ON THREE SPECTRAL REGULARIZATION METHODS FOR A BACKWARD HEAT CONDUCTION PROBLEM

Xiang-Tuan Xiong, Chu-Li Fu, and Zhi Qian

ABSTRACT. We introduce three spectral regularization methods for solving a backward heat conduction problem (BHCP). For the three spectral regularization methods, we give the stability error estimates with optimal order under an a-priori and an a-posteriori regularization parameter choice rule. Numerical results show that our theoretical results are effective.

1. Introduction

The backward heat conduction problem (BHCP) is also referred to as final boundary value problem. In general no solution which satisfies the heat conduction equation with final data and the boundary conditions exists. Even if a solution exists, it will not be continuously dependent on the final data. The BHCP is a typical example of an ill-posed problem which is unstable by numerical methods and requires special regularization methods. In the context of approximation method for this problem, many approaches have been investigated. Such authors as R. Lattes and J. L. Lions [9], R. E. Showalter [14], K. A. Ames [1], K. Miller [13] have approximated the BHCP by quasi-reversibility methods. In [17], T. Schröter and U. Tautenhahn established an optimal error estimate for a special BHCP. N. S. Mera and M. Jourhmane used many numerical methods with regularization techniques to approximate the problem in [7, 11, 12], etc. A mollification method has been studied by D. N. Hào in [5]. S. M. Kirkup and M. Wadsworth used an operator-splitting method in [8].

Spectral methods for solving sideways heat equation have been studied by U. Tautenhahn in [15], however he don't give the a-posterior parameter choice rule, which is more important in practice.

In this paper, we will use the spectral methods to study a backward heat equation under an a-priori and a-posteriori parameter choice rules. This is a remedy for the reference [15]. Of course, our a-posteriori parameter choice rule can be applied to the sideways heat equation.

The paper is organized as follows. In the next section, we review some spectral methods in the general regularization theory and an a-posteriori parameter choice rule is given; in Section 3, a spectral regularization method together with an error estimate is provided for solving the BHCP ; in Section 4, a numerical example is tested to verify the validity of the parameter choice rules.

2. An a-posteriori parameter choice rule

Now let us review some results on general regularization theory.

Consider an ill-posed operator equation [3, 6, 10, 16]

$$(2.1) \qquad\qquad Ax^\dagger = y,$$

where $A : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y.

Most regularization operators can be written in the form,

$$(2.2) \qquad\qquad R_\alpha := g_\alpha(A^*A)A^*$$

with some function g_α satisfying

$$(2.3) \qquad\qquad \lim_{\alpha \to 0} g_\alpha(\lambda) = \frac{1}{\lambda}.$$

Then for the regularization solution with unperturbed data, we have $x_\alpha := R_\alpha y$ and $x^\dagger - x_\alpha = r_\alpha(A^*A)x^\dagger$ with $r_\alpha(\lambda) = 1 - \lambda g_\alpha(\lambda)$. For example,

Spectral method 1.

$$g_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ \frac{1}{\alpha}, & \lambda < \alpha. \end{cases}$$

$$r_\alpha(\lambda) = \begin{cases} 0, & \lambda \geq \alpha, \\ 1 - \frac{\lambda}{\alpha}, & \lambda < \alpha. \end{cases}$$

Spectral method 2.

$$g_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ \frac{1}{\sqrt{\alpha\lambda}}, & \lambda < \alpha. \end{cases}$$

$$r_\alpha(\lambda) = \begin{cases} 0, & \lambda \geq \alpha, \\ 1 - \sqrt{\frac{\lambda}{\alpha}}, & \lambda < \alpha. \end{cases}$$

Spectral method 3 (TSVD method).

$$g_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ 0, & \lambda < \alpha. \end{cases}$$

$$r_\alpha(\lambda) = \begin{cases} 0, & \lambda \geq \alpha, \\ 1, & \lambda < \alpha. \end{cases}$$

