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WEIGHTED INEQUALITIES FOR COMMUTATORS OF POTENTIAL TYPE OPERATORS

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Abstract. We derive a kind of weighted norm inequalities which relate the commutators of potential type operators to the corresponding maximal operators.

1. Introduction

For a nonnegative, locally integrable function Φ on $\mathbb{R}^n$, define the potential type operator T_Φ by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy.$$

The basic example is provided by the Riesz potentials or fractional integrals I_α, defined by the kernel $\Phi(x) = |x|^{\alpha-n}$, $0 < \alpha < n$.

Now assume that the kernel Φ satisfies the following weak growth condition: there are constants $\delta, c > 0, 0 \le \varepsilon < 1$ with the property that for all $k \in \mathbb{Z}$,

$$\sup_{2^k < |x| \le 2^{k+1}} \Phi(x) \le \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \le 2\delta(1+\varepsilon)2^k} \Phi(y) dy. \tag{1}$$

Associated to any kernel Φ we denote by $\widetilde{\Phi}$ the positive function defined for $t \ge 0$,

$$\widetilde{\Phi}(t) = \int_{|z|<t} \Phi(z) dz.$$

For the two-weight problem of general potential operator T_Φ with Φ satisfying the condition (1), in [5] Pérez gave some sufficient conditions on weights (u, v) such that $T_\Phi : L^p(v^p) \to L^q(u^q)$, $1 < p \le q < \infty$. Pérez [4] derived weighted norm inequalities which relate the potential type operators to the corresponding maximal operators.

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Let $b \in BMO$, and Φ satisfy (1), the higher order commutators of T_Φ are defined by

$$T_\Phi^{b,m} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m \Phi(x-y) f(y) dy,$$

where $m = 0, 1, 2, \ldots$.

In this paper, using the technique developed in [4], [5] and [7], we derive weighted norm inequalities which relate the commutators of potential type operators to the corresponding maximal operators.

2. Preliminaries and main results

We will need the following facts about Orlicz spaces, for further information see [1]. A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex and increasing, and if $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. A Young function B is doubling if there exists a positive constant C such that $B(2t) \leq CB(t)$ for all $t > 0$. If A, B are two Young functions, we write $A(t) \approx B(t)$ if there are constants $c, c_1, c_2 > 0$ with $c_1 A(t) \leq B(t) \leq c_2 A(t)$ for $t > c$. Given a Young function B, define the mean Luxemburg norm of f on a cube Q by

$$\|f\|_{B,Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|} \int_Q B\Big(\frac{|f(y)|}{\lambda}\Big) dy \leq 1\Big\}.$$

For a given Young function B, there exists a complementary Young function $\overline{B}$ such that

$$t \leq B^{-1}(t) \overline{B}^{-1}(t) \leq 2t.$$

There is another characterization of the Luxemburg norm, which we will need:

$$\|f\|_{B,Q} \leq \inf_{s>0} \Big\{s + \frac{s}{|Q|} \int_Q B\Big(\frac{|f(x)|}{s}\Big) dx\Big\} \leq 2\|f\|_{B,Q}. \tag{2}$$

Given three Young functions A, B and C such that for all $t > 0$,

$$A^{-1}(t) C^{-1}(t) \leq B^{-1}(t),$$

then we have the following generalized Hölder inequality due to O'Neil [3]: for all functions f and g and for any cube Q,

$$\|fg\|_{B,Q} \leq 2\|f\|_{A,Q} \|g\|_{C,Q}. \tag{3}$$

In particular, given any Young function B,

$$\frac{1}{|Q|} \int_Q |f(x) g(x)| dx \leq 2\|f\|_{B,Q} \|g\|_{\overline{B},Q}. \tag{4}$$

Given a Young function B, define the associated Orlicz maximal operator by

$$M_B f(x) = \sup_{Q \ni x} \|f\|_{B,Q}.$$

The dyadic maximal operator M_B^d is defined similarly, except the supremum is restricted to dyadic cubes containing x.

For an increasing function $\phi : [0,\infty) \to [0,\infty)$ and a Young function B, we define the maximal operator associated to ϕ and B by

$$M_{B,\phi}f(x) = \sup_{Q \ni x} \phi(|Q|^{1/n})\|f\|_{B,Q}.$$

For the case $B(t) = t$, we write $M_{B,\phi}$ simply as M_ϕ. For $\alpha > 0$, $\phi(t) = t^\alpha$, the maximal operator associated to ϕ and B is denoted by $M_{B,\alpha}$. When $B(t) = t$, we denote M_B and $M_{B,\alpha}$ simply as M and M_α respectively. Clearly M is the well-known Hardy-Littlewood maximal operator and M_α is the fractional maximal operator. It is easy to prove that $M_{B,\alpha}f(y) \le M_\alpha M_B f(y)$ for any Young function B.

