J. Korean Math. Soc. **44** (2007), No. 6, pp. 1339–1350

ON THE GEOMETRY OF BIHYPERELLIPTIC CURVES

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ABSTRACT. Here we consider bihyperelliptic curves, i.e., double covers of hyperelliptic curves. By applying the theory of quadruple covers, among other things we prove that the bihyperelliptic locus in the moduli space of smooth curves is irreducible and unirational for $g \geq 4\gamma + 2 \geq 10$.

0. Introduction and notation

Let $\mathbb{C}$ be the complex field and let $\mathcal{M}_g$ be the coarse moduli space of smooth projective curves of genus g over $\mathbb{C}$. The aim of this paper is to deal with the bihyperelliptic locus $\mathcal{B}_g^\gamma \subseteq \mathcal{M}_g$ i.e., the locus of curves which are double covers of hyperelliptic curves of genus γ.

When $\gamma = 0, 1$ such loci reduce to the well known hyperelliptic and bielliptic loci respectively, which are irreducible and unirational of respective dimensions $2g - 1$ and $2g - 2$ (they are actually rational for $\gamma = 0$ and for $\gamma = 1$ and $g = 3, 4, 5$: see [3], [14] and the references cited there).

The case $\gamma \geq 2$ seems to be still widely open despite its interest in some different contexts. Indeed, $\mathcal{B}_g^\gamma$ are among the components of the singular locus of $\mathcal{M}_g$ when $\gamma = 0, 1, 2$ (see [6]). In [11] the authors describe the locus inside $\mathcal{M}_g$ corresponding to bihyperelliptic curves with more bihyperelliptic structures, proving it has many connected components. Moreover, bihyperelliptic curves arise naturally in the setup of Prym and Jacobian varieties (see [10], Section 3). Finally, even more general double covers have been recently addressed in [7] from the point of view of the slope of fibrations.

The main result of the present paper is the following

Theorem. *For $g \geq 4\gamma + 2 \geq 10$ the loci $\mathcal{B}_g^\gamma$ are irreducible and unirational of dimension $g - \gamma + 2[(g - \gamma)/2] + 1$.*

In order to prove the above statement we observe that bihyperelliptic curves are particular tetragonal curves, so that we can apply the theory of quadruple covers (see [4]). Therefore we are able to associate to each bihyperelliptic curve two locally free sheaves on $\mathbb{P}^1_{\mathbb{C}}$ and a section of a particular tensor product of them. We are thus in the position to compute the splitting type of such

Received March 8, 2006.
2000 *Mathematics Subject Classification.* 14H10, 14H45, 14E20.
Key words and phrases. curves, moduli, cover, unirationality.

sheaves and to identify exactly the sections defining bihyperelliptic curves (see Proposition 2.5 and Remark 2.6). Via such a description we are also able to define a stratification of $\mathcal{B}_g^\gamma$, whose strata are quotients of suitable projective spaces with respect to the action of some explicit matrix groups, in particular we show that each stratum is irreducible and unirational (see Proposition 3.4).

The question about the rationality of $\mathcal{B}_g^\gamma$ is rather intriguing, but it seems to be quite hard, as well as the case of double coverings of arbitrary smooth curves of genus $\gamma \geq 3$. We leave both these points as open problems worth of further investigation.

Notation

As usual we denote by $\mathcal{O}_X$ and ω_X the structure sheaf and the canonical sheaf of the irreducible, smooth, projective variety X. As customary, we denote by $\mathcal{M}_g$ the moduli space of smooth curves of genus g.

If $\mathcal{F}$ is a locally free $\mathcal{O}_X$–sheaf and $s \in H^0(X, \mathcal{F})$ we denote by $D_0(s)$ the subscheme of X locally defined by the vanishing of s, i.e., set–theoretically $D_0(s) := \{x \in X |\ s(x) = 0\}$. If $\mathcal{E}$ is any $\mathcal{O}_X$–sheaf then we denote its dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ by $\check{\mathcal{E}}$.

If g is an element of a certain group G then $\langle g \rangle$ denotes the subgroup of G generated by g.

In the sequel, we will also introduce the shorthands $\alpha_{\min} := [(g - 2\gamma + 1)/2]$ and $\alpha_{\max} := [(g - \gamma + 2)/2]$.

For other notations and definitions we always refer to [12].

1. Generalities about tetragonal curves

We begin by summarizing some results about covers of $\mathbb{P}^1_{\mathbb{C}}$. Recall that if C is a smooth curve, then a morphism $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ is a cover of degree 4 if it is quasi–finite of degree 4. We refer to [4] for results about covers of degree 4.

Let $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ be a cover of degree 4. There exists a natural exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \xrightarrow{\varrho^\#} \varrho_* \mathcal{O}_C \to \check{\mathcal{E}} \to 0,$$

where $\check{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$–sheaf of rank 3 called the *Tschirhausen module of* ϱ. The sequence above splits (see [4]) and we obtain a decomposition $\varrho_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \check{\mathcal{E}}$. In particular, for each $n \in \mathbb{Z}$,

$$h^i\big(C, \varrho^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\big) = h^i\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\big) + h^i\big(\mathbb{P}^1_{\mathbb{C}}, \check{\mathcal{E}}(n)\big). \tag{1.2}$$

Moreover the relative dualizing sheaf $\omega_{C|\mathbb{P}^1_{\mathbb{C}}} \cong \omega_C \otimes \varrho^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)$ is defined and invertible. By relative duality $\varrho_* \omega_{C|\mathbb{P}^1_{\mathbb{C}}} \cong \big(\varrho_* \mathcal{O}_C\big)^\vee \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{E}$ (see [12], exercise III 6.10 b)).