In general, the exact solution $x^\dagger \in X$ is required to satisfy a so-called source condition, otherwise the convergence of the regularization method approximating the problem can be arbitrarily slow. For most ill-posed problems, the source condition is chosen as

$$(2.4) \qquad x^\dagger = [\varphi(A^*A)]^{1/2}\omega, \quad \|\omega\| \leq E,$$

i.e., $x^\dagger$ belongs to the source set

$$(2.5) \qquad M_{\varphi,E} = \{[\varphi(A^*A)]^{1/2}\omega, \quad \omega \in X \text{ and } \|\omega\| \leq E\},$$

where $\varphi(\lambda)$ satisfies some properties: $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ and $\varphi(\lambda)$ is strict monotonically increasing.

The following similar results on the a-posteriori regularization parameter can be found in [6].

Choice Rule. Choose $\alpha = \alpha(\delta, y^\delta)$ such that the following conditions are satisfied with $C > 1$:

$$(2.6) \qquad \|Ax_\alpha^\delta - y^\delta\| \leq C\delta,$$

$$(2.7) \qquad \alpha \leq 1,$$

$$(2.8) \qquad \alpha \leq 1 \Rightarrow \exists \alpha' \in [\alpha, 2\alpha] \ s.t. \|Ax_{\alpha'}^\delta - y^\delta\| \geq C\delta.$$

For most regularization methods a parameter α satisfying (2.6)-(2.8) exists and can be found by a simple algorithm.

Lemma 2.1. *Assume* $\|y - y^\delta\| \leq \delta$, *(2.3) and* $|r_\alpha(\lambda)| \leq C_r$ *with* $C_r = 1$ *hold and that* $\delta > 0$. *If* (α_n) *is some sequence of positive numbers converging monotonically decreasing to 0 with* $\alpha_0 = 1$ *and* $\alpha_n/\alpha_{n+1} \leq 2$, *then the algorithm*

$$n := 0$$

$$while \ \|Ax_{\alpha_n}^\delta - y^\delta\| > C\delta$$

$$n := n + 1$$

terminates after number N *of steps and yields a number* $\alpha = \alpha_N$ *satisfying* *(2.6)-(2.8) with* $\alpha' = \alpha_{N-1}$ *if* $N > 0$.

Proposition 2.1. *Assume that (2.3),* $|g_\alpha(\lambda)| \leq C_g/\alpha$, $|r_\alpha(\lambda)| \leq 1$ *hold for some constant* C_g, *suppose that* $\alpha = \alpha(\delta, y^\delta)$ *is chosen by (2.6)-(2.8) and* $x^\dagger$ *satisfies source condition (2.4). Then the error estimate*

$$(2.9) \qquad \|x_\alpha^\delta - x^\dagger\| \leq E\rho^{-1}((C+1)\delta^2/E^2)(1 + o(1)) \ for \ \delta \to 0.$$

holds, where $\varphi(\lambda) = \lambda^s(d \ln \frac{1}{\lambda})^{-r}$, *where* $s \geq 0, r \geq 0, d > 0$, $\rho^{-1}(\lambda) = \lambda^{\frac{s}{s+1}}[\frac{d}{s+1} \ln \frac{1}{\lambda}]^{-\frac{r}{s+1}}(1 + o(1))$ *for* $\lambda \to 0$.

It is easy to verify the fact that the above three spectral methods satisfy all the assumptions in Proposition 2.1. As for the a-priori parameter choice, we will prove the error estimate for one of the spectral methods.

3. Error estimate on an a-priori parameter choice rule for a BHCP

In this paper, we consider a special BHCP [5]:

$$
\text{(3.1)} \qquad \begin{aligned} u_t(x,\, t) &= u_{xx}(x,\, t) \quad x \in \mathbb{R}, 0 < t < T, \\ u(x, T) &= g(x), \qquad x \in \mathbb{R}. \end{aligned}
$$

We want to obtain the temperature distribution $u(x, t)$ for $0 < t < T$. Since the data $g(\cdot)$ are based on (physical) observations and are not known with complete accuracy, we assume that $g(\cdot)$ and $g^\delta(\cdot)$ satisfy

$$
\text{(3.2)} \qquad \|g(\cdot) - g^\delta(\cdot)\| \le \delta,
$$

where $g(\cdot)$ and $g^\delta(\cdot)$ belong to $L^2(\mathbb{R})$, $g^\delta(\cdot)$ denotes the measured data and δ denotes the noise level.