The main example that we are going to be using is $\Psi_m(t) = t\log(e+t)^m$, $m = 1, 2, \ldots$. For the Young function Ψ_m, we denote the mean Luxemburg norm of f on a cube Q by $\|f\|_{L(\log L)^m,Q}$, the maximal function by $M_{L(\log L)^m}f$. For this maximal function, Pérez [6] obtains that $M_{L(\log L)^m}f(y) \approx M^{m+1}f(y)$ whenever $m = 0, 1, 2, \ldots$, where M^k denotes the Hardy-Littlewood maximal operator M iterated k times. The complementary Young function of Ψ_m is given by $\overline{\Psi}_m(t) \approx e^{t^{1/m}}$, with the corresponding mean Luxemburg norm and the maximal function denoted by $\|f\|_{\exp L^{\frac{1}{m}},Q}$ and $M_{\exp L^{\frac{1}{m}}}f$ respectively. Otherwise, we need the following lemmas about the maximal functions.

Lemma 2.1 ([2]). *Let $0 < \alpha < n$ and Mf, $M_\alpha f$ be locally integrable. Then there are constants $C > 0, C_1 > 0$ and $C_2 > 0$, independent of f and y, such that*

(i) $MM_\alpha f(y) \le CM_\alpha Mf(y)$,

(ii) $C_1 M_\alpha M^m f(y) \le M_{L(\log L)^m,\alpha} f(y) \le C_2 M_\alpha M^m f(y), \quad m = 0, 1, 2, \ldots$.

Let $1 < p < \infty$. We say that a doubling Young function B satisfies the B_p-condition if there is a positive constant c such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} < \infty.$$

Lemma 2.2 ([6]). *For $1 < p < \infty$, let B be a doubling Young function. Then the following are equivalent:*

(i) $B \in B_p$;

(ii) *there is a constant C such that*

$$\int_{\mathbb{R}^n} M_B f(y)^p dy \le C \int_{\mathbb{R}^n} f(y)^p dy$$

for all nonnegative functions f;

(iii) *there is a constant C such that*

$$\int_{\mathbb{R}^n} M_B f(y)^p w(y) dy \le C \int_{\mathbb{R}^n} f(y)^p Mw(y) dy$$

for all nonnegative functions f and w.

Our main results are the following Theorem and Corollary. Here and below, by weights we mean nonnegative locally integrable functions.

Theorem 2.3. *Given $m \geq 0$ an integer, $\Psi_m(t) = t\log(e+t)^m$. Let $T_\Phi^{b,m}$ be the commutator of potential type operator T_Φ with Φ satisfies (1) and $b \in BMO$.*

(i) *There is a constant C such that for any weight w and all f,*

$$\int_{\mathbb{R}^n} |T_\Phi^{b,m} f(y)| w(y) dy \leq C \int_{\mathbb{R}^n} M_{L(\log L)^m} f(y) M_{L(\log L)^m, \widetilde{\Phi}} w(y) dy. \tag{5}$$

(ii) *For $1 < p < \infty$, there is a constant C such that for any weight w and all f,*

$$\int_{\mathbb{R}^n} |T_\Phi^{b,m} f(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M(M_{L(\log L)^{[(m+1)p]}, \widetilde{\Phi}^p}) w(y) dy. \tag{6}$$

For $0 < \alpha < n$ the case $\Phi(x) = |x|^{\alpha-n}$ corresponds to the Riesz potential of order α. In this case $\widetilde{\Phi}(t) \cong t^\alpha$. By Lemma 2.1 and Theorem 2.3, we have the following corollary:

Corollary 2.4. *Given $m \geq 0$ an integer, $0 < \alpha < n$ and $\Psi_m(t) = t\log(e+t)^m$. Let*

$$I_\alpha^{b,m} f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x-y|^{n-\alpha}} f(y) dy$$

be the commutator of order m for I_α and $b \in BMO$.

(i) *There is a constant C such that for any weight w and all f,*

$$\int_{\mathbb{R}^n} |I_\alpha^{b,m} f(y)| w(y) dy \leq C \int_{\mathbb{R}^n} M_{L(\log L)^m} f(y) M_{L(\log L)^m, \alpha} w(y) dy.$$

(ii) *If $1 < p < \infty$, there is a constant C such that for any weight w and all f,*

$$\int_{\mathbb{R}^n} |I_\alpha^{b,m} f(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M_{L(\log L)^{[(m+1)p]+1}, p\alpha} w(y) dy.$$

3. The proof of Theorem 2.3

In order to prove Theorem 2.3, we need the following generalized Calderón-Zygmund decomposition.

Lemma 3.1 ([6]). *Given a doubling Young function B, suppose f is a nonnegative function such that $\|f\|_{B,Q}$ tends to zero as $l(Q)$ tends to infinity. Then for each $t > 0$ there exists a disjoint collection of maximal dyadic cubes $\{Q_{t,j}\}$ such that for each j, $t < \|f\|