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THE BONNESEN-TYPE INEQUALITIES IN A PLANE OF CONSTANT CURVATURE*

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ABSTRACT. We investigate the containment measure of one domain to contain in another domain in a plane X^κ of constant curvature. We obtain some Bonnesen-type inequalities involving the area, length, radius of the inscribed and the circumscribed disc of a domain D in X^κ.

1. Introduction

A geometric inequality describes the relation between invariants of geometric subjects. Perhaps the best and the most remarkable one is the classical isoperimetric inequality that relates volume to area of a plane domain: Among domains with fixed areas the disc has the shortest circumlength. That is, the domain D with area A and length L satisfies

$$L^2 - 4\pi A \geq 0,$$

with equality if and only if D is a disc.

The isoperimetric inequality has been generalized to higher dimensions, which has been the object of much research in the last century, still going on today. Its applications reach algebra, differential geometry, differential equations and many mathematical areas. One can find the literature from references [1], [4], [5], [6].

The following inequalities are known (see [1], [4], [5], [9]).

Proposition 1. *Let D be a domain of area A and bonded by a simple closed curve of length L in the Euclidean plane R^2. Let r_i and r_e be, respectively, the radius of the inscribed disc and the circumscribed disc. Then for any disc of*

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radius r ($r_i \le r \le r_e$), we have the following inequalities:

$$
\begin{array}{lll}
 & L^2 - 4\pi A \ge 0; & L^2 - 4\pi A \ge \pi^2 (r_e - r_i)^2; \\
 & \pi r^2 - Lr + A \le 0; & L^2 - 4\pi A \ge (L - 2\pi r)^2; \\
 & L^2 - 4\pi A \ge \left(L - \frac{2A}{r}\right)^2; & L^2 - 4\pi A \ge \left(\frac{A}{r} - \pi r\right)^2; \\
(1.1) & L^2 - 4\pi A \ge A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2; & L^2 - 4\pi A \ge L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2; \\
 & L^2 - 4\pi A \ge A^2 \left(\frac{1}{r_i} - \frac{1}{r}\right)^2; & L^2 - 4\pi A \ge L^2 \left(\frac{r - r_i}{r + r_i}\right)^2; \\
 & L^2 - 4\pi A \ge A^2 \left(\frac{1}{r} - \frac{1}{r_e}\right)^2; & L^2 - 4\pi A \ge L^2 \left(\frac{r_e - r}{r_e + r}\right)^2; \\
 & \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \le r_i \le r \le r_e \le \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}; &
\end{array}
$$

Any one equality of above holds when and only when D is a disc. The second inequality of (1.1) *is called the Bonnesen isoperimetric inequality.*

We define the *isoperimetric deficit* of D as

$$\Delta(D) \equiv L^2 - 4\pi A.$$

Then we can see the geometric meaning of inequalities (1.1). $\Delta(D)$ measures the deficit between a domain D and the disc.

One hope to obtain the Bonnesen-type inequalities for domains in the higher dimension spaces. Refer to [3, 7, 8, 9, 10, 11, 12] for more results about the containment measures. In this paper, we hope to obtain the Bonnesen-type inequalities for domains in a plane X^κ of constant curvature κ. The methods could result isoperimetric inequalities for higher dimensions.

Let D_k ($k = i, j$) be a domain in the ambient space X^κ, the plane of constant curvature κ. Thus X^κ is either the Euclidean plane R^2 ($\kappa = 0$), the projective plane RP^2 ($\kappa > 0$), or the hyperbolic plane H^2 ($\kappa < 0$). We assume that ∂D_k is a rectifiable simple closed curve. The area and perimeter length of D_k is denoted by A_k and L_k, respectively, or simply A and L.

Let G_κ be the group of isometry in X^κ and dg be the kinematic measure (Haar measure in measure theory) on G_κ. We consider the following containment measure

$$
\begin{aligned}
m\{g \in G_\kappa : gD_j \subset D_i \text{ or } gD_j \supset D_i\} &= \int_{\{g \in G_\kappa : gD_j \subset D_i \text{ or } gD_j \supset D_i\}} dg \\
&= \int_{\{g \in G_\kappa : D_i \bigcap gD_j \neq \emptyset\}} dg - \int_{\{g \in G_\kappa : \partial D_i \bigcap g\partial D_j \neq \emptyset\}} dg.
\end{aligned} \tag{1.2}
$$

If we can estimate the last integral from above and the integral

$$\int_{\{g \in G_\kappa : D_i \bigcap gD_j \neq \emptyset\}} dg$$

from below in terms of geometric invariants of D_k, then we obtain an inequality of the form

$$m\{g \in G_\kappa : gD_j \supset D_i \text{ or } gD_j \subset D_i\} \ge f(I_i^1, \ldots, I_i^l; I_j^1, \ldots, I_j^l), \tag{1.3}$$

where each of I_k^α ($k = i, j$; $1 \le \alpha \le l$) is an integral geometric invariant of D_k.