Theorem 1.3. *Let C be a smooth, integral curve and $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ a cover of degree* 4.

There exists an embedding $i\colon C \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{E})$ ($\check{\mathcal{E}}$ *is the Tschirnhausen module of* ϱ) *such that* $\omega_{C|\mathbb{P}^1_{\mathbb{C}}} \cong i^*\mathcal{O}_{\mathbb{P}}(1)$ *and* $\varrho = \pi \circ i$, $\pi\colon \mathbb{P} \to \mathbb{P}^1_{\mathbb{C}}$ *being the natural projection.*

There exists a locally free $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$*-sheaf* $\mathcal{F}$ *of rank* 2 *such that* $\det \mathcal{F} \cong \det \mathcal{E}$, *and fitting into an exact sequence of the form*

$$0 \to \pi^* \det \mathcal{E}(-4) \to \pi^* \mathcal{F}(-2) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_C \to 0. \tag{1.3.1}$$

Sequence (1.3.1) *is unique up to isomorphism.*

The restriction to $\mathbb{P}_y := \pi^{-1}(y) \cong \mathbb{P}^2_{\mathbb{C}}$ *of sequence* (1.3.1) *is a minimal free resolution of the structure sheaf of* $C_y := \varrho^{-1}(y)$. *In particular* $C_y \subseteq \mathbb{P}_y$ *is the complete intersection of two conics, hence* C_y *is not contained in any line* $r \in \check{\mathbb{P}}_y$.

Proof. See [4], Theorem 2.1. □

Twisting sequence (1.3.1) by $\mathcal{O}_{\mathbb{P}}(2)$ and applying π_* we obtain

$$0 \to \mathcal{F} \to \mathcal{S}^2 \mathcal{E} \to \varrho_* \omega^2_{C|\mathbb{P}^1_{\mathbb{C}}} \to 0. \tag{1.4}$$

Since every locally free $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$–sheaf splits into a direct sum of invertible sheaves we can introduce the following

Definition 1.5. Let $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ be a cover of degree 4, and fix decompositions $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha_i)$, $\alpha_1 \le \alpha_2 \le \alpha_3$, $\mathcal{F} \cong \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\beta_j)$, $\beta_1 \le \beta_2$.

The sequence $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$ is called the scrollar sequence of the cover ϱ.

We have three monomorphisms $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha_i) \rightarrowtail \mathcal{E}$ into the three summands, hence three fibrewise independent sections $u \in H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\alpha_1)\big) \cong H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-\alpha_1)\big)$, $v \in H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\alpha_2)\big) \cong H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-\alpha_2)\big)$, $w \in H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\alpha_3)\big) \cong H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-\alpha_3)\big)$. We set $U := D_0(u)$, $V := D_0(v)$, $W := D_0(w)$.

Notice that $\delta \in \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}}}\big(\pi^*\mathcal{F}(-2), \mathcal{O}_{\mathbb{P}}\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\check{\mathcal{F}}\big)$. Via the isomorphism

$$\begin{aligned} H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\check{\mathcal{F}}\big) \cong\ & H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\beta_1)\big) \\ & \oplus H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\beta_2)\big), \end{aligned}$$

we can identify δ with a pair (a, b) where

$$\begin{aligned} a(u,v,w) &= a_{2\alpha_1-\beta_1}u^2 + 2a_{\alpha_1+\alpha_2-\beta_1}uv + 2a_{\alpha_1+\alpha_3-\beta_1}uw + \\ &\quad + a_{2\alpha_2-\beta_1}v^2 + 2a_{\alpha_2+\alpha_3-\beta_1}vw + a_{2\alpha_3-\beta_1}w^2, \\ b(u,v,w) &= b_{2\alpha_1-\beta_2}u^2 + 2b_{\alpha_1+\alpha_2-\beta_2}uv + 2b_{\alpha_1+\alpha_3-\beta_2}uw + \\ &\quad + b_{2\alpha_2-\beta_2}v^2 + 2b_{\alpha_2+\alpha_3-\beta_2}vw + b_{2\alpha_3-\beta_2}w^2, \end{aligned} \tag{1.6}$$

$a_h,\ b_h \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h)\big)$. In particular $C = A \cap B$ where $A := D_0(a)$, $B := D_0(b)$.

Here we list some helpful remarks that will be used in the following sections.

Remark 1.7. a) Let $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ be a cover of degree 4 having $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$ as scrollar sequence. Then $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 = g + 3$.

b) If C is smooth and irreducible then $\alpha_1 \geq 1$ (see [13], Proposition 1.2) and $\beta_1 \leq 2\alpha_1$, otherwise $V \cap W \subseteq C$ in contradiction with the irreducibility of C. Conversely if $\alpha_1 \geq 1$, then C is connected (formula (1.2) with $i = n = 0$), hence, if C is smooth, then it is also irreducible.

c) Finally, since C is complete intersection locally over $\mathbb{P}^1_{\mathbb{C}}$ of relative conics, then its ideal is generated locally over $\mathbb{P}^1_{\mathbb{C}}$ by any two relative conics containing C without common components. Since $\beta_1 \leq \beta_2$ one can take any two relative conics linearly equivalent to A.

