J. Korean Math. Soc. **44** (2007), No. 6, pp. 1397–1415

SEMI-DISCRETE CENTRAL DIFFERENCE METHOD FOR DETERMINING SURFACE HEAT FLUX OF IHCP

ZHI QIAN AND CHU-LI FU

ABSTRACT. We consider an inverse heat conduction problem(IHCP) in a quarter plane which appears in some applied subjects. We want to determine the heat flux on the surface of a body from a measured temperature history at a fixed location inside the body. This is a severely ill-posed problem in the sense that arbitrarily "small" differences in the input temperature data may lead to arbitrarily "large" differences in the surface flux. A semi-discrete central difference scheme in time is employed to deal with the ill posed problem. We obtain some error estimates which also give the information about how to choose the step length in time. Some numerical examples illustrate the effects of the proposed method.

1. Introduction

In some industrial applications one wishes to determine the temperature or heat flux on the surface of a body, where the surface itself is inaccessible for measurements [1]. In this case it is necessary to determine the surface temperature or heat flux from a measured temperature history at a fixed location inside the body. This problem is called inverse heat conduction problem (IHCP). The following equation in a quarter plane is a model of this situation in a one-dimensional setting:

$$\begin{aligned} &u_{xx} = u_t, && x > 0, \quad t > 0, \\ &u(x,0) = 0, && x \geq 0, \\ &u(1,t) = g(t), && t \geq 0, \quad u(x,t)|_{x\to\infty} \text{ bounded}. \end{aligned} \tag{1.1}$$

We want to determine the heat flux $u_x(x,t)$ for $0 \leq x < 1$. Carasso [2] considered the problem and obtained a good result using a special Tikhonov regularization method. However, he left behind the "zero point" problem (i.e., the case $x = 0$). In [5], we also considered the problem using a wavelet regularization method and obtained a better result. Especially, we solved the "zero point" problem. It is a pity that we did not consider the numerical aspects in that paper.

Received March 31, 2006.

2000 *Mathematics Subject Classification.* 65M30, 35R25, 35R30.

Key words and phrases. inverse problem, regularization, central difference scheme, error estimate.

In the present paper we develop and perfect Eldén's work [3] in which he studied the temperature distribution of problem (1.1) using the difference schemes in time. Our main aim here is to investigate the heat flux distribution of problem (1.1) using a central difference scheme in time. We do some convergence estimates in section 3, study the stability of space march difference in section 4 and give some interesting numerical examples to test the effects of the method in section 5.

2. Ill-posedness of problem (1.1) and central difference schemes

It is well known that the problem (1.1) is well-posed for $x > 1$, so we can easily obtain the solution $u(x,t)$ and its gradient $u_x(x,t)$ for $x > 1$ using classical numerical methods (e.g., see [6]). Therefore the heat flux data $u_x(1,t)$ is also easily obtained. As this reason we can formulate (1.1) as a Cauchy problem with appropriate Cauchy data $[u, u_x]$ given on the line $x = 1$. But for $0 \leq x < 1$ the problem is severely ill-posed in the sense that the solution, if it exists, does not depend continuously on the data. Even arbitrarily "small" differences in the input temperature data may lead to arbitrarily "large" differences in the solution. In this section, we simply analyze the ill-posedness of the problem (1.1) and propose an appropriate method, for the ill-posed problem.

But before doing that, we need to define all functions appearing in the paper to be zero for $t < 0$, since we will consider our problem in $L^2(\mathbb{R})$ with respect to the variable t. Note that, in the problem (1.1), the input temperature data $g(t)$ can only be measured, there must be measurement error. Thus we would actually have as data some function $g_\delta(t) \in L^2(\mathbb{R})$, for which

$$\|g_\delta - g\| = \|g_\delta(\cdot) - u(1,\cdot)\| \leq \delta, \tag{2.1}$$

where $\|\cdot\|$ denotes L^2-norm and the constant $\delta > 0$ represents a bound on the measurement error. Let

$$\widehat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\xi t} dt$$

be the Fourier transform of the exact data function.

Now the problem (1.1) can be formulated, in frequency space, as below:

$$\begin{array}{ll} \widehat{u}_{xx}(x,\xi) = i\xi\widehat{u}(x,\xi), & x > 0, \xi \in \mathbb{R} \\ \widehat{u}(1,\xi) = \widehat{g}(\xi), & \xi \in \mathbb{R} \\ \widehat{u}(x,\xi)|_{x\to\infty}, & \text{bounded.} \end{array} \tag{2.2}$$

The solution to this problem is given by

$$\widehat{u}(x,\xi) = e^{(1-x)\theta(\xi)}\widehat{g}(\xi), \tag{2.3}$$

and naturally

$$\widehat{u}_x(x,\xi) = -\theta(\xi)e^{(1-x)\theta(\xi)}\widehat{g}(\xi), \tag{2.4}$$

where $\theta(\xi)$ is the principal value of $\sqrt{i\xi}$:

$$\theta(\xi) = (1 + \sigma i)\sqrt{|\xi|/2}, \quad \sigma = \text{sign}(\xi), \quad \xi \in \mathbb{R}. \tag{2.5}$$

Since the real part of θ is nonnegative, $\widehat{u}(x,\xi)$ and $\widehat{u}_x(x,\xi)$ are in $L^2(\mathbb{R})$ with respect to the variable ξ, we see from (2.3) and (2.4) that the exact data function $\widehat{g}(\xi)$ must decay rapidly as $\xi \to \infty$, small errors in high-frequency components can blow up and completely destroy the solution for $0 \le x < 1$. As the measured data $g_\delta(t)$, its Fourier transform $\widehat{g}_\delta(\xi)$ is merely in $L^2(\mathbb{R})$.

To obtain continuous dependence on the data, we assume, as for any ill-posed problem, there exists an *a priori* bound,

$$\|f(\cdot)\| := \|u(0,\cdot)\| \le E. \tag{2.6}$$

This is essentially necessary in order to obtain any meaningful error estimates for approximating the exact solution. Additionally, in order to get a procedure that can be implemented numerically, it is necessary to somehow modify the problem. Often, the dependence on δ and E is included by choosing the value of some parameter in the numerical procedure.

In the present paper, as a regularized approximation of problem (1.1) we consider the following problem

$$\begin{cases} v_{xx}(x,t) = \frac{1}{2k}(v(x,t+k) - v(x,t-k)), & x>0, t>0, \\ v(x,0)=0, & x \ge 0, \\ v(1,t) = g_\delta(t), & t \ge 0, \\ v(x,t)|_{x\to\infty} \quad \text{bounded}, & \end{cases} \tag{2.7}$$

where we have replaced the time derivative by a central difference with the step length k. The advantage of not discretizing in the space variable is that we can use Fourier transform techniques. Before further discussing, we must claim that the problem (2.7) was proposed by Eldén [3] for the first time. He studied the temperature distribution for $0 < x < 1$ and obtained some valuable convergence estimates. A remedied work for the "zero point"(i.e., $x = 0$) can be found in [7]. In the present paper, we borrow his idea to consider the heat flux distribution which is also of practical interest, for $0 \le x < 1$. Furthermore, we discuss the stability of space marching difference and give some interesting numerical examples to test the effects of the proposed method. In this sense, we develop Eldén's work [3]. For further development on the semi-discrete central method, refer to [8], [9].