The problem (3.1) has a uniqueness solution according to [4]. In order to use Fourier transform technique, we define the Fourier transform of function $f(x)$ $(x \in \mathbb{R})$ as the following:

$$
\text{(3.3)} \qquad \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx,\, \xi \in \mathbb{R}.
$$

We consider the problem (3.1) in L^2-space with respect to the variable x. Then taking Fourier transform with respect to x, the problem (3.1) can be reformulated in frequency space as follows:

$$
\text{(3.4)} \qquad \begin{aligned} \hat{u}_t(\xi,\, t) &= (i\xi)^2 \hat{u}(\xi,\, t), \\ \hat{u}(\xi,\, T) &= \hat{g}(\xi), \quad \xi \in \mathbb{R}. \end{aligned}
$$

The solution to equation (3.4) is given by

$$
\text{(3.5)} \qquad \hat{u}(\xi,\, t) = e^{\xi^2(T-t)} \hat{g}(\xi).
$$

From (3.5), we can easily see that

$$
\text{(3.6)} \qquad \hat{u}(\xi,\, 0) = e^{T\xi^2} \hat{g}(\xi).
$$

By the similar method in [15], (3.1) can be formulated as an operator equation in frequency

$$
\text{(3.7)} \qquad \hat{A}(t)\hat{u}(\xi, t) = \hat{g}(\xi) \text{ with } \hat{A}(t) = FA(t)F^{-1},
$$

where $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier operator. Similarly the multiplication operator $\hat{A}(t)$ is given by

$$
\text{(3.8)} \qquad \hat{A}(t) = e^{-\xi^2(T-t)},
$$

and $\hat{A}^*(t)\hat{A}(t) = e^{-2\xi^2(T-t)}$.

As usual, assume that there exists an a-priori condition for our problem (3.1):

$$
\text{(3.9)} \qquad \|u(\cdot,\, 0)\|_p \le E,
$$

where $\|v\|_p = \|(1+s^2)^{p/2}\hat{v}(s)\|_{L^2(\mathbb{R})}$ is the norm of Sobolev space $H^p = \{v(t) \in L^2(\mathbb{R}) : \|v\|_p < \infty\}$, then this condition can be reformulated into an equivalent condition $u(x,t) \in M_{\varphi,E}$ (see (2.5)) with φ given by

$$(3.10) \qquad \varphi(\lambda) = \lambda^{\frac{t}{T-t}}\Big[\frac{1}{2(T-t)}\ln\frac{1}{\lambda}\Big]^{-p}, \ 0 \le t < T, \ p \ge 0.$$

Now we use the three spectral methods for solving the backward heat equation

Spectral method 1.

$$(3.11) \qquad \hat{u}_{\alpha}^{\delta}(\xi) = \begin{cases} e^{\xi^2(T-t)}\hat{g}^{\delta}(\xi), & e^{-2\xi^2(T-t)} \ge \alpha, \\ \frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g}^{\delta}(\xi), & e^{-2\xi^2(T-t)} < \alpha. \end{cases}$$

Spectral method 2.

$$(3.12) \qquad \hat{u}_{\alpha}^{\delta}(\xi) = \begin{cases} e^{\xi^2(T-t)}\hat{g}^{\delta}(\xi), & e^{-2\xi^2(T-t)} \ge \alpha, \\ \frac{1}{\sqrt{\alpha}}\hat{g}^{\delta}(\xi), & e^{-2\xi^2(T-t)} < \alpha. \end{cases}$$

Spectral method 3.

$$(3.13) \qquad \hat{u}_{\alpha}^{\delta}(\xi) = \begin{cases} e^{\xi^2(T-t)}\hat{g}^{\delta}(\xi), & e^{-2\xi^2(T-t)} \ge \alpha, \\ 0, & e^{-2\xi^2(T-t)} < \alpha. \end{cases}$$

Because the three spectral methods are very similar, we only give the properties of the first spectral method.