Conversely we point out the following fact.

Theorem 1.8. *Let $\mathcal{E}$, $\mathcal{F}$ be as above. we denote by $\mathbb{P}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)}$ the projective space associated to $H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\check{\mathcal{F}}\big)$. The subsets*

$$\begin{aligned}&\mathcal{U}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)}\\ &:= \{(a,b) \in \mathbb{P}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)} \mid D_0(a,b) \text{ is a smooth and connected}\}\end{aligned}$$

are open inside $\mathbb{P}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)}$.

Proof. See [4], Theorem 4.5 and [2], Theorem 2.5. □

2. Bihyperelliptic curves

In the following C will always denote a fixed curve of genus g endowed with an $i_C \in \mathrm{Aut}(C)$ of order two such that $\Gamma := C/\langle i_C\rangle$ is a smooth projective curve of genus $\gamma \geq 0$. The automorphism i_C is said to be an *involution of genus γ*.

Definition 2.1. With the above notations we say that C is bihyperelliptic if Γ is hyperelliptic. The locus inside $\mathcal{M}_g$ of points representing bihyperelliptic curves which are double covers of a curve of genus γ will be denoted by $\mathcal{B}^\gamma_g$.

Clearly each bihyperelliptic curve is trivially tetragonal. The composition of the natural morphism $\psi\colon C \to \Gamma := C/\langle i_C\rangle$ with any morphism $\varphi\colon \Gamma := C/\langle i_C\rangle \to \mathbb{P}^1_{\mathbb{C}}$ of degree 2, is a cover $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ of degree 4.

Lemma 2.2. *Let $C \in \mathcal{M}_g$ be a tetragonal curve and let $i_C \in \mathrm{Aut}(C)$ be an involution of genus γ. If $g \geq 2\gamma + 4$ then each g^1_4 on C is composed with i_C and $C \in \mathcal{B}^\gamma_g$. If $\gamma \geq 2$ then $C \notin B^1_g$. If $g \geq 4\gamma + 2$ then i_C is the unique involution of genus at most γ in $\mathrm{Aut}(C)$.*

Proof. Each cover $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ of degree 4 factorizes through $C \to \Gamma := C/\langle i_C\rangle$ by the Castelnuovo–Severi inequality (see [1], Theorem 3.5). The same inequality also guarantees that $\mathcal{B}^\gamma_g \cap \mathcal{B}^1_g = \emptyset$ if $\gamma \geq 2$. If C carries also another involution of genus $\gamma' \leq \gamma$ then, again by Castelnuovo–Severi, $g \leq 4\gamma + 1$. □

Clearly $\mathcal{B}_g^0$ is the hyperelliptic locus, so we will assume $\gamma \geq 1$ from now on. Notice that $\mathcal{B}_g^1$ is the bielliptic locus which has been already studied from the scrollar sequence viewpoint (see Section 3 of [8]: see also [9] and [3]).

Again by Castelnuovo–Severi each curve $C \in \mathcal{M}_g$, $g \geq 10$, carries at most one g_4^1 unless $C \in \mathcal{B}_g^1$. Thus if $C \in \mathcal{B}_g^\gamma$ for $g \geq 4\gamma + 2 \geq 10$ the following definition makes sense.

Definition 2.3. Let $C \in \mathcal{B}_g^\gamma$. If $g \geq 4\gamma + 2 \geq 10$, the scrollar sequence of C is the scrollar sequence of the uniquely associated cover $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ of degree 4.

From now on we will always assume that $C \in \mathcal{B}_g^\gamma$ with $g \geq 2\gamma + 4 \geq 6$ and we will denote by $\psi\colon C \to \Gamma$ the morphism of degree 2 induced by i_C, the involution of genus γ. Thus $\psi_* \mathcal{O}_C \cong \mathcal{O}_\Gamma \oplus \mathcal{L}^{-1}$ for some $\mathcal{L} \in \mathrm{Pic}(\Gamma)$ such that $H^0(\Gamma, \mathcal{L}^2)$ has a section with reduced zero–locus on Γ. For each morphism $\varphi\colon \Gamma \to \mathbb{P}^1_{\mathbb{C}}$ of degree 2 we have $\varphi_* \mathcal{O}_\Gamma \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\gamma - 1)$ by the Hurwitz formula. We conclude that for each cover $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ of degree 4 we have the following isomorphism

$$\varrho_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\gamma - 1) \oplus \varphi_* \mathcal{L}^{-1}. \tag{2.4}$$

Proposition 2.5. *Let $C \in \mathcal{M}_g$ and let γ be a positive integer such that $g \geq 4\gamma + 2 \geq 6$.*

i) *If $C \in \mathcal{B}_g^\gamma$ then its scrollar sequence is*

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) = (\gamma + 1, \alpha, g - \gamma + 2 - \alpha, 2\gamma + 2, g - 2\gamma + 1), \tag{2.5.1}$$

where $\gamma + 1 < [(g - 2\gamma + 2)/2] =: \alpha_{\min} \leq \alpha \leq \alpha_{\max} := [(g - \gamma + 2)/2]$ and

$$\alpha = \min\{\, h \in \mathbb{Z} \mid \varphi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}{}^*(h) \otimes \mathcal{L}^{-1} \text{ is effective } \}$$

where $\mathcal{L}$ and φ are the ones defined above. Moreover $b_{\alpha_1+\alpha_2-\beta_2} = b_{\alpha-g+3\gamma} = 0$, $b_{\alpha_1+\alpha_3-\beta_2} = b_{2\gamma+2-\alpha} = 0$ in Equations (1.6).