By taking the Fourier transform for variable t in (2.7) we have

$$\begin{cases} \widehat{v}_{xx}(x,\xi) = i\frac{\sin k\xi}{k}\widehat{v}(x,\xi), \\ \widehat{v}(1,\xi) = \widehat{g}_\delta(\xi), \\ \widehat{v}(x,\xi)|_{x\to\infty} \quad \text{bounded}. \end{cases} \tag{2.8}$$

The formal solution of (2.8) can be easily obtained

$$\widehat{v}(x,\xi) = e^{(1-x)\rho(k,\xi)}\widehat{g}_\delta(\xi), \tag{2.9}$$

and naturally

$$\widehat{v}_x(x,\xi) = -\rho(k,\xi)e^{(1-x)\rho(k,\xi)}\widehat{g}_\delta(\xi), \tag{2.10}$$

where $\rho(k,\xi)$ is the principal value of $\sqrt{i\frac{\sin k\xi}{k}}$:

$$\rho(k,\xi) = (1+\nu i)\sqrt{|\sin k\xi|/2k}, \quad \nu = \mathrm{sign}(\sin k\xi), \quad \xi \in \mathbb{R}. \tag{2.11}$$

Note that if k is small, then for small ξ, ρ in (2.11) is close to θ in (2.5). Further, $\rho(k,\xi)$ is a period bounded function with respect to ξ, therefore, $e^{(1-x)\rho(k,\xi)}$ and $\rho(k,\xi)e^{(1-x)\rho(k,\xi)}$ is bounded even if $0 \le x < 1$.

3. Error estimate

In this section we will discuss the difference of the heat flux solution between problems (2.7) and (1.1) for $0 \le x < 1$. Theorem 3.1 gives the error estimate for the case $0 < x < 1$ and Theorem 3.4 for the case $x = 0$.

Theorem 3.1. *Let $u_x(x,t)$, whose Fourier transform is given by (2.4), be the exact heat flux. Let $v_x(x,t)$, whose Fourier transform is given by (2.10), be the regularization approximation of $u_x(x,t)$. The regularized parameter or the step length $k \in (0,1)$ is chosen*

$$k = \frac{1}{2(\ln(E/\delta))^2}. \tag{3.1}$$

Let the measured temperature history at $x = 1$, $g_\delta(t)$, satisfy (2.1), and let the a priori *assumption (2.6) holds. Then*

(1) *If $\frac{\delta}{E} \ge e^{-5}$, for fixed $x \in (0,1)$ we have*

$$\|u_x(x,\cdot) - v_x(x,\cdot)\| \le CE^{1-x}\delta^x \tag{3.2}$$

where $C = 5\sqrt{2} + \max\left\{\frac{2\sqrt{2}}{x}, 10\sqrt{2}\right\}$.

(2) *If $\frac{\delta}{E} < e^{-5}$,*

$(2-1)$ *for $\frac{5}{\ln(E/\delta)} < x < 1$, we have*

$$\|u_x(x,\cdot) - v_x(x,\cdot)\| \le \sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x + \varepsilon_1, \tag{3.3}$$

where $\varepsilon_1 = \max\left\{\frac{2\sqrt{2}}{x}E^{1-x}\delta^x, 2\sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x, \frac{\sqrt{2}}{6}\left(1+2\ln\frac{E}{\delta}\right)\left(\frac{5}{xe}\right)^5\frac{E}{(\ln(E/\delta))^4}\right\}$,

$(2-2)$ *for $0 < x \le \frac{5}{\ln(E/\delta)}$, we have*

$$\|u_x(x,\cdot) - v_x(x,\cdot)\| \le \sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x + \varepsilon_2, \tag{3.4}$$

where $\varepsilon_2 = \max\left\{\frac{2\sqrt{2}}{x}, 2\sqrt{2}\ln\frac{E}{\delta}, \frac{\sqrt{2}}{6}\left(1+2\ln\frac{E}{\delta}\right)\ln\frac{E}{\delta}\right\}E^{1-x}\delta^x$.

Firstly, we want to claim that, in the following proof (including the proof of Corollary 3.2), we have borrowed some results of Eldén [3], but for completeness of the presentation we still give a detailed proof here. Since "the heat flux is more difficult to calculate accurately than the surface temperature", the proof of theoretic error estimate on the heat flux is more complex than that of Eldén [3].

Proof. By the Parseval formula, (2.4) and (2.10), we have
(3.5)
$$\begin{aligned}\|u_x(x,\cdot)-v_x(x,\cdot)\| &= \|\widehat{u}_x(x,\cdot)-\widehat{v}_x(x,\cdot)\| \\ &= \|\theta(\xi)e^{\theta(\xi)(1-x)}\widehat{g}(\xi)-\rho(k,\xi)e^{\rho(k,\xi)(1-x)}\widehat{g}_\delta(\xi)\|.\end{aligned}$$

For abbreviation, we denote

$$\theta := \theta(\xi),\ \rho := \rho(k,\xi),\ \widehat{g} := \widehat{g}(\xi),\ \text{etc.}$$

Note that (2.3), we have

$$\widehat{f}(\xi) := \widehat{u}(0,\xi) = e^{\theta(\xi)}\widehat{g}(\xi). \tag{3.6}$$

So (3.5) becomes, due to (3.6), (2.1) and (2.6),

$$\begin{aligned}\|u_x(x,\cdot)-v_x(x,\cdot)\| &= \|\theta e^{\theta(1-x)}\widehat{g}-\rho e^{\rho(1-x)}\widehat{g}+\rho e^{\rho(1-x)}\widehat{g}-\rho e^{\rho(1-x)}\widehat{g}_\delta\| \\ &\le \left\|\left(\theta e^{-\theta x}-\rho e^{\rho(1-x)-\theta}\right)\widehat{f}\right\|+\left\|\rho e^{\rho(1-x)}(\widehat{g}-\widehat{g}_\delta)\right\| \\ &\le \sup_{\xi\in\mathbb{R}}A(\xi)E+\sup_{\xi\in\mathbb{R}}B(\xi)\delta, \qquad (3.7)\end{aligned}$$

where

$$A(\xi) := \left|\theta e^{-\theta x}-\rho e^{\rho(1-x)-\theta}\right|,\quad B(\xi) := \left|\rho e^{\rho(1-x)}\right|.$$

We start by estimating the second term on the right hand side of (3.7). Note that ρ is given in (2.11), and k is chosen in (3.1),

$$B(\xi)\delta = |\rho|e^{\mathrm{Re}(\rho)(1-x)}\delta \le \sqrt{\frac{1}{k}}e^{\sqrt{\frac{1}{2k}}(1-x)}\delta = \sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x. \tag{3.8}$$

To estimate the first term on the right hand side of (3.7), we rewrite $A(\xi)$ as

$$A(\xi) = \left|e^{-\theta x}\right|\left|\theta-\rho e^{-\tau(1-x)}\right|, \tag{3.9}$$

where

$$\tau := \theta-\rho = \frac{1+\sigma i}{\sqrt{2}}\sqrt{|\xi|}-\frac{1+\nu i}{\sqrt{2}}\left(\frac{|\sin k\xi|}{k}\right)^{1/2}, \tag{3.10}$$

and $\sigma = \mathrm{sign}(\xi)$, $\nu = \mathrm{sign}(\sin k\xi)$.