Theorem 3.1. *Supposed that $u(x, t)$ is exact solution with exact data g and that $u_{\alpha}^{\delta}(x, t)$ is approximate solution by spectral method 1 with noisy data g^{δ}. If we have an a-priori bound $\|u(\cdot, 0)\|_p \le E$ and the data functions satisfy $\|g - g^{\delta}\| \le \delta$, and if we choose $\alpha = [\ln\frac{E}{\delta}]^{p(1-t/T)}(\frac{\delta}{E})^{2(1-t/T)}$, then we can obtain the following error estimate:*

$$(3.14) \qquad \begin{aligned} &\|u(\cdot, t) - u_{\alpha}^{\delta}(\cdot, t)\| \\ &\le (2 + T^{\frac{p}{2}})\delta^{t/T}E^{1-t/T}[\ln\frac{E}{\delta}]^{-\frac{p}{2}(1-t/T)}(1 + o(1)) \quad for \ \delta \to 0. \end{aligned}$$

Proof. First define two sets $A = \{\xi | e^{-2\xi^2(T-t)} \geq \alpha\}$ and $B = \{\xi | e^{-2\xi^2(T-t)} < \alpha\}$, due to Parseval relation and the formulas (3.5), (3.11), we have
(3.15)
$$\|u(\cdot,t) - u_\alpha^\delta(\cdot,t)\| = \|\hat{u}(\cdot,t) - \hat{u}_\alpha^\delta(\cdot,t)\|$$

$$= \left(\int_A |e^{\xi^2(T-t)}(\hat{g} - \hat{g}^\delta)|^2 d\xi + \int_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g}^\delta - e^{\xi^2(T-t)}\hat{g}|^2 d\xi \right)^{1/2}$$

$$\leq \left(\int_A |e^{\xi^2(T-t)}(\hat{g} - \hat{g}^\delta)|^2 d\xi \right)^{1/2} + \left(\int_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g}^\delta - e^{\xi^2(T-t)}\hat{g}|^2 d\xi \right)^{1/2}$$

$$\leq \sup_A |e^{\xi^2(T-t)}| \left(\int_A |(\hat{g} - \hat{g}^\delta)|^2 d\xi \right)^{1/2}$$

$$+ \left(\int_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g}^\delta - \frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g}|^2 d\xi \right)^{1/2}$$

$$+ \left(\int_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}\hat{g} - e^{\xi^2(T-t)}\hat{g}|^2 d\xi \right)^{1/2}$$

$$\leq \sup_A |e^{\xi^2(T-t)}| \cdot \delta + \sup_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}| \cdot \delta$$

$$+ \left(\int_B |(\frac{1}{\alpha}e^{-\xi^2(T-t)} - e^{\xi^2(T-t)})\hat{g}|^2 d\xi \right)^{1/2}$$

$$= \sup_A |e^{\xi^2(T-t)}| \cdot \delta + \sup_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}| \cdot \delta$$

$$+ \left(\int_B |(\frac{1}{\alpha}e^{-\xi^2(T-t)} - e^{\xi^2(T-t)})(1+\xi^2)^{-\frac{p}{2}}(1+\xi^2)^{\frac{p}{2}}e^{-\xi^2 T}\hat{u}(\cdot,0)|^2 d\xi \right)^{1/2}$$

$$\leq \sup_A |e^{\xi^2(T-t)}| \cdot \delta + \sup_B |\frac{1}{\alpha}e^{-\xi^2(T-t)}| \cdot \delta$$

$$+ \sup_B |\xi|^{-p}|\frac{e^{-\xi^2(2T-t)} - \alpha e^{-\xi^2 t}}{\alpha}| \left(\int_B |((1+\xi^2)^{\frac{p}{2}}\hat{u}(\cdot,0)|^2 d\xi \right)^{1/2}$$

$$\leq \frac{2\delta}{\sqrt{\alpha}} + \sup_B |\xi|^{-p} e^{-\xi^2 t} E$$

$$\leq \frac{2\delta}{\sqrt{\alpha}} + \left[(\ln(\frac{1}{\alpha})^{\frac{1}{2(T-t)}}) \right]^{-\frac{p}{2}} \alpha^{\frac{t}{2(T-t)}} E.$$