ii) *If $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ is a cover of degree 4 with scrollar sequence (2.5.1) and $b_{\alpha_1+\alpha_2-\beta_2} = b_{\alpha-g+3\gamma} = 0$, $b_{\alpha_1+\alpha_3-\beta_2} = b_{2\gamma+2-\alpha} = 0$ in Equations (1.6) then $C \in \mathcal{B}_g^\gamma$.*

Proof. The case $\gamma = 1$ has been already described in Section 3 of [8] (see also [9] and [3]), thus we can assume $\gamma \geq 2$, hence $g \geq 4\gamma + 2 \geq 10$, so that the hyperelliptic involution on Γ is unique. Then $\varrho = \varphi \circ \psi$ where $\psi\colon C \to \Gamma$ is induced by i_C and $\varphi\colon \Gamma \to \mathbb{P}^1_{\mathbb{C}}$ is an hyperelliptic involution on Γ.

Formula 2.4 implies that one of the scrollar invariants of ϱ is $\gamma + 1$, hence the sum of the remaining two is $g - \gamma + 2 \geq 3\gamma + 4$ by Remark 1.7 a). Thus either $\gamma + 1 = \alpha_1$ and $\alpha_2 \geq \gamma + 1$ or $\alpha_2 \leq \gamma$. In the second case $\alpha_3 = g - \gamma + 2 - \alpha_2 \geq g - 2\gamma + 2$: since $\alpha_1 + \alpha_2 \geq \alpha_3$ (see [5], Proposition 1.1) this is not possible due to the hypothesis $g \geq 4\gamma + 2$, hence

$$\varphi_* \mathcal{L}^{-1} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\alpha_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\alpha_3). \tag{2.5.2}$$

Consider the pair of points on C conjugated with respect to i_C. Such points generate in the general fibre of π a pair of lines, thus we obtain a relative degenerate conic $T \subseteq \mathbb{P}$ containing C. Consider the closure T_0 of the singular locus of B in the general fibre. Then Remark 1.7 c) yields $T_0 \cap C = \emptyset$ since C is smooth, hence we can assume that $A \cap T_0 = \emptyset$, again due to Remark 1.7 c).

Since T_0 intersects each fibre of π in a point, we obtain $0 = A \cdot T_0 = 2\xi \cdot T_0 - \beta_1$, ξ being the tautological class of $A(\mathbb{P})$, whence β_1 is even. Since $\beta_1 \leq 2\alpha_1$ by Remark 1.7 b), then $\left|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\beta_1/2)\right|$ is base–point–free, hence each general member $H \in \left|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-\beta_1/2)\right|$ intersects T along a smooth curve and the projection from T_0 makes $H \cap T$ and Γ isomorphic.

Let $\mathcal{O}_{\mathbb{P}}(T) \cong \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-n)$: necessarily $n \leq \beta_2$ since the ideal of C is globally generated by A and B. Adjuntion on $\mathbb{P}$ yields $n = g - \gamma + 2 - \beta_1/2$. The hypothesis $g \geq 4\gamma + 2$ and the restriction $\beta_1 \leq 2\gamma + 2$ (see Remark 1.7) imply $2g - 2\gamma + 4 > 6\gamma + 6 \geq 3\beta_1$, whence we obtain $n = g - \gamma + 2 - \beta_1/2 > \beta_1$. It follows that T must be a multiple of B: since it does not contain fibres by construction, it follows $T = B$, i.e., $n = \beta_2$. Since $H \cap T \cong \Gamma$, adjunction on $\mathbb{P}$ and the equality $\beta_1 + \beta_2 = g + 3$ (see Remark 1.7 a)) yield $\beta_1 = 2\gamma + 2$ and $\beta_2 = g - 2\gamma + 1$.

In particular the equation

$$\delta := \begin{vmatrix} 0 & b_{\alpha-g+3\gamma} & b_{2\gamma+2-\alpha} \\ b_{\alpha-g+3\gamma} & b_{2\alpha-g+2\gamma-1} & b_{\gamma+1} \\ b_{2\gamma+2-\alpha} & b_{\gamma+1} & b_{g+3-2\alpha} \end{vmatrix}$$

of the discriminant Δ of B must be identically zero.

If $b_{\alpha-g+3\gamma} \neq 0$, then it has to be proportional to $b_{2\gamma+2-\alpha}$ since $\delta \equiv 0$. Thus up to a proper automorphism of $\mathcal{E}$ of the form $(u, v, w) \mapsto (u, v + \lambda w, w)$, λ being a suitable linear form of degree $g - \gamma + 2 - 2\alpha$, we can assume $b_{2\gamma+2-\alpha} = 0$, hence $\delta = b^2_{\alpha-g+3\gamma} b_{g+3-2\alpha}$. It would follow that $b_{g+3-2\alpha} = 0$. If this were the case B, hence C, would be reducible, whence we conclude that $b_{\alpha-g+3\gamma} = 0$. A similar argument now shows that $b_{2\gamma+2-\alpha} = 0$ too.