For estimating $A(\xi)$, we shall distinguish between two cases.

Case I: for large $|\xi|$, i.e., $|\xi| \ge \xi_0 := \frac{1}{k} = 2\left(\ln\frac{E}{\delta}\right)^2$, note that the real part of τ is non-negative, we can estimate (3.9)

$$\begin{aligned}A(\xi) &\le \left|e^{-\theta x}\right|(|\theta|+|\rho|) \\ &= e^{-x\mathrm{Re}(\theta)}\left(\sqrt{|\xi|}+\sqrt{|\sin k\xi|/k}\right) \qquad (3.11)\\ &\le 2\sqrt{|\xi|}e^{-x\sqrt{|\xi|/2}}.\end{aligned}$$

Let

$$H(t) := 2te^{-xt/\sqrt{2}},\quad t := \sqrt{|\xi|} \ge \sqrt{\xi_0}\,. \tag{3.12}$$

It is easy to know that for $x \in (0,1)$, $H'(t_0) = 0$ when $t_0 = \frac{\sqrt{2}}{x}$, and $H(t)$ increases for $t \le t_0$ and decreases for $t > t_0$, so

(i) $t_0 \geq \sqrt{\xi_0}$, $H_{\max} = 2t_0 e^{-xt_0/\sqrt{2}} = \frac{2\sqrt{2}}{x} e^{-xt_0/\sqrt{2}} \leq \frac{2\sqrt{2}}{x} e^{-x\sqrt{\xi_0/2}}$.
(ii) $t_0 < \sqrt{\xi_0}$, $H_{\max} = 2\sqrt{\xi_0} e^{-x\sqrt{\xi_0/2}}$.
Combining (i), (ii) and (3.11), we have
(3.13)
$$A(\xi)E \leq 2\max\left\{\frac{\sqrt{2}}{x}, \sqrt{\xi_0}\right\} e^{-x\sqrt{\xi_0/2}} E \leq 2\sqrt{2}\max\left\{\frac{1}{x}, \ln\frac{E}{\delta}\right\} E^{1-x}\delta^{x}.$$

Case II: for $|\xi| < \xi_0$, i.e., $|k\xi| < 1$. We could observe that for $|\xi|$ in this interval, $\sigma = \mathrm{sign}(\xi) = \mathrm{sign}(\sin k\xi) = \nu$, which means that we can rewrite (3.10) as

$$\tau = \tau_1(1+\sigma i), \quad \tau_1 = \frac{1}{\sqrt{2}}\left(\sqrt{|\xi|} - \left(\frac{|\sin k\xi|}{k}\right)^{1/2}\right). \tag{3.14}$$

Note that, using the triangle inequality, we can estimate (3.9)

$$\begin{aligned} A(\xi) &= \left|e^{-\theta x}\right| \Big|\theta - \theta e^{-i\sigma\tau_1(1-x)} + \theta e^{-i\sigma\tau_1(1-x)} - \rho e^{-i\sigma\tau_1(1-x)} \\ &\quad + \rho e^{-i\sigma\tau_1(1-x)} - \rho e^{-\tau_1(1+\sigma i)(1-x)}\Big| \\ &\leq \left|e^{-\theta x}\right| (|\theta| A_1 + |\tau| + |\rho| A_2), \end{aligned} \tag{3.15}$$

where

$$A_1 := \left|1 - e^{-i\sigma\tau_1(1-x)}\right|, \quad A_2 := \left|1 - e^{-\tau_1(1-x)}\right|.$$

Since $\tau_1 \geq 0$, $0 < x < 1$, we have

$$A_1 = 2|\sin(\sigma\tau_1(1-x)/2)| \leq \tau_1(1-x) \leq \tau_1\,. \tag{3.16}$$

Similarly, note that the inequality $1 - e^{-y} \leq y (y \geq 0)$ holds, we have

$$A_2 \leq \tau_1(1-x) \leq \tau_1. \tag{3.17}$$

Obviously,

$$|\tau| = \sqrt{2}\tau_1. \tag{3.18}$$

Thus, combining the inequalities (3.15)-(3.18), we have

$$\begin{aligned} A(\xi) &\leq \left|e^{-\theta x}\right| (|\theta|\tau_1 + \sqrt{2}\tau_1 + |\rho|\tau_1) \\ &\leq (\sqrt{2} + 2\sqrt{|\xi|}) e^{-x\sqrt{|\xi|/2}} \tau_1 \\ &\leq (1 + \sqrt{2}\sqrt{\xi_0}) e^{-x\sqrt{|\xi|/2}} \left(\sqrt{|\xi|} - \left(\frac{|\sin k\xi|}{k}\right)^{1/2}\right) \\ &= \left(k^{-\frac{1}{2}} + \sqrt{2}k^{-1}\right) e^{-x\sqrt{|k\xi|}/\sqrt{2k}} (\sqrt{|k\xi|} - \sqrt{|\sin k\xi|}). \end{aligned} \tag{3.19}$$

Introducing a new variable $r = \sqrt{|k\xi|}$, we shall find an upper bound of function $h(r)$ for $0 \leq r < 1$, where

$$h(r) := e^{-xr/\sqrt{2k}}(r - \sqrt{\sin(r^2)}).$$

Using the inequalities

$$\sqrt{\sin(r^2)} \geq (r^2 - r^6/6)^{1/2} \geq r(1 - r^4/6) \quad \text{for} \quad 0 \leq r < 1, \tag{3.20}$$

we have

$$h(r) \leq e^{-xr/\sqrt{2k}} \frac{r^5}{6} =: \bar{h}(r).$$

By an elementary calculation we find that if

$$r_0 := \frac{5\sqrt{2k}}{x} < 1, \tag{3.21}$$

then

$$\bar{h}_{\max} = \frac{1}{6} e^{-xr_0/\sqrt{2k}} r_0^5,$$

which leads to the estimate

$$\begin{aligned} A(\xi)E &\leq \left(k^{-\frac{1}{2}} + \sqrt{2}k^{-1}\right) \bar{h}_{\max} E \\ &\leq \tfrac{\sqrt{2}}{6}\left(1 + 2\ln \tfrac{E}{\delta}\right)\left(\tfrac{5}{xe}\right)^5 \tfrac{E}{(\ln(E/\delta))^4}. \end{aligned} \tag{3.22}$$

If (3.21) is not satisfied then the maximum of $\bar{h}(r)$ is attained at $r = 1$, and we have

$$\begin{aligned} A(\xi)E &\leq \frac{1}{6}\left(k^{-\frac{1}{2}} + \sqrt{2}k^{-1}\right) e^{-x/\sqrt{2k}} E \\ &= \frac{\sqrt{2}}{6}\left(1 + 2\ln \frac{E}{\delta}\right) \ln \frac{E}{\delta} E^{1-x}\delta^x. \end{aligned} \tag{3.23}$$

We now analyse the relation between the range of r_0 and the "signal-to-noise ratio" $\frac{E}{\delta}$:

(1) $\frac{\delta}{E} \geq e^{-5}$, then $\ln \frac{E}{\delta} \leq 5$, $r_0 = \frac{5\sqrt{2k}}{x} = \frac{5}{x \ln(E/\delta)} > \frac{1}{x} \geq 1$ for any $x \in (0, 1]$. Hence, for $|\xi| < \xi_0$, the inequality (3.23) is valid

$$A(\xi)E \leq \frac{\sqrt{2}}{6}\left(1 + 2\ln \frac{E}{\delta}\right) \ln \frac{E}{\delta} E^{1-x}\delta^x \leq 10\sqrt{2}E^{1-x}\delta^x.$$