One can then immediately state the following conclusions:

1. If $f(I_i^1,\ldots,I_i^l;I_j^1,\ldots,I_j^l)>0$ then there is an isometry $g\in G_\kappa$ such that either gD_j contains or is contained in D_i.

2. If one let $D_i\equiv D_j$ ($\equiv D$), then there is no $g\in G_\kappa$ such that $gD\subset D$ or $gD\supset D$. Hence we have

$$f(I^1(D),\ldots,I^l(D))\leq 0. \tag{1.4}$$

This is an geometric inequality of domain D.

3. Let D_i be, respectively, the in-disc and the out-disc of domain D_j ($\equiv D$), that is, the largest inscribed disc contained in D and the smallest circumscribed disc containing D. Then there is no $g\in G_\kappa$ such that $gD\subset D_i$ or $gD\supset D_i$. Therefore we have

$$f(I^1(D),\ldots,I^l(D),r_e)\leq 0,\ f(I^1(D),\ldots,I^l(D),r_i)\leq 0, \tag{1.5}$$

where r_e and r_i are, respectively, the circumscribed radius and inscribed radius of D. From these inequalities one will obtain the Bonnesen inequality (the second inequality in (1.1)) in a plane X^κ of constant curvature.

4. If one let D_i be a disc of radius r between the inscribed disc of radius r_i and the circumscribed disc of radius r_e of D_j ($\equiv D$). Then repeating the same procedure of above will lead to following inequality

$$f(I^1(D),\ldots,I^l(D),r)\leq 0;\qquad i\leq r\leq r_e. \tag{1.6}$$

It is usually called the Bonnesen-type inequality.

Above ideas is due to the first author (see [9, 10, 11, 12, 13, 14, 15]) and he obtain some Bonnesen-type inequalities in Proposition 1 for domain D in the Euclidean plane.

In this paper, we follow Zhou's idea and use the containment measure of Grinbeg, Ren and Zhou (see [2]) for a plane X^κ of constant curvature κ. We obtain some Bonnesen-type inequalities for domains in either a hyperbolic plane or a projective plane. Zhou's idea could result more Bonnesen-type inequalities for higher dimensions if appropriate containment measure of domains are achieved (see [10, 11, 12, 13, 14, 15]).

2. Bonnesen-type inequalities

Let D_k ($k=i,j$) be domains in a plane X^κ of constant curvature κ. For $g\in G_\kappa$ the group of isometry of X^κ. Grinberg, Ren and Zhou have the following containment measure inequality (see [2]):

$$\begin{aligned}&m\{g\in G_\kappa : gD_j\subset D_i \text{ or } gD_j\supset D_i\}\\ &=\int_{\{g\in G_\kappa:gD_j\subset D_i \text{ or } gD_j\supset D_i\}} dg\geq 2\pi(A_i+A_j)-L_iL_j-\kappa A_iA_j.\end{aligned} \tag{2.1}$$

If we let $D_i\equiv D_j\equiv D$, then there is no $g\in G_\kappa$ such that $gD\subset D$ or $gD\supset D$ and the containment measure inequality (2.1) immediately result in the following isoperimetric inequality in X^κ

$$L^2 - 4\pi A + \kappa A^2 \geq 0. \tag{2.2}$$

For a disc of radius r in the hyperbolic plane H^2, that is, $\kappa = \frac{-1}{\rho^2}$, we have

$$L = 2\pi\rho \sinh\frac{r}{\rho}, \quad A = 4\pi^2\rho^2 \sinh^2\frac{r}{2\rho}. \tag{2.3}$$

Therefore let $D_i = D$ and let D_j be a disc of radius r between the inscribed disc of radius r_i and the circumscribed disc of radius r_e of D. We have neither $gD_j \supset D$ nor $gD_j \subset D$ for any $g \in G_\kappa$. Then the measure $m\{g \in G_\kappa : gD_j \supset D \text{ or } gD_j \subset D\} = 0$ and the inequality (2.1) leads to

$$L\rho \sinh\frac{r}{\rho} - (4\pi\rho^2 + 2A)\sinh^2\frac{r}{2\rho} - A \geq 0, \qquad (r_i \leq r \leq r_e). \tag{2.4}$$