If the α in (3.15) is chosen by $\alpha = [\ln \frac{E}{\delta}]^{p(1-t/T)}(\frac{\delta}{E})^{2(1-t/T)}$, then

$$\frac{2\delta}{\sqrt{\alpha}} = 2\delta^{t/T}E^{1-t/T}[\ln \frac{E}{\delta}]^{-\frac{p}{2}(1-t/T)},$$

$$\left[(\ln(\frac{1}{\alpha})^{\frac{1}{2(T-t)}})\right]^{-\frac{p}{2}} = \left[\frac{1}{T}\ln \frac{E}{\delta}\right]^{-\frac{p}{2}}(1+o(1)) \quad \text{for } \delta \to 0,$$

$$\alpha^{\frac{t}{2(T-t)}}E = \delta^{t/T}E^{1-t/T}[\ln \frac{E}{\delta}]^{\frac{p}{2}\frac{t}{T}}.$$

So, the error estimate (3.14) holds. $\square$

Remark 3.1. When $t = 0$, the error estimate (3.14) becomes

$$\|u(\cdot,t) - u_\alpha^\delta(\cdot,t)\| \le (2 + T^{\frac{p}{2}})E[\ln \frac{E}{\delta}]^{-\frac{p}{2}}(1+o(1)) \quad \text{for } \delta \to 0.$$

This solve the problem of convergence at $t = 0$.

Remark 3.2. For the other two spectral method, we can establish the error estimates similar to (3.14). Here we omit them.

As for the a-posteriori parameter choice case, Proposition 2.1 guarantees the convergence of the method, we can give the algorithm according to the formulas (2.6)-(2.8) and Lemma 2.1.

$$n := 0, \ \alpha_0 = 1;$$

$$\alpha_n = \frac{1}{n+1};$$

$$\text{while } \left(\int_{|\xi|>\xi_{max}^{(n)}} (\frac{e^{-2\xi^2(T-t)}}{\alpha_n} - 1)^2 |g^\delta(\xi)|^2 d\xi \right)^{1/2} > C\delta$$

$$\text{with } \xi_{max}^{(n)} := \left[\ln[(\frac{1}{\alpha_n})^{\frac{1}{2(T-t)}}]\right]^{1/2};$$

$$n := n + 1;$$

terminates after number N of steps and yields a number $\alpha = \alpha_N$ satisfying (2.6)-(2.8) with $\alpha' = \alpha_{N-1}$ if $N > 0$.

We will do a numerical experiment for the two parameter choice rules in the next section.

4. Numerical experiment

The aim of this section is to verify the theoretical results. It is easy to see that the function

$$(4.1) \qquad\qquad u(x,t) = \frac{1}{\sqrt{1+4t}}e^{-\frac{x^2}{1+4t}}$$

is the unique solution of the initial problem

$$(4.2) \qquad \begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

Hence, $u(x,t)$ given by (4.1) is also the solution of the following backward heat equation for $0 \le t < 1$:

$$(4.3) \qquad \begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, \ 0 \le t < 1, \\ g(x) = u(x,1) = \frac{1}{\sqrt{5}} e^{-\frac{x^2}{5}}, & x \in \mathbb{R}. \end{cases}$$

By simple calculation, we can get $\|u(\cdot,0)\|_0 = 1.11$ and $\|u(\cdot,0)\|_{1/2} = 1.30$. So we take $E = 1.11$ for $p = 0$ and $E = 1.30$ for $p = 1/2$, respectively. We also can get the Fourier transform $\hat{g}(\xi) = \frac{1}{\sqrt{2}} e^{-\frac{5\xi^2}{4}}$ of $g(t)$ and we take $\hat{g}^{\delta}(\xi) = \frac{1}{\sqrt{2}} e^{-\frac{5\xi^2}{4} - \delta}$. Thus, if $\delta = 0.006$, then $\|g(\cdot) - g^{\delta}(\cdot)\| = 0.0045$. In the following numerical experiment, we fix the $\delta = 0.006$.