In this case, the inequality (3.13) for $|\xi| \geq \xi_0$ becomes

$$\begin{aligned} A(\xi)E &\leq 2\sqrt{2}\max\left\{\tfrac{1}{x}, \ln \tfrac{E}{\delta}\right\} E^{1-x}\delta^x \\ &\leq 2\sqrt{2}\max\left\{\tfrac{1}{x}, 5\right\} E^{1-x}\delta^x, \end{aligned}$$

and the inequality (5.5) becomes

$$B(\xi)\delta \leq \sqrt{2}\ln \frac{E}{\delta} E^{1-x}\delta^x \leq 5\sqrt{2}E^{1-x}\delta^x. \tag{3.24}$$

Therefore, there holds

$$\|u_x(x, \cdot) - v_x(x, \cdot)\| \leq CE^{1-x}\delta^x,$$

where $C = 5\sqrt{2} + \max\left\{\frac{2\sqrt{2}}{x}, 10\sqrt{2}\right\}$. This just is the estimate (3.2).

(2) $\frac{\delta}{E} < e^{-5}$, $r_0 = \frac{5\sqrt{2k}}{x} = \frac{5}{x}\frac{1}{\ln(E/\delta)}$.

(2-1) If $\frac{5}{\ln(E/\delta)} < x < 1$, then $r_0 < 1$. So the inequality (3.22) holds for $|\xi| < \xi_0$. Combining this with (3.13), we have

$$\begin{aligned} & A(\xi)E \\ \leq & \max\left\{\tfrac{2\sqrt{2}}{x}E^{1-x}\delta^x, 2\sqrt{2}\ln\tfrac{E}{\delta}E^{1-x}\delta^x, \tfrac{\sqrt{2}}{6}\left(1+2\ln\tfrac{E}{\delta}\right)\left(\tfrac{5}{xe}\right)^5\tfrac{E}{(\ln(E/\delta))^4}\right\} \\ = & :\varepsilon_1, \end{aligned}$$

and furthermore, note that (5.5), there holds

$$\|u_x(x,\cdot) - v_x(x,\cdot)\| \leq \sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x + \varepsilon_1.$$

This just is the estimate (3.3).

(2-2) If $0 < x \leq \frac{5}{\ln(E/\delta)}$, Then $r_0 \geq 1$ and the inequality (3.23) holds for $|\xi| < \xi_0$. Combining it with (3.13), we have

$$A(\xi)E \leq \max\left\{\frac{2\sqrt{2}}{x}, 2\sqrt{2}\ln\frac{E}{\delta}, \frac{\sqrt{2}}{6}\left(1+2\ln\frac{E}{\delta}\right)\ln\frac{E}{\delta}\right\}E^{1-x}\delta^x =: \varepsilon_2.$$

Note that (5.5), we get

$$\|u_x(x,\cdot) - v_x(x,\cdot)\| \leq \sqrt{2}\ln\frac{E}{\delta}E^{1-x}\delta^x + \varepsilon_2.$$

This just is the estimate (3.4).

The proof of Theorem 3.1 is now complete. □

Theorem 3.1 indicates how to choose the step length k. It also shows that when the "signal-to-noise ratio" E/δ is relatively low, the error between the exact "heat flux" solution and its approximation by the time difference is not significantly large. But, when we let δ tend to zero, the rate of convergence of the time-discrete scheme is only logarithmic.

Corollary 3.2. *Let x be fixed in $(0,1)$. Then, asymptotically, as $\delta \to 0$,*

$$\|u(x,\cdot) - v(x,\cdot)\| \sim \frac{E}{(\ln(E/\delta))^3}. \tag{3.25}$$

Proof. If δ is small enough, then $5/\ln(E/\delta) < x < 1$, and the largest term in (3.3) dominates. □

Remark 3.3. The error estimates in Theorem 3.1 do not give any useful information on the continuous dependence of the solution at $x = 0$. Moreover, as $x \to 0^+$, the accuracy of the regularized solution becomes progressively lower. Carasso [2] and Eldén [3] met the same trouble. Actually, this is common in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $x = 0$, we introduce a stronger *a priori* assumption

$$\|f(\cdot)\|_p := \|u(0,\cdot)\|_p \leq E, \quad p \geq 0, \tag{3.26}$$

where $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(\mathbb{R})$ defined by

$$\|f\|_p := \left(\int_{-\infty}^{\infty}(1+\xi^2)^p|\widehat{f}(\xi)|^2d\xi\right)^{1/2}.$$

Theorem 3.4. *Let the conditions* (2.1) *and* (3.26) *hold.* u_x, v_x *are given by* (2.4) *and* (2.10) *respectively. The step length* $k \in (0,1)$ *is chosen*

$$k = \frac{1}{2\left(\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\right)^2}. \tag{3.27}$$

Then, if $p > \frac{1}{2}$, *we have*

$$\|u_x(0,\cdot)-v_x(0,\cdot)\| \le \sqrt{2}E\left(\ln\frac{E}{\delta}\right)^{1-2p} + \sqrt{2}E\left(\ln\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\left(\ln\frac{E}{\delta}\right)^{-2p} + \widetilde{\varepsilon},$$

where $\widetilde{\varepsilon} = \max\left\{2k^{\frac{4}{5}(p-\frac{1}{2})},\ \frac{1+\sqrt{2}}{6}k^2\right\}E$.

Proof. As the proof of Theorem 3.1 and note that the first inequality of (3.7), we know

$$\begin{aligned}\|u_x(0,\cdot)-v_x(0,\cdot)\| &\le \|(\theta-\rho e^{\rho-\theta})\widehat{f}\| + \|\rho e^{\rho}(\widehat{g}-\widehat{g}_\delta)\| \\ &= \|(\theta-\rho e^{\rho-\theta})(1+\xi^2)^{-p/2}(1+\xi^2)^{p/2}\widehat{f}\| + \|\rho e^{\rho}(\widehat{g}-\widehat{g}_\delta)\|.\end{aligned}$$

Now the conditions (2.1) and (3.26) lead to

$$\|u_x(0,\cdot)-v_x(0,\cdot)\| \le \sup_{\xi\in\mathbb{R}}\widetilde{A}(\xi)E + \sup_{\xi\in\mathbb{R}}\widetilde{B}(\xi)\delta, \tag{3.28}$$

where

$$\widetilde{A}(\xi) := |(\theta-\rho e^{\rho-\theta})(1+\xi^2)^{-p/2}|, \quad \widetilde{B}(\xi) := |\rho e^{\rho}|. \tag{3.29}$$

We also start by estimating the second term on the right hand side of (3.28). Due to (2.11) and (3.27), we have

$$\begin{aligned}\widetilde{B}(\xi)\delta &\le \sqrt{\frac{1}{k}}e^{\sqrt{1/2k}}\delta = \sqrt{2}\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\cdot\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\cdot\delta \\ &= \sqrt{2}E\left(\ln\frac{E}{\delta}\right)^{1-2p} + \sqrt{2}E\left(\ln\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\left(\ln\frac{E}{\delta}\right)^{-2p}.\end{aligned} \tag{3.30}$$

To estimate the first term on the right hand side of (3.28), we rewrite

$$\widetilde{A}(\xi) = |\theta-\rho e^{-\tau}|(1+\xi^2)^{-p/2}, \tag{3.31}$$

where τ is given by (3.10).