Using the equalities

$$\sinh 2x = 2\sinh x \cosh x, \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x} \tag{2.5}$$

and the formula (2.4) we have

$$2\rho L \tanh\frac{r}{2\rho} - A - (4\pi\rho^2 + A)\tanh^2\frac{r}{2\rho} \geq 0. \tag{2.6}$$

Letting

$$\psi(r) = 2\rho L \tanh\frac{r}{2\rho} - A - (4\pi\rho^2 + A)\tanh^2\frac{r}{2\rho} \qquad (r_i \leq r \leq r_e)$$

immediately gives

$$\frac{(2\rho L)^2}{4(4\pi\rho^2 + A)} - A = (4\pi\rho^2 + A)\left[\frac{2\rho L}{2(4\pi\rho^2 + A)} - \tanh\frac{r}{2\rho}\right]^2 + \psi(r). \tag{2.7}$$

In special cases when $r = r_i$ and r_e, respectively, the equality (2.7) also hold, that is,

$$\begin{cases} \frac{(2\rho L)^2}{4(4\pi\rho^2+A)} - A = (4\pi\rho^2 + A)\left[\frac{2\rho L}{2(4\pi\rho^2+A)} - \tanh\frac{r_i}{2\rho}\right]^2 + \psi(r_i), \\ \frac{(2\rho L)^2}{4(4\pi\rho^2+A)} - A = (4\pi\rho^2 + A)\left[\frac{2\rho L}{2(4\pi\rho^2+A)} - \tanh\frac{r_e}{2\rho}\right]^2 + \psi(r_e). \end{cases} \tag{2.8}$$

Since $\psi(r) \geq 0 \ \ (r_i \leq r \leq r_e)$, we have

$$\begin{cases} \frac{(2\rho L)^2}{4(4\pi\rho^2+A)} - A \geq (4\pi\rho^2 + A)\left[\frac{2\rho L}{2(4\pi\rho^2+A)} - \tanh\frac{r_i}{2\rho}\right]^2, \\ \frac{(2\rho L)^2}{4(4\pi\rho^2+A)} - A \geq (4\pi\rho^2 + A)\left[\tanh\frac{r_e}{2\rho} - \frac{2\rho L}{2(4\pi\rho^2+A)}\right]^2. \end{cases} \tag{2.9}$$

By adding two inequalities of (2.9) we have

$$\frac{(2\rho L)^2}{4(4\pi\rho^2 + A)} - A \geq \frac{(4\pi\rho^2 + A)}{4}\left(\tanh\frac{r_e}{2\rho} - \tanh\frac{r_i}{2\rho}\right)^2, \tag{2.10}$$

that is,

$$L^2 - 4\pi A - \frac{A^2}{\rho^2} \geq \frac{(4\pi\rho^2 + A)^2}{4\rho^2}\left(\tanh\frac{r_e}{2\rho} - \tanh\frac{r_i}{2\rho}\right)^2. \tag{2.11}$$

We proved the following:

Theorem 1. *Let D be a domain of area A and bonded by a simple closed curve of length L in the hyperbolic plane H^2. Let r_i and r_e be, respectively, the radius of the inscribed disc and the circumscribed disc. Then for any disc of radius r $(r_i \leq r \leq r_e)$, we have the following inequalities:*

$$\begin{aligned} &L^2 - 4\pi A - \tfrac{A^2}{\rho^2} \geq 0;\\ &L^2 - 4\pi A - \tfrac{A^2}{\rho^2} \geq \tfrac{(4\pi\rho^2+A)^2}{4\rho^2}\left(\tanh\tfrac{r_e}{2\rho} - \tanh\tfrac{r_i}{2\rho}\right)^2;\\ &(4\pi\rho^2 + A)\tanh^2\tfrac{r}{2\rho} - 2L\rho\tanh\tfrac{r}{2\rho} + A \leq 0. \end{aligned} \tag{2.12}$$

The second inequality of (2.12) can be rewritten in several equivalent forms:

Theorem 2. *Let D be a domain of area A and bonded by a simple closed curve of length L in the hyperbolic plane H^2. Let r_i and r_e be, respectively, the radius of the inscribed disc and the circumscribed disc. Then for any disc of radius r $(r_i \leq r \leq r_e)$, we have the following inequalities:*

$$\begin{aligned} &L^2 - 4\pi A - \tfrac{A^2}{\rho^2} \geq \left(L - \tfrac{4\pi\rho^2+A}{\rho}\tanh\tfrac{r}{2\rho}\right)^2;\\ &L^2 - 4\pi A - \tfrac{A^2}{\rho^2} \geq \left(L - \tfrac{A}{\rho\tanh\frac{r}{2\rho}}\right)^2;\\ &L^2 - 4\pi A - \tfrac{A^2}{\rho^2} \geq \left[\tfrac{A}{2\rho\tanh\frac{r}{2\rho}} - (2\pi\rho + \tfrac{A}{2\rho})\tanh\tfrac{r}{2\rho}\right]^2. \end{aligned} \tag{2.13}$$

From the second formula of (2.13) we have

$$\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}} \geq L - \frac{A}{\rho\tanh\frac{r_e}{2\rho}}; \qquad \sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}} \geq \frac{A}{\rho\tanh\frac{r_i}{2\rho}} - L. \tag{2.14}$$

Adding inequalities (2.14) yields

$$L^2 - 4\pi A - \frac{A^2}{\rho^2} \geq \frac{A^2}{4\rho^2}\left(\frac{1}{\tanh\frac{r_i}{2\rho}} - \frac{1}{\tanh\frac{r_e}{2\rho}}\right)^2. \tag{2.15}$$

Adding inequalities (2.14) after multiplied by $\frac{1}{\rho\tanh\frac{r_i}{2\rho}}$ and $\frac{1}{\rho\tanh\frac{r_e}{2\rho}}$, respectively, gives

$$L^2 - 4\pi A - \frac{A^2}{\rho^2} \geq L^2\left(\frac{\tanh\frac{r_e}{2\rho} - \tanh\frac{r_i}{2\rho}}{\tanh\frac{r_e}{2\rho} + \tanh\frac{r_i}{2\rho}}\right)^2. \tag{2.16}$$

Notice that the equation $2\rho L\tanh\frac{r}{2\rho} - A - (4\pi\rho^2 + A)\tanh^2\frac{r}{2\rho} = 0$ has two roots

$$\tanh\frac{r_k}{2\rho} = \frac{\rho L \pm \rho\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}}}{4\pi\rho^2 + A}; \quad k = i, e. \tag{2.17}$$

So we obtain
(2.18)

$$\frac{\rho L - \rho\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}}}{4\pi\rho^2 + A} \leq \tanh\frac{r_i}{2\rho} \leq \tanh\frac{r_e}{2\rho} \leq \frac{\rho L + \rho\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}}}{4\pi\rho^2 + A},$$

and we proved the following

Theorem 3. *Let D be a domain of area A and bonded by a simple closed curve of length L in the hyperbolic plane H^2. Let r_i and r_e be, respectively, the radius of the inscribed disc and the circumscribed disc. Then we have*

$$\begin{aligned} &L^2 - 4\pi A - \frac{A^2}{\rho^2} \geq \frac{A^2}{4\rho^2}\left(\frac{1}{\tanh\frac{r_i}{2\rho}} - \frac{1}{\tanh\frac{r_e}{2\rho}}\right)^2; \\ &L^2 - 4\pi A - \frac{A^2}{\rho^2} \geq L^2\left(\frac{\tanh\frac{r_e}{2\rho} - \tanh\frac{r_i}{2\rho}}{\tanh\frac{r_e}{2\rho} + \tanh\frac{r_i}{2\rho}}\right)^2; \\ &\frac{\rho L - \rho\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}}}{4\pi\rho^2 + A} \leq \tanh\frac{r_i}{2\rho} \leq \tanh\frac{r_e}{2\rho} \leq \frac{\rho L + \rho\sqrt{L^2 - 4\pi A - \frac{A^2}{\rho^2}}}{4\pi\rho^2 + A}. \end{aligned} \tag{2.19}$$

Each equality holds when and only when D is a disc.