We do the numerical experiment in the intervals $x \in [-20, 20]$ and $t \in [0, 1]$. This is reasonable in that the initial data at the points $x = -20, 20$ in (4.3) can be considered to be 0 in the computation by noting that the final value $u(x,1) \to 0$ in (4.3) when $x \to \pm\infty$.

Fig. 1 and Fig. 3 are based on the a-posteriori parameter choice rule in Lemma 2.1.

Fig. 2 and Fig. 4 are based on the a-priori parameter choice rule in Theorem 3.1.
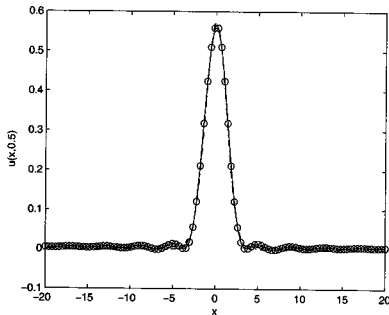
The numerical results are presented as follows.
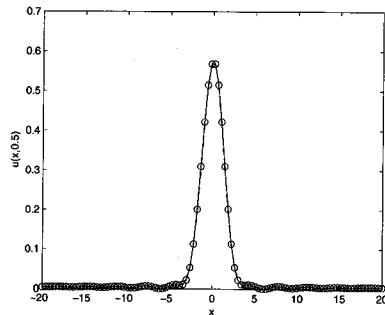

FIGURE 1.
$C = 1.2, \alpha = 1/10$

FIGURE 2.
$p = 0, \alpha = 0.045$

5. Conclusions

In this paper, we discussed three spectral regularization methods for the one-dimensional backward heat conduction problem. We can see that the spectral
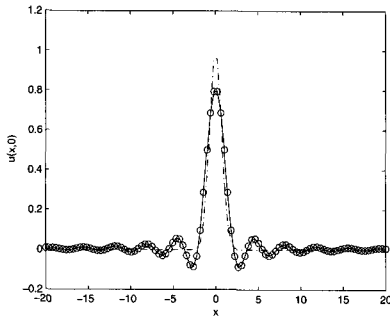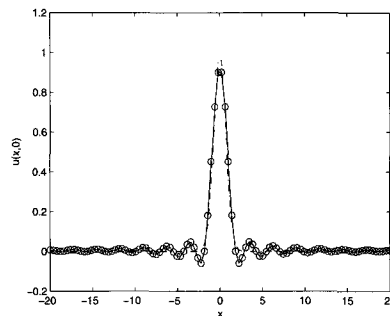
FIGURE 3.
$C = 1.1, \alpha = 1/283$

FIGURE 4.
$p = 1/2, \alpha = 4.8*10^{-5}$

methods stabilize the ill-posed problem by cutting off the high frequency. The third method just is the Fourier method in [2]. Numerical results show that these method are effective with appropriately chosen regularization parameters. The methods can be easily generalized to two-dimensional case.

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XIANG-TUAN XIONG
DEPARTMENT OF MATHEMATICS
LANZHOU UNIVERSITY
LANZHOU 730000, P. R. CHINA
E-mail address: xiongxt@fudan.edu.cn

CHU-LI FU
DEPARTMENT OF MATHEMATICS
LANZHOU UNIVERSITY
LANZHOU 730000, P. R. CHINA
E-mail address: fuchuli@lzu.edu.cn

ZHI QIAN
DEPARTMENT OF MATHEMATICS
LANZHOU UNIVERSITY
LANZHOU 730000, P. R. CHINA
E-mail address: qianzh03@163.com