For estimating $\widetilde{A}(\xi)$ in (3.31), we will distinguish three cases.

Case I: for large values of $|\xi|$, i.e., for $|\xi| \geq \widetilde{\xi}_0 := k^{-\frac{4}{5}}$. Note that $\mathrm{Re}(\tau) \geq 0$, there holds

$$\widetilde{A}(\xi) \leq (|\theta| + |\rho|)|\xi|^{-p} \leq 2|\xi|^{\frac{1}{2}-p} \leq 2\widetilde{\xi}_0^{\frac{1}{2}-p} = 2k^{\frac{4}{5}(p-\frac{1}{2})}, \quad p > \frac{1}{2}. \tag{3.32}$$

Case II: for $1 < |\xi| < \widetilde{\xi}_0$, taking the similar procedure of (3.14)-(3.18), we get

$$\begin{aligned} |\theta - \rho e^{-\tau}| &= |\theta - \theta e^{-i\sigma\tau_1} + \theta e^{-i\sigma\tau_1} - \rho e^{-i\sigma\tau_1} + \rho e^{-i\sigma\tau_1} - \rho e^{-\tau_1(1+i\sigma)}| \\ &\leq |\theta|\tau_1 + \sqrt{2}\tau_1 + |\rho|\tau_1 \\ &\leq (\sqrt{2} + 2\sqrt{|\xi|})\tau_1. \end{aligned} \tag{3.33}$$

Therefore $\widetilde{A}(\xi)$ in (3.31), can be estimated as bellow (note that $|k\xi| < k^{\frac{1}{5}} < 1$, τ_1 is given in (3.14), and the inequality (3.20) holds):

$$\begin{aligned} \widetilde{A}(\xi) &\leq (\sqrt{2} + 2\sqrt{|\xi|})\tau_1 \left(1+\xi^2\right)^{-p/2} \\ &\leq \frac{k^2}{6}\left(|\xi|^{\frac{5}{2}} + \sqrt{2}|\xi|^3\right)\left(1+\xi^2\right)^{-p/2}. \end{aligned} \tag{3.34}$$

If $\frac{1}{2} < p < \frac{5}{2}$, note that $|\xi| < \widetilde{\xi}_0$, we have

$$\begin{aligned} \widetilde{A}(\xi) &\leq \frac{k^2}{6}\left(|\xi|^{\frac{5}{2}-p} + \sqrt{2}|\xi|^{3-p}\right) \leq \frac{k^2}{6}\left(\widetilde{\xi}_0^{\frac{5}{2}-p} + \sqrt{2}\widetilde{\xi}_0^{3-p}\right) \\ &= \frac{1}{6}(k^{\frac{4}{5}p} + \sqrt{2}k^{\frac{4}{5}p-\frac{2}{5}}) \leq \frac{1+\sqrt{2}}{6}k^{\frac{4}{5}(p-\frac{1}{2})}, \end{aligned} \tag{3.35}$$

else if $\frac{5}{2} \leq p < 3$, note that $1 < |\xi| < \widetilde{\xi}_0$, we have

$$\widetilde{A}(\xi) \leq \frac{k^2}{6}\left(|\xi|^{\frac{5}{2}-p} + \sqrt{2}|\xi|^{3-p}\right) \leq \frac{k^2}{6}\left(1 + \sqrt{2}\widetilde{\xi}_0^{3-p}\right) \leq \frac{1+\sqrt{2}}{6}k^{\frac{4}{5}(p-\frac{1}{2})}, \tag{3.36}$$

else if $p \geq 3$, note that $|\xi| > 1$, we have

$$\widetilde{A}(\xi) \leq \frac{k^2}{6}\left(|\xi|^{\frac{5}{2}-p} + \sqrt{2}|\xi|^{3-p}\right) \leq \frac{1+\sqrt{2}}{6}k^2. \tag{3.37}$$

Case III: $|\xi| \leq 1$, following the procedure of (3.33) and (3.34), we can also estimate (3.31) as

$$\widetilde{A}(\xi) \leq \frac{k^2}{6}\left(|\xi|^{\frac{5}{2}} + \sqrt{2}|\xi|^3\right)\left(1+\xi^2\right)^{-p/2} \leq \frac{1+\sqrt{2}}{6}k^2. \tag{3.38}$$

Summarizing (3.32), (3.35)-(3.39), we complete the estimate of the first term on the right hand side of (3.28)

$$\widetilde{A}(\xi)E \leq \max\left\{2k^{\frac{4}{5}(p-\frac{1}{2})}, \frac{1+\sqrt{2}}{6}k^2\right\}E =: \widetilde{\varepsilon}, \quad p > \frac{1}{2}. \tag{3.39}$$

The theorem now follows by combining (3.39) and (3.30). □

Remark 3.5. For $p > \frac{1}{2}$, since the step length k tends to zero as the data error $\delta \to 0$, we can easily find that $\widetilde{\varepsilon} \to 0(\delta \to 0)$ too. Hence,

$$\lim_{\delta\to 0} \|u_x(0,\cdot) - v_x(0,\cdot)\| = 0, \qquad \text{for} \quad p > \frac{1}{2}.$$

Remark 3.6. In the practical applications $\|f\|_p$ is usually not known, therefore we have no the exact *a priori* bound E. However, if we select

$$k = \frac{1}{2\left(\ln\left(\frac{1}{\delta}\left(\ln\frac{1}{\delta}\right)^{-2}\right)\right)^2}, \tag{3.40}$$

i.e., let $p = 1$, $E = 1$ in (3.27), we can also obtain

$$\|u_x(0,\cdot) - v_x(0,\cdot)\| \leq \sqrt{2}\left(\ln\frac{1}{\delta}\right)^{-1} + \sqrt{2}\left(\ln\left(\ln\frac{1}{\delta}\right)^{-2}\right)\left(\ln\frac{1}{\delta}\right)^{-2} + \widetilde{\varepsilon}_1,$$

where $\widetilde{\varepsilon}_1 = \max\left\{2k^{\frac{2}{5}}, \frac{1+\sqrt{2}}{6}k^2\right\}E$. This may be helpful in our realistic computation.

Remark 3.7. We separately consider the case $0 < x < 1$ (Theorem 3.1) and the case $x = 0$ (Theorem 3.3), in order to emphasize the following facts. For the case $0 < x < 1$, the *a priori* bound for $\|u(\cdot,0)\|$ is sufficient. However, for the case $x = 0$, the stronger *a priori* bound for $\|u(\cdot,0)\|_p$ $(p > \frac{1}{2})$ must be imposed. For the unification of both cases, one can see our previous paper [5].

4. Stability of space march difference for the regularized problem

The approximation $v(x,t)$ and $v_x(x,t)$ in (2.9), (2.10), may be viewed as the solutions to an initial value problem for a second order partial differential equation, with appropriate initial data. This point of view is the basis for a powerful computational procedure for obtaining $v(x,t)$ and $v_x(x,t)$ from the data at $x = 1$.