Since $\alpha = \alpha_2 \leq \alpha_3 = g - \gamma + 2 - \alpha$ then $\alpha \leq \alpha_{\max}$. Moreover Formula (2.5.2) yields

$$H^0\big(\Gamma, \varphi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}{}^*(h) \otimes \mathcal{L}^{-1}\big) \cong H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h - \alpha_2)\big) \oplus H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h - \alpha_3)\big).$$

In particular if $h = \alpha$ then $\varphi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}{}^*(h) \otimes \mathcal{L}^{-1}$ is effective, whereas if $h < \alpha \leq g - \gamma + 2 - \alpha$ then $\varphi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}{}^*(h) \otimes \mathcal{L}^{-1}$ is non–effective. Finally notice that if $h \leq [(g - 2\gamma)/2]$ then $\deg(\varphi^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}{}^*(h) \otimes \mathcal{L}^{-1}) \leq -1$, hence $\alpha \geq \alpha_{\min}$.

Conversely if $b_{\alpha-g+3\gamma} = b_{2\gamma+2-\alpha} = 0$ in Equations (2.6.2), then B splits on the general fibre of π in a union of two distinct lines each of them containing two points of C. Thus the projection from $V \cap W$ onto a general $U_0 \in |U|$ induces a cover $\psi \colon C \to \Gamma := U \cap B$ of degree 2. Since B is smooth outside $V \cap W$ and $|U|$ is base–point–free then Γ is smooth and connected. Since $\Gamma = U \cap B$,

adjunction on $\mathbb{P}$ shows that its genus is γ. Since the map $\varphi := \pi_{|\Gamma}$ is necessarily a cover of degree 2 then Γ is hyperelliptic. We conclude that $C \in \mathcal{B}_g^\gamma$. □

Remark 2.6. Notice that Proposition 2.5 asserts that the two sheaves $\mathcal{E}$ and $\mathcal{F}$ defined in Section 1 and naturally associated to the induced cover of degree 4 are

$$\mathcal{E}_g^\gamma(\alpha) := \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\gamma+1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(g-\gamma+2-\alpha),$$
$$\mathcal{F}_g^\gamma := \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\gamma+2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(g-2\gamma+1)$$

for some $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. Moreover C is bihyperelliptic if and only if $C = D_0(a,b) \subseteq \mathbb{P}(\mathcal{E}_g^\gamma(\alpha))$ where

$$\begin{aligned} a(u,v,w) &= a_0u^2 + 2a_{\alpha-\gamma-1}uv + 2a_{g-2\gamma+1-\alpha}uw + \\ &\quad + a_{2\alpha-2\gamma-2}v^2 + 2a_{g-3\gamma}vw + a_{2g-4\gamma+2-2\alpha}w^2, \\ b(u,v,w) &= b_{2\alpha-g+2\gamma-1}v^2 + 2b_{\gamma+1}vw + b_{g+3-2\alpha}w^2, \end{aligned} \tag{2.6.1}$$

$a_h,\ b_h \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h)\big)$.

3. Bihyperelliptic curves with fixed scrollar sequence

In the previous Section we proved that once the two genera g and γ are fixed, then the scrollar sequence of each point in $\mathcal{B}_g^\gamma$ depends only on the number α. Thus it is natural to stratify the locus $\mathcal{B}_g^\gamma$ with respect to such a number α.

Definition 3.1. Let $g \geq 4\gamma+2$. For each integer α in the range $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$ we denote by $\mathcal{B}_g^\gamma(\alpha)$ the locus of $C \in \mathcal{B}_g^\gamma$ having scrollar sequence $(\gamma+1, \alpha, g-\gamma+2-\alpha, 2\gamma+2, g-2\gamma+1)$.

The loci $\mathcal{B}_g^\gamma$ with $\gamma = 0,1,2$ are components of the singular locus of $\mathcal{M}_g$ (see [6]). In particular they are non–empty and irreducible of dimension $2g-2\gamma+1$. We now want to extend such properties also to all the loci $\mathcal{B}_g^\gamma$ and $\mathcal{B}_g^\gamma(\alpha)$ for each values of α and γ.

First of all notice that $\mathcal{B}_g^\gamma(\alpha) \neq \emptyset$ in the considered range. Indeed $\varphi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha)$ is very ample for each $\alpha \geq \alpha_{\min}$: let $\Gamma \to \mathbb{P}^{2\alpha-\gamma}_{\mathbb{C}}$ be the corresponding embedding as curve of degree $2\alpha \geq g-2\gamma+1$. Since $\alpha \leq \alpha_{\min}$ then $g-2\gamma+1 > 2\alpha-\gamma$, hence we can find $g-2\gamma+1$ points of Γ which are in general position in a fixed hyperplane. The corresponding invertible sheaf $\mathcal{L}$ is such that $\varphi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha) \otimes \mathcal{L}^{-1}$ is effective but $\varphi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\alpha-1) \otimes \mathcal{L}^{-1}$ is non–effective. Moreover $\mathcal{L}$ is very ample since $g-2\gamma+1 \geq 2\gamma+3$, thus we can find a section $s \in H^0\big(\Gamma, \mathcal{L}^2\big)$ with smooth zero locus, which thus defines a point $C \in \mathcal{B}_g^\gamma(\alpha)$.