The equation $2\rho L\tanh\frac{r}{2\rho} - A - (4\pi\rho^2 + A)\tanh^2\frac{r}{2\rho} = 0$ has an unique root when and only when $L^2 - 4\pi A - \frac{A^2}{\rho^2} = 0$. This leads to $\tanh\frac{r_i}{2\rho} = \tanh\frac{r_i}{2\rho}$. We conclude that *each equality of those inequalities in Theorem 1, Theorem 2 and Theorem 3 holds when and only when D is a geodesic disc.*

In the case of that D is a domain in the projective plane PR^2, that is, $\kappa = \frac{1}{\rho^2}$. For a geodesic disc of radio r we have

$$L = 2\pi\rho\sin(\frac{r}{\rho}), \quad A = 4\pi\rho^2\sin^2(\frac{r}{2\rho}), \quad (\text{where} \quad r \leq \frac{\pi}{2}\rho). \tag{2.20}$$

We use the same method we just used in hyperbolic plane, let $D_i \equiv D$ and let D_j be a disc of radius r between the inscribed disc of radius r_i and the circumscribed disc of radius r_e of D (where we assume that the $r_e \leq \frac{\pi}{2}\rho$). We have neither $gD_j \supset D$ nor $gD_j \subset D$ for any $g \in G_\kappa$. Then the measure $m\{g \in G_\kappa : gD_j \supset D \text{ or } gD_j \subset D\} = 0$ and the inequality (2.1) leads to

$$2\rho L\tan\frac{r}{2\rho} - A - (4\pi\rho^2 - A)\tan^2\frac{r}{2\rho} \geq 0. \tag{2.21}$$

Let

$$\phi(r) = 2\rho L\tan\frac{r}{2\rho} - A - (4\pi\rho^2 - A)\tan^2\frac{r}{2\rho} \geq 0. \tag{2.22}$$

Then we obtain

$$\frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A = (4\pi\rho^2 - A)\left[\frac{2\rho L}{2(4\pi\rho^2 - A)} - \tan\frac{r}{2\rho}\right]^2 + \phi(r).$$

In special cases when $r = r_i$ and r_e, respectively, the equality (2.22) also hold, that is,

$$\text{(2.23)}\quad \begin{cases} \frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A = (4\pi\rho^2 - A)\left[\frac{2\rho L}{2(4\pi\rho^2 - A)} - \tan\frac{r_i}{2\rho}\right]^2 + \phi(r_i), \\ \frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A = (4\pi\rho^2 - A)\left[\frac{2\rho L}{2(4\pi\rho^2 - A)} - \tan\frac{r_e}{2\rho}\right]^2 + \phi(r_e). \end{cases}$$

Since $\phi(r) \geq 0$ $(r_i \leq r \leq r_e)$, we have

$$\text{(2.24)}\quad \begin{cases} \frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A \geq (4\pi\rho^2 - A)\left[\frac{2\rho L}{2(4\pi\rho^2 - A)} - \tan\frac{r_i}{2\rho}\right]^2, \\ \frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A \geq (4\pi\rho^2 - A)\left[\tan\frac{r_e}{2\rho} - \frac{2\rho L}{2(4\pi\rho^2 - A)}\right]^2. \end{cases}$$

By adding two inequalities of (2.24) we have

$$\text{(2.25)}\qquad \frac{(2\rho L)^2}{4(4\pi\rho^2 - A)} - A \geq \frac{(4\pi\rho^2 - A)}{4}\left(\tan\frac{r_e}{2\rho} - \tan\frac{r_i}{2\rho}\right)^2,$$

that is,

$$\text{(2.26)}\qquad L^2 - 4\pi A + \frac{A^2}{\rho^2} \geq \frac{(4\pi\rho^2 - A)^2}{4\rho^2}\left(\tan\frac{r_e}{2\rho} - \tan\frac{r_i}{2\rho}\right)^2.$$

If we let $D_i \equiv D_j \equiv D$, then the containment measure inequality for the case of projective plan PR^2 gives

$$\text{(2.27)}\qquad L^2 - 4\pi A + \frac{A^2}{\rho^2} \geq 0.$$

Since

$$\phi(r) = 2\rho L \tan\frac{r}{2\rho} - A - (4\pi\rho^2 - A)\tan^2\frac{r}{2\rho} = 0,$$

has two roots for $\tan\frac{r}{2\rho}$:

$$\text{(2.28)}\qquad \tan\frac{r_k}{2\rho} = \frac{\rho L \pm \rho\sqrt{L^2 - 4\pi A + \frac{A^2}{\rho^2}}}{4\pi\rho^2 - A};\quad k = i,\, e,$$

therefore we have

(2.29)

$$\frac{\rho L - \rho\sqrt{L^2 - 4\pi A + \frac{A^2}{\rho^2}}}{4\pi\rho^2 - A} \leq \tan\frac{r_i}{2\rho} \leq \tan\frac{r_e}{2\rho} \leq \frac{\rho L + \rho\sqrt{L^2 - 4\pi A + \frac{A^2}{\rho^2}}}{4\pi\rho^2 - A}.$$

The equation $\phi(r) = 0$ has a unique root when and only when $L^2 - 4\pi A + \frac{A^2}{\rho} = 0$ hence $\tan\frac{r_i}{2\rho} = \tan\frac{r_e}{2\rho}$. This means that D is domain bounded by a geodesic circle ∂D.