Let $g_\delta(t)$ be the measured temperature data at $x = 1$ (extended by zero for $t < 0$). Define

$$g_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi t}\widehat{g}_\delta(\xi)d\xi \tag{4.1}$$

$$g_2(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi t}\rho(k,\xi)\widehat{g}_\delta(\xi)d\xi, \tag{4.2}$$

where ρ is given by (2.11).

Let us make a change of variable to reflect integration in the direction of decreasing x, and so that the origin corresponds to $x = 1$, i.e., put

$$z = 1 - x \tag{4.3}$$

and let

$$w(z,t) = v(x,t). \tag{4.4}$$

Then $w_z(z,t) = -v_x(x,t)$, and we will consider the following Cauchy problem

$$(4.5)\quad \begin{aligned} &w_{zz}(z,t) = \tfrac{1}{2k}(w(z,t+k) - w(z,t-k)), \ 0 < z < 1, \ -\infty < t < \infty,\\ &w(0,t) = g_1(t), \quad -\infty < t < \infty,\\ &w_z(0,t) = g_2(t), \quad -\infty < t < \infty. \end{aligned}$$

Theorem 4.1. *The initial problem* (4.5) *has a unique solution* $w(z,t)$. *Moreover, if* $0 \le z \le 1$, *then*

$$(4.6)\qquad w(z,t) = v(1-z,t),$$

$$(4.7)\qquad w_z(z,t) = -v_x(1-z,t),$$

where $v(x,t)$, $v_x(x,t)$ *are the approximations defined in* (2.9), (2.10).

Proof. Taking Fourier transform for t-variable in (4.5), for each frequency ξ, we obtain an initial value problem for an ordinary differential equation, i.e.,

$$\begin{aligned} \widehat{w}_{zz}(z,\xi) &= i\frac{\sin(k\xi)}{k}\widehat{w}(z,\xi),\\ \widehat{w}(0,\xi) &= \widehat{g}_1(\xi) = \widehat{g}_\delta(\xi),\\ \widehat{w}_z(0,\xi) &= \widehat{g}_2(\xi) = \rho(k,\xi)\widehat{g}_\delta(\xi). \end{aligned}$$

The above problem has the unique solution

$$(4.8)\qquad \widehat{w}(z,\xi) = e^{\rho z}\widehat{g}_\delta(\xi),$$

and clearly

$$(4.9)\qquad \widehat{w}_z(z,\xi) = \rho e^{\rho z}\widehat{g}_\delta(\xi).$$

An inverse Fourier transform in (4.8) and (4.9) now yields the unique solution $w(z,t)$ of (4.5) and its gradient $w_z(z,t)$. If we now compare $w(z,t)$, $w_z(z,t)$ with the approximations in (2.9), (2.10), we verify that (4.6), (4.7) are true. □

We will show that, for problem (4.5), there are explicit and unconditionally convergent difference schemes for appropriately filtered initial data such as $g_1(t)$ and $g_2(t)$ in (4.1), (4.2).

Rewrite the initial value problem (4.5) as an equivalent first order system

$$(4.10)\quad \begin{aligned} &w_z(z,t) = h(z,t), \quad 0 < z < 1, \quad -\infty < t < \infty,\\ &h_z = \tfrac{1}{2k}(w(z,t+k) - w(z,t-k)), \quad 0 < z < 1, \quad -\infty < t < \infty,\\ &w(0,t) = g_1(t), h(0,t) = g_2(t), \quad -\infty < t < \infty. \end{aligned}$$

Let Δz be a small increment in the z-variable and let $(N+1)\Delta z = 1$. We consider a difference scheme where only the z-variable is discretized, while the t-variable is left continuous. Let $w^n(t)$, $h^n(t)$ denote, respectively, $w(n\Delta z,t)$, $h(n\Delta z,t)$, for $0 \le n \le N+1$. The following difference approximation is explicit, consistent, and second order accurate in Δz.

$$w^{n+1}(t) = w(n\Delta z,t) + \Delta z w_z(n\Delta z,t) + \frac{\Delta z^2}{2}w_{zz}(n\Delta z,t)$$

$$= w^n(t) + \Delta z h^n(t) + \frac{\Delta z^2}{2}\frac{1}{2k}\left(w^n(t+k) - w^n(t-k)\right), \quad 0 \le n \le N, \tag{4.11}$$

$$\begin{aligned} h^{n+1}(t) &= h(n\Delta z, t) + \Delta z h_z(n\Delta z, t) + \frac{\Delta z^2}{2} h_{zz}(n\Delta z, t) \\ &= h(n\Delta z, t) + \Delta z \frac{1}{2k}(w(n\Delta z, t+k) - w(n\Delta z, t-k)) \\ &\quad + \frac{\Delta z^2}{2}\frac{1}{2k}(w_z(n\Delta z, t+k) - w_z(n\Delta z, t-k)) \\ &= h(n\Delta z, t) + \Delta z \frac{1}{2k}(w(n\Delta z, t+k) - w(n\Delta z, t-k)) \\ &\quad + \frac{\Delta z^2}{2}\frac{1}{2k}(h(n\Delta z, t+k) - h(n\Delta z, t-k)) \\ &= h^n(t) + \Delta z \frac{1}{2k}(w^n(t+k) - w^n(t-k)) \\ &\quad + \frac{\Delta z^2}{2}\frac{1}{2k}(h^n(t+k) - h^n(t-k)), \quad 0 \le n \le N, \end{aligned} \tag{4.12}$$

$$w^0(t) = g_1(t), \quad h^0(t) = g_2(t). \tag{4.13}$$

Now the following convergence result holds.

Theorem 4.2. *Let $w^n(t)$, $h^n(t)$ be the solutions of the difference scheme (4.11)-(4.13) and let $w(z,t)$, $h(z,t)$ be the solutions of (4.10). Then, as $n \to \infty$, $\Delta z \to 0$ and $n\Delta z = z$, we have, for each t,*

$$w^n(t) \to w(z,t), \quad h^n(t) \to h(z,t). \tag{4.14}$$

Proof. Taking Fourier transform about the t-variable in (4.11)-(4.13), we have

$$\widehat{w}^{n+1}(\xi) = \left(1 + \frac{\Delta z^2}{2}\frac{i\sin(k\xi)}{k}\right)\widehat{w}^n(\xi) + \Delta z \widehat{h}^n(\xi), \tag{4.15}$$

$$\widehat{h}^{n+1}(\xi) = \Delta z \frac{i\sin(k\xi)}{k}\widehat{w}^n(\xi) + \left(1 + \frac{\Delta z^2}{2}\frac{i\sin(k\xi)}{k}\right)\widehat{h}^n(\xi), \tag{4.16}$$

$$\widehat{w}^0(\xi) = \widehat{g}_\delta(\xi), \quad \widehat{h}^0(\xi) = \rho\widehat{g}_\delta(\xi).$$

Since ρ is given by (2.11), i.e.,

$$\rho = \sqrt{\frac{i\sin(k\xi)}{k}},$$

we get

$$\widehat{w}^1(\xi) = \left(1 + \frac{(\rho\Delta z)^2}{2}\right)\widehat{w}^0(\xi) + \Delta z \widehat{h}^0(\xi).$$