For each α in the range $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$ we define by $\mathcal{U}_g^\gamma(\alpha)$ the set of $(a,b) \in \mathcal{U}_{(\gamma+1,\alpha,g-\gamma+2-\alpha,2\gamma+2,g-2\gamma+1)}$ of the form (2.6.1). We have natural surjective rational maps $\vartheta_g^\gamma(\alpha)\colon \mathcal{U}_g^\gamma(\alpha) \dashrightarrow \mathcal{B}_g^\gamma(\alpha)$. We now want to deal with $\vartheta_g^\gamma(\alpha)$ proving that they are equivariant with respect to the action of a suitable algebraic group, and we wish to identify their generic fibres. To this purpose we recall the following

Definition 3.2. Two covers $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ and $\varrho'\colon C' \to \mathbb{P}^1_{\mathbb{C}}$ are said to be isomorphic if there exists a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\varrho} & \mathbb{P}^1_{\mathbb{C}} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \lambda} \\ C' & \xrightarrow{\varrho'} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

for some suitable $\lambda \in \mathrm{PGL}_2$ and isomorphism $\theta\colon C \to C'$.

Let (a,b), $(a',b') \in \mathcal{U}^\gamma_g(\alpha)$ be sections giving rise to isomorphic covers $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ and $\varrho'\colon C' \to \mathbb{P}^1_{\mathbb{C}}$. Since the Tschirnhausen modules of ϱ' and $\lambda \circ \varrho$ coincide with $\mathcal{E}^\gamma_g(\alpha)$ we then obtain an automorphism $\mu \in E^\gamma_g(\alpha) := \mathrm{Aut}(\mathcal{E}^\gamma_g(\alpha))/\mathbb{C}^*$ ($\mathbb{C}^*$ is identified with the normal subgroup of scalar matrices). Finally consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}^\gamma_g & \to & \mathcal{S}^2\,\mathcal{E}^\gamma_g(\alpha) & \to & \varrho_*\omega^2_{C|\mathbb{P}^1_{\mathbb{C}}} & \to & 0 \\ & & & & \downarrow{\scriptstyle \mathcal{S}^2\,\widehat{\mu}} & & \| & & \\ 0 & \to & \mathcal{F}^\gamma_g & \to & \mathcal{S}^2\,\mathcal{E}^\gamma_g(\alpha) & \to & \varrho'_*\omega^2_{C'|\mathbb{P}^1_{\mathbb{C}}} & \to & 0 \end{array}$$

where $\widehat{\mu}$ is any lifting of μ. In particular we obtain $\nu \in F^\gamma_g := \mathrm{Aut}(\mathcal{F}^\gamma_g)/\mathbb{C}^*$ (again $\mathbb{C}^*$ is identified with the normal subgroup of scalar matrices). Thus if (a,b) and (a',b') define isomorphic covers there is $(\lambda,\mu,\nu) \in G^\gamma_g(\alpha) := \mathrm{PGL}_2 \ltimes (E^\gamma_g(\alpha) \times F^\gamma_g)$ such that $(\lambda,\mu,\nu)(a,b) = (a',b')$. Conversely if this happens it is clear that the induced covers are isomorphic. This proves the first assertion of the following

Lemma 3.3. *Let γ be an integer such that $g \geq 4\gamma+2 \geq 10$ and let $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. Then $G^\gamma_g(\alpha)$ acts on $\mathcal{U}^\gamma_g(\alpha)$ with finite generic stabilizer and the fibres of $\vartheta^\gamma_g(\alpha)$ are exactly the orbits of its action.*

Proof. Let $C \in \mathcal{B}^\gamma_g(\alpha)$. Assume that $C \cong D_0(a,b) \subseteq \mathbb{P}(\mathcal{E}^\gamma_g(\alpha))$. Due to the above description the map onto the fibre defined by $(\lambda,\mu,\nu) \mapsto (\lambda,\mu,\nu)(a,b)$ surjects onto the fibre of $\vartheta^\gamma_g(\alpha)$ over C. This proves the second part of the statement.

We claim that the stabilizer $G^\gamma_g(\alpha)_{(a,b)}$ is finite for a general $(a,b) \in \mathcal{U}^\gamma_g(\alpha)$. Indeed, due to the semicontinuity of the dimension, it suffices to check that the pair (a,b) where

$$\begin{aligned} a(u,v,w) &= a_0u^2 + a_{2\alpha-2\gamma-2}v^2 + a_{2g-4\gamma+2-2\alpha}w^2, \\ b(u,v,w) &= 2b_{\gamma+1}vw, \end{aligned}$$

has finite stabilizer for general a_h, $b_h \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h)\big)$. Each $(\lambda,\mu,\nu) \in G^\gamma_g(\alpha)_{(a,b)}$ must fix both a and b, hence the product $a_{2\alpha-2\gamma-2}a_{2g-4\gamma+2-2\alpha}$: since it has degree at least 8 then $\lambda = id$ for a general choice of (a,b).

