i-1)}_{p_{i+1}} \cdots L^{(N)}_{p_r}[K^{(\vec{p};N)}_{\vec{n}}],$$

where $L^{(s)}_{p_i}[\cdot]$*'s are raising operators as in* (2.2).

Proof. Since

$$L^{(s)}_{q_i} L^{(s-1)}_{q_j}[y] = L^{(s)}_{q_j} L^{(s-1)}_{q_i}[y], \quad q_i, q_j, s \in \mathbb{R},$$

we obtain inductively that for $i = 1, 2, \ldots, r$,

$$L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N-1)}_{p_r}[K^{(\vec{p};N-1)}_{\vec{n}-e_i}]$$

$$= L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N+r-i)}_{p_{i-1}} L^{(N+r-i-1)}_{p_{i+1}} L^{(N+r-i-2)}_{p_i} \cdots L^{(N-1)}_{p_r}[K^{(\vec{p};N-1)}_{\vec{n}-e_i}]$$

$$= L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N+r-i)}_{p_{i-1}} L^{(N+r-i-1)}_{p_{i+1}} \cdots L^{(N)}_{p_r} L^{(N-1)}_{p_i}[K^{(\vec{p};N-1)}_{\vec{n}-e_i}]$$

$$= -L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N+r-i)}_{p_{i-1}} L^{(N+r-i-1)}_{p_{i+1}} \cdots L^{(N)}_{p_r}[K^{(\vec{p};N)}_{\vec{n}}].$$

Applying $L^{(N+r-2)}_{p_1} L^{(N+r-3)}_{p_2} \cdots L^{(N-1)}_{p_r}$ on both sides of the equation (2.4), the conclusion follows. □

Theorem 2.4. *The multiple Kravchuk polynomial* $\{K^{(p_1,p_2;N)}_{(n_1,n_2)}(x)\}$ *(for* $r=2$*) satisfies a difference equation*

$$(1-p_1)(1-p_2)x(x-1)\Delta\nabla^2 y + x[N(p_1+p_2-2p_1p_2)$$

$$+ (2-p_1-p_2)(1-x)]\Delta\nabla y + [(Np_1-x)(Np_2-x) + Np_1p_2 - x]\Delta y$$

$$+ [(1-p_2)n_1 + (1-p_1)n_2]x\nabla y$$

$$+ [n_1(Np_2+p_2-x) + n_2(Np_1+p_1-x)]y = 0.$$

Proof. By Theorem 2.3, $\{K^{(p_1,p_2;N)}_{(n_1,n_2)}(x)\}$ satisfies a difference equation

$$L^{(N)}_{p_1} L^{(N-1)}_{p_2}[\Delta y] = -n_1 L^{(N)}_{p_2}[y] - n_2 L^{(N)}_{p_1}[y],$$

where

$$L_{p_1}^{(N)}[y] = (1-p_1)x\nabla y + [(N+1)p_1 - x]y$$

and

$$L_{p_2}^{(N)}[y] = (1-p_2)x\nabla y + [(N+1)p_2 - x]y.$$

Since

$$\begin{aligned}
L_{p_1}^{(N)}L_{p_2}^{(N-1)}[y] &= (1-p_1)x\nabla L_{p_2}^{(N-1)}[y] + [(N+1)p_1 - x)]L_{p_2}^{(N-1)}[y] \\
&= (1-p_1)x\nabla[(1-p_2)x\nabla y + (Np_2 - x)y] \\
&\quad + ((N+1)p_1 - x))[(1-p_2)x\nabla y + (Np_2 - x)y] \\
&= (1-p_1)(1-p_2)x(x-1)\nabla^2 y \\
&\quad + x[(1-p_1)(1-p_2) + (1-p_1p_2) + (1-p_1)(Np_2 - x) \\
&\qquad + (1-p_2)(Np_1 - x)]\nabla y \\
&\quad + [(Np_1 - x)(Np_2 - x) + Np_1p_1 - x]y,
\end{aligned}$$

the conclusion follows. □

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