Note that $\widehat{h}^0(\xi) = \rho\widehat{g}_\delta(\xi) = \rho\widehat{w}^0(\xi)$, therefore

$$\widehat{w}^1(\xi) = \left(1 + \rho\Delta z + \frac{(\rho\Delta z)^2}{2}\right)\widehat{w}^0(\xi).$$

Similarly, we have

$$\widehat{h}^1(\xi) = \left(1 + \rho\Delta z + \frac{(\rho\Delta z)^2}{2}\right)\widehat{h}^0(\xi).$$

Denote

$$q(\xi) = 1 + \rho\Delta z + \frac{(\rho\Delta z)^2}{2}.$$

An simple induction argument using (4.15) and (4.16) shows that

$$\widehat{w}^n(\xi) = q^n(\xi)\widehat{w}^0(\xi),$$
$$\widehat{h}^n(\xi) = q^n(\xi)\widehat{h}^0(\xi).$$

Now doing the inverse Fourier transform, we get

$$w^n(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi t} q^n(\xi)\widehat{g}_\delta(\xi)d\xi, \tag{4.17}$$

$$h^n(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi t} q^n(\xi)\rho\widehat{g}_\delta(\xi)d\xi. \tag{4.18}$$

Now, $q^n(\xi) \to e^{\rho z}$ as $n \to \infty$. Furthermore,

$$|q^n(\xi)| \le \left(1 + \frac{|\rho|z}{n} + \frac{(|\rho|z)}{2n^2}\right)^n \le e^{|\rho|z} \le e^{|\rho|} = e^{1/\sqrt{k}}.$$

Hence, the Lebesgue dominated convergence theorem now shows that $w^n(t) \to w(z,t)$, $w_z^n(t) \to w_z(z,t)$ for each t. □

This theorem shows that the space marching scheme is stable for appropriate initial data. A similar result can be found in Carasso's famous paper [2].

5. Numerical tests

In this section we do some numerical tests intended to illustrate the effects of the proposed method. The tests are performed using MATLAB6.5.

The numerical examples are constructed in the following way: First we select the exact solution $f(t)$ of problem (1.1) at $x = 0$, and compute the data $u_x(0,t) = d(t)$, $u(1,t) = g(t)$ and $u_x(1,t) = p(t)$, by solving a well-posed problem (e.g., see [6]). Since these computed solutions are very close to the exact solutions, we will use them as the exact data in the following discussion. Then we add a normally distributed perturbation to each data function giving vectors g^δ and p^δ. Finally we solve the system (2.7) by the method of lines [4], i.e., we discretized the problem only with respect to the time variable t and left the spatial variable x continuous, and then we get a system of ordinary differential equations. The space marching is performed using a Runge-Kutta method (*ode45* in MATLAB) with automatic step size control, where the basic method is of order 4 and the embedded method is of order 5. In all tests the required accuracy in the R-K method is 10^{-4}.

We create $(g_i^\delta) \in \mathbb{R}^n$, which is a perturbation vector (normal distributed random error is added) of the discrete vector $(g_i) = (g(t_i))$ and $t_i = (i-1)\Delta t$, $\Delta t = \frac{1}{n-1}$, $i = 1, 2, \ldots, n$, such that

$$\|g^\delta - g\|_l := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(g_i^\delta - g_i)^2} = \delta.$$

We similarly created the perturbation vector $(h_i^\delta) \in \mathbb{R}^n$.

In the following, we first give an example which has the exact expression of the solution $u(x,t)$. This will be convenient when a reader tries to verify it.

Example 1. It is easy to verify that the function

$$u(x,t) = \begin{cases} \frac{x+1}{t^{3/2}} \exp\left\{-\frac{(x+1)^2}{4t}\right\}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{5.1}$$

is the exact solution of problem (1.1) with data

$$g(t) = \begin{cases} \frac{2}{t^{3/2}} \exp\left\{-\frac{1}{t}\right\}, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{5.2}$$

So

$$f(t) := u(0,t) = \begin{cases} \frac{1}{t^{3/2}} \exp\left\{-\frac{1}{4t}\right\}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{5.3}$$

and $\|f\|_{L^2(\mathbb{R})} \doteq 1.9999976$. We might as well take $E = 2$ in (2.6).

Test 1. We chose the perturbation $\delta \doteq 0.001$, the a priori bound $E = 2$, the regularization parameter k, i.e., the step length Δt of t-variable is $\Delta t = k = \frac{1}{(\ln(E/\delta))^2} \doteq \frac{1}{58}$, so $n = 59$. From Figure 1 we could find that the computed results are satisfactory when x is not small. However, the results are not very good as x closes to 0. This is consistent with Theorem 3.1.

In the following, we are interested in the "zero point" ($x = 0$), i.e., we only illustrate the computed results $v_x(0,t)$. The errors of the recovered v_x at $x = 0$ are measured by the weighted l^2-norm defined as

$$E(u_x) := \left(\frac{1}{n}\sum_{i=1}^{n}|v_x(0,t_i) - u_x(0,t_i)|^2\right)^{1/2}. \tag{5.4}$$

Test 2. Considering also Example 1, we take the perturbation $\delta \doteq 10^{-4}$ to compute the l^2 errors $E(u_x)$ by choosing different step length (see Table 1). Let $k_0 = \frac{1}{2\left(\ln\left(\frac{1}{\delta}\left(\ln\frac{1}{\delta}\right)^{-2}\right)\right)^2} \doteq 0.0223$. From Table 1, we can at least find two useful information. Firstly, the step length of t-variable k has a regularization effect. A better parameter choice is $k = 0.7k_0$. Secondly, Table 1 can also show

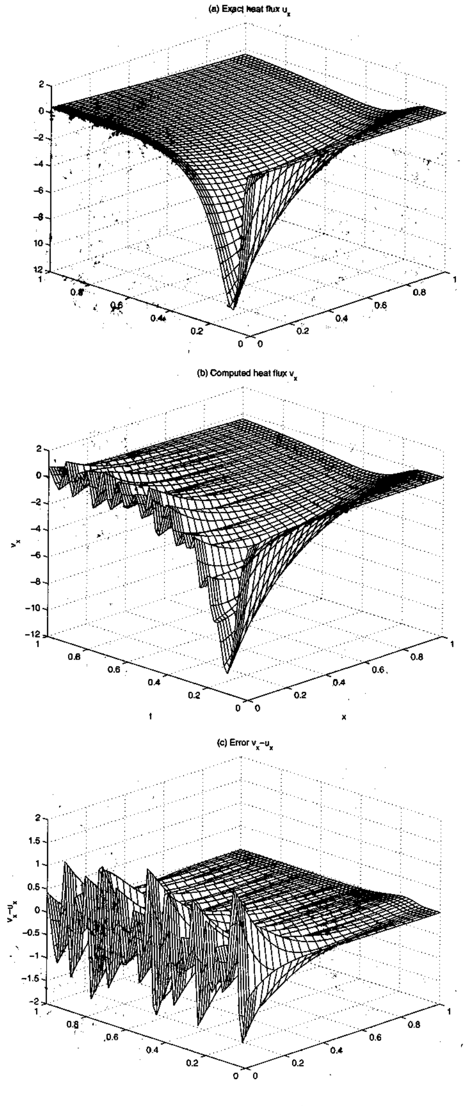

FIGURE 1. Example 1: (a) the exact heat flux u_x; (b) the computed heat flux v_x; (c) the error $v_x - u_x$.

that the quality of the computed heat flux v_x is not very sensitive to variations of the step length. Thus, in practice, it is relatively easy to find an appropriate value for k.