Let μ and ν be represented by the matrices $(\mu_{i,j})_{0\leq i,j\leq 2}$ and $(\nu_{i,j})_{0\leq i,j\leq 1}$ respectively. Due to our numerical restrictions on α, g, γ we have $\mu_{1,0} = \mu_{2,0} =$

$\nu_{1,0} = 0$. Since b does not contain terms in u and a is fixed by (λ, μ, ν), it also follows that $\mu_{0,1} = \mu_{0,2} = 0$. We have the system

$$\begin{cases} \nu_{0,0}\mu_{0,0}^2 = \nu_{1,1}\mu_{1,1}\mu_{2,2} + \nu_{1,1}\mu_{1,2}\mu_{2,1} = p \\ \nu_{0,0}\mu_{1,1}^2 a_1 + \nu_{0,0}\mu_{2,1}^2 a_2 + 2\nu_{0,1}\mu_{1,1}\mu_{2,1} b = pa_1 \\ \nu_{0,0}\mu_{1,2}^2 a_1 + \nu_{0,0}\mu_{2,2}^2 a_2 + 2\nu_{0,1}\mu_{1,2}\mu_{2,2} b = pa_2 \\ \nu_{0,0}\mu_{1,1}\mu_{1,2} a_1 + \nu_{0,0}\mu_{2,1}\mu_{2,2} a_2 + \nu_{0,1}\mu_{1,1}\mu_{2,2} b + \nu_{0,1}\mu_{1,2}\mu_{2,1} b = 0 \\ \nu_{1,1}\mu_{1,1}\mu_{2,1} = \nu_{1,1}\mu_{1,2}\mu_{2,2} = 0 \end{cases}$$

for some non–zero $p \in k$. If either $\mu_{1,2} \neq 0$ or $\mu_{2,2} \neq 0$ we get an absurd. Thus we obtain $\mu_{0,0}^2 = \mu_{1,1}^2 = \mu_{2,2}^2$, $\nu_{0,1} = 0$ and $\nu_{0,0}^2 = \nu_{1,1}^2$. Thus we obtain a finite number of elements in $E_g^\gamma(\alpha)$ and in $F_g^\gamma(\alpha)$. □

Finally we can prove the main result of this Section.

Proposition 3.4. *Let γ be an integer such that $g \geq 4\gamma + 2 \geq 10$ and let $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. Then the locus $\mathcal{B}_g^\gamma(\alpha)$ is non–empty, irreducible, unirational and*

$$\dim(\mathcal{B}_g^\gamma(\alpha)) = \begin{cases} g - \gamma + 2\alpha - 1 & \text{if } \alpha = \alpha_{\max}, \\ g - \gamma + 2\alpha & \text{otherwise.} \end{cases}$$

Moreover $\mathcal{B}_g^\gamma(\alpha_{\max})$ is dense inside $\mathcal{B}_g^\gamma$. In particular $\mathcal{B}_g^\gamma$ is non–empty, irreducible and unirational of dimension $2g - 2\gamma + 1$.

Proof. The non–emptyness of $\mathcal{B}_g^\gamma(\alpha)$ has been checked above (see the discussion after Definition 3.1). The surjectivity of $\vartheta_g^\gamma(\alpha)\colon \mathcal{U}_g^\gamma(\alpha) \dashrightarrow \mathcal{B}_g^\gamma(\alpha)$ yields the irreducibility and unirationality of $\mathcal{B}_g^\gamma(\alpha)$. Moreover Lemma 3.3 above shows that

$$\dim(\mathcal{B}_g^\gamma(\alpha)) = \dim(\mathcal{U}_g^\gamma(\alpha)) - \dim(G_g^\gamma(\alpha)) = 7 - 5\gamma + 3g - \dim(E_g^\gamma(\alpha))).$$

The statement about the dimension of $\mathcal{B}_g^\gamma(\alpha)$ then follows since

$$\dim(E_g^\gamma(\alpha))) = \begin{cases} 7 - 4\gamma + 2g - 2\alpha + 1 & \text{if } \alpha = \alpha_{\max}, \\ 7 - 4\gamma + 2g - 2\alpha & \text{otherwise.} \end{cases}$$

In order to prove the assertion on $\mathcal{B}_g^\gamma$, it suffices to check it is irreducible and $\dim(\mathcal{B}_g^\gamma) = \dim(\mathcal{B}_g^\gamma(\alpha_{\max}))$. Let $\mathcal{H}_{\gamma,n} \subseteq \mathcal{M}_{\gamma,n}$ be the locus of n–pointed hyperelliptic curves of genus γ.

Let $\Gamma \in \mathcal{M}_\gamma$ be hyperelliptic and let $q_1, \ldots, q_{2\gamma+2}$ be the fixed points of the involution on Γ. Then $|q_1 + \cdots + q_{2\gamma+2}|$ embeds Γ in $\mathbb{P}_{\mathbb{C}}^{\gamma+2}$ as the intersection of the cone V on the Veronese curve in $\mathbb{P}_{\mathbb{C}}^{\gamma+1}$ with a quadric. Let $H \subseteq H^0\big(V, \mathcal{O}_V(2)\big)$ be the set of sections s such that $D_0(s)$ is a smooth and integral curve. The projection

$$\mathcal{I} := \{\ (s, p_1, \ldots, p_n) \in H \times V^n \mid s(p_i) = 0,\ i = 1, \ldots, n\ \} \to H$$

is proper, surjective and its fibres are integral varieties of dimension n, thus $\mathcal{I}$ is irreducible. Since we have a natural surjective morphism $\mathcal{I} \to \mathcal{H}_{\gamma,n}$, it follows that $\mathcal{H}_{\gamma,n}$ is irreducible too.