We proved the following

Theorem 4. *Let D be a domain of area A and bonded by a simple closed curve of length L in the projective plane PR^2. Let r_i and r_e ($r_e \le \frac{\pi}{2}\rho$) be, respectively, radius of the inscribed disc and the circumscribed disc. Then for any disc of radius r ($r_i \le r \le r_e$), we have the following inequalities*:

$$\begin{aligned}
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge 0;\\
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge \tfrac{(4\pi\rho^2 - A)^2}{4\rho^2}\left(\tan\tfrac{r_e}{2\rho} - \tan\tfrac{r_i}{2\rho}\right)^2;\\
&A - 2\rho L \tan\tfrac{r}{2\rho} + (4\pi\rho^2 - A)\tan^2\tfrac{r}{2\rho} \le 0;\\
&\tfrac{\rho L - \rho\sqrt{L^2 - 4\pi A + \frac{A^2}{\rho^2}}}{4\pi\rho^2 - A} \le \tan\tfrac{r_i}{2\rho} \le \tan\tfrac{r_e}{2\rho} \le \tfrac{\rho L + \rho\sqrt{L^2 - 4\pi A + \frac{A^2}{\rho^2}}}{4\pi\rho^2 - A}.
\end{aligned} \tag{2.30}$$

Each equality holds when and only when D is a geodesic disc.

The second inequality of (2.30) in Theorem 4 can be rewritten in several equivalent forms, that is:

Theorem 5. *Let D be a domain of area A and bonded by a simple closed curve of length L in the projective plane PR^2. Let r_i and r_e ($r_e \le \frac{\pi}{2}\rho$) be, respectively, radius of the inscribed disc and the circumscribed disc. Then for any disc of radius r ($r_i \le r \le r_e$), we have*

$$\begin{aligned}
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge \left(L - \tfrac{4\pi\rho^2 - A}{\rho}\tan\tfrac{r}{2\rho}\right)^2;\\
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge \left(L - \tfrac{A}{\rho\tan\frac{r}{2\rho}}\right)^2;\\
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge \left[\tfrac{A}{2\rho\tan\frac{r}{2\rho}} - (2\pi\rho - \tfrac{A}{2\rho})\tan\tfrac{r}{2\rho}\right]^2.
\end{aligned} \tag{2.31}$$

From the second formula of (2.31) we have

$$L^2 - 4\pi A + \frac{A^2}{\rho^2} \ge \frac{A^2}{4\rho^2}\left(\frac{1}{\tan\frac{r_i}{2\rho}} - \frac{1}{\tan\frac{r_e}{2\rho}}\right)^2, \tag{2.32}$$

and

$$L^2 - 4\pi A + \frac{A^2}{\rho^2} \ge L^2\left(\frac{\tan\frac{r_e}{2} - \tan\frac{r_i}{2}}{\tan\frac{r_e}{2} + \tan\frac{r_i}{2}}\right)^2, \tag{2.33}$$

that is

Theorem 6. *Let D be a domain of area A and bonded by a simple closed curve of length L in the projective plane PR^2. Let r_i and r_e ($r_e \le \frac{\pi}{2}\rho$) be, respectively, radius of the inscribed disc and the circumscribed disc. Then we have*

$$\begin{aligned}
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge \tfrac{A^2}{4\rho^2}\left(\tfrac{1}{\tan\frac{r_i}{2\rho}} - \tfrac{1}{\tan\frac{r_e}{2\rho}}\right)^2;\\
&L^2 - 4\pi A + \tfrac{A^2}{\rho^2} \ge L^2\left(\tfrac{\tan\frac{r_e}{2} - \tan\frac{r_i}{2}}{\tan\frac{r_e}{2} + \tan\frac{r_i}{2}}\right)^2.
\end{aligned} \tag{2.34}$$

Each equality holds when and only when D is a geodesic disc.

The Bonnesen-type inequalities in higher dimensional space are still unknown for many cases. Zhang [8] has some results for convex domain D. The Willmore functional inequalities of Bonnesen-type is investigated by the first author (see [10, 11, 12, 13, 14, 15] for more details).

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