Test 3. In Figure 2, we give the comparison of the exact heat flux and computed approximation at $x = 0$ for perturbations $\delta = 10^{-3}$ and $\delta = 10^{-4}$.

TABLE 1. The l^2 errors of the computed heat flux $E(u_x)$, $k_0 \doteq 0.0223$

k	$0.3k_0$	$0.4k_0$	$0.5k_0$	$0.6k_0$	$0.7k_0$	$0.8k_0$	$1k_0$	$1.2k_0$	$1.5k_0$	$2k_0$
$E(u_x)$	2.071	0.729	0.505	0.432	0.338	0.533	0.809	0.841	1.034	1.391

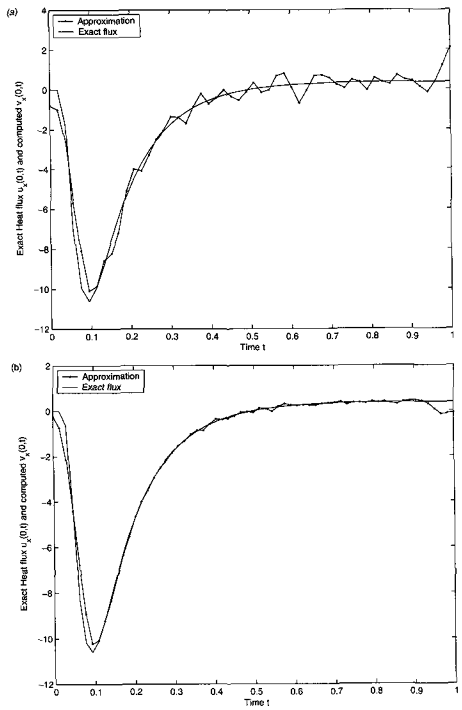

FIGURE 2. Example 1: (a) the exact $u_x(0,t)$ and computed $v_x(0,t)$, the perturbation $\delta \doteq 10^{-3}$, the step length $k \doteq \frac{1}{52}$; (b) the exact $u_x(0,t)$ and computed $v_x(0,t)$, $\delta \doteq 10^{-4}$, the step length $k \doteq \frac{1}{65}$.

Example 2. We consider a function that is not infinitely smooth:

$$f(t) = \begin{cases} 0, & 0 \le t \le 0.2, \\ 4t - 0.8, & 0.2 \le t \le 0.5, \\ 3.2 - 4t, & 0.5 \le t \le 0.8, \\ 0, & 0.8 \le t \le 1. \end{cases} \tag{5.5}$$

Test 4. Considering Example 2, in Figure 3, we also give the comparison of the exact heat flux and computed approximation at $x = 0$ for perturbations $\delta = 10^{-4}$ and $\delta = 10^{-5}$.

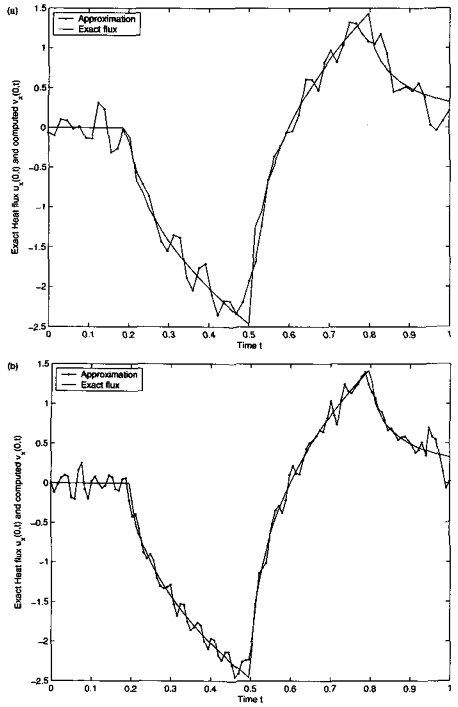

FIGURE 3. Example 2: (a) the exact $u_x(0,t)$ and computed $v_x(0,t)$, the perturbation $\delta \doteq 10^{-4}$, the step length $k \doteq \frac{1}{65}$; (b) the exact $u_x(0,t)$ and computed $v_x(0,t)$, $\delta \doteq 10^{-5}$, the step length $k \doteq \frac{1}{118}$.

From these tests, we conclude that the semi-discrete central difference method in time works well for an appropriate step length k which plays the role of regularization parameter. The idea will be taken into account when we consider other ill-posed problems.

6. Conclusion

In this paper, we developed Eldén's work [3]. We discussed the convergence of the surface heat flux of IHCP with respect to the perturbation data and obtained an explicit error estimate. With a stronger assumption on the regularity of the solution, the convergence estimate was obtained for the whole domain (i.e., including $x = 0$). We also discussed the stability of space marching difference and gave some interesting numerical examples to test the effects of the proposed method. The limitation of the proposed method we found in our numerical experiments was that the results are not satisfactory for problems with higher error levels. Fortunately, the problem could be solved when space-marching schemes were combined with initial filtering of the data, which reduces the sensitivity to perturbations in the data (e.g., [2] and [6]).

Acknowledgements. We thank the anonymous referee for her/his careful reading and helpful suggestions and comments. The project is supported by the NNSF of China (No. 10671085), the NSF of Gansu Province of China (No. 3ZS051-A25-015) and the Fundamental Research Fund for Physics and Mathematic of Lanzhou University (No. Lzu05005).

References

[1] J. V. Beck, B. Blackwell, and S. R. Clair, *Inverse Heat Conduction: Ill-posed problems*, John Wiley and Sons, Inc., 1985.

[2] A. Carasso, *Determining surface temperatures from interior observations*, SIAM J. Appl. Math. **42** (1982), no. 3, 558–574.

[3] L. Eldén, *Numerical solution of the sideways heat equation by difference approximation in time*, Inverse Problems **11** (1995), no. 4, 913–923.

[4] ———, *Solving the sideways heat equation by a 'method of lines'*, J. Heat Trans. ASME **119** (1997), 406–412.

[5] C. L. Fu and C. Y. Qiu, *Wavelet and error estimation of surface heat flux*, J. Comput. Appl. Math. **150** (2003), no. 1, 143–155.

[6] D. N. Hào, H.-J. Reinhardt, and A. Schneider, *Numerical solution to a sideways parabolic equation*, Internat. J. Numer. Methods Engrg. **50** (2001), no. 5, 1253–1267.

[7] Z. Qian, C. L. Fu, and X. T. Xiong, *Semidiscrete central difference method in time for determining surface temperatures*, Int. J. Math. Math. Sci. (2005), no. 3, 393–400.

[8] X. T. Xiong, C. L. Fu, and H. F. Li, *Central difference method of a non-standard inverse heat conduction problem for determining surface heat flux from interior observations*, Appl. Math. Comput. **173** (2006), no. 2, 1265–1287.

[9] ———, *Central difference schemes in time and error estimate on a non-standard inverse heat conduction problem*, Appl. Math. Comput. **157** (2004), no. 1, 77–91.

Zhi Qian
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000, P. R. China
E-mail address: qianzh03@163.com

Chu-Li Fu
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000, P. R. China
E-mail address: fuchuli@lzu.edu.cn