In order to check the irreducibility of $\mathcal{B}_g^\gamma$, let $\mathcal{B}_{g,2g-4\gamma+2}^\gamma \subseteq \mathcal{M}_{g,2g-4\gamma+2}$ be the locus of $(2g-4\gamma+2)$-pointed smooth curves of genus g carrying a $2:1$ map over a smooth hyperelliptic curve of genus γ ramified at the $2g-4\gamma+2$ marked points. We have a diagram

$$\begin{array}{ccc} \mathcal{B}_{g,2g-4\gamma+2}^\gamma & \to & \mathcal{B}_g^\gamma \\ \downarrow p & & \\ \mathcal{H}_{\gamma,2g-4\gamma+2} & & \end{array}$$

where the horizontal arrow is the natural forgetful morphism, which has finite fibres due to the definition of $\mathcal{B}_{g,2g-4\gamma+2}^\gamma$, and the vertical one is the morphism associating to each curve the hyperelliptic curve it covers.

Since $\mathcal{H}_{\gamma,2g-4\gamma+2}$ is irreducible, from [6], Remark 1, it follows that the morphism p is a topological covering whose monodromy acts transitively on the fibres. Hence we deduce that $\mathcal{B}_{g,2g-4\gamma+2}^\gamma$ is smooth and connected, in particular it is irreducible. □

Remark 3.5. We conclude this Section with a remark on the automorphism group $\mathrm{Aut}(C)$ of $C \in \mathcal{B}_g^\gamma(\alpha)$ when $g \geq 4\gamma+2 \geq 10$.

On one hand, as pointed out in the introduction, bihyperelliptic curves have been recently studied in [11]. In the paper the authors study bihyperelliptic curves C with at least two commuting involutions, hence such that $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{Aut}(C)$, proving that the locus of such curves in $\mathcal{M}_g$ is highly disconnected. On the other hand we have just proved above that the locus of bihyperelliptic curves is irreducible for $g \geq 4\gamma+2 \geq 10$. These two results are not in contradiction since the general bihyperelliptic curve C of genus $g \geq 4\gamma+2 \geq 10$ satisfies $\mathrm{Aut}(C) = \mathbb{Z}_2$.

In order to check this assertion we first notice that in any case $i_C \in \mathrm{Aut}(C)$, thus $\mathbb{Z}_2 \subseteq \mathrm{Aut}(C)$. Let $\varepsilon \in \mathrm{Aut}(C)$. Since the g_4^1 on C is unique it follows that ε fixes this linear series, thus ε induces two elements $\lambda \in \mathrm{PGL}_2$ and $\mu \in E_g^\gamma(\alpha)$. In particular we obtain an element $(\lambda,\mu) \in \mathrm{PGL}_2 \ltimes E_g^\gamma(\alpha) \in \mathrm{Aut}(\mathbb{P}(\mathcal{E}_g^\gamma(\alpha)))$ fixing $C = D_0(a,b) \subseteq \mathbb{P}(\mathcal{E}_g^\gamma(\alpha))$ and whose restriction to $D_0(a,b)$ coincides with ε.

Since λ must fix the branch locus of the cover $\varrho\colon C \to \mathbb{P}^1_{\mathbb{C}}$ which has degree $2g+6$, then $\lambda = id$ for a general choice of C. Moreover C carries a unique involution of genus γ (see Lemma 2.2), thus $(\mu_{i,j})_{0\leq i,j\leq 2}$ must fix the divisor $B := D_0(b)$, hence b up to scalars. If $2\alpha < g-\gamma+2$ then $b_{2,1} = 0$ and we can assume

$$\begin{cases} \mu_{1,1}^2 b_{2\alpha-g+2\gamma-1} = b_{2\alpha-g+2\gamma-1} \\ \mu_{1,1}\mu_{1,2} b_{2\alpha-g+2\gamma-1} + \mu_{1,1}\mu_{2,2} b_{\gamma+1} = b_{\gamma+1} \\ \mu_{1,2}^2 b_{2\alpha-g+2\gamma-1} + 2\mu_{1,2}\mu_{2,2} b_{\gamma+1} + \mu_{2,2}^2 b_{g+3-2\alpha} = b_{g+3-2\alpha} \end{cases}$$

whence $\mu_{1,2} = 0$ and $\mu_{1,1} = \mu_{2,2} = \pm 1$. If $2\alpha = g - \gamma + 2$ then $\mu_{i,j} \in \mathbb{C}$, $1 \leq i, j \leq 2$ and the vectors of $\mathbb{C}^3$ representing the coefficients of the same monomials (if any) in $b_{2\alpha-g+2\gamma-1}$, $b_{\gamma+1}$, $b_{g+3-2\alpha}$ must be eigenvectors of $\mathcal{S}^2\,\mu$. These vectors are at least $\gamma + 2 \geq 4$, hence μ acts as the identity on v, w. Thus again $\mu_{1,1} = \mu_{2,2} = \pm 1$, $\mu_{1,0} = \mu_{0,1} = 0$. Since $A := D_0(a)$ must be transformed by μ into a conic in the pencil generated by A and B then we also obtain $\mu_{0,1} = \mu_{1,0}$ and $\mu_{0,0}^2 = 1$, thus $\mathbb{Z}_2 = \mathrm{Aut}(C)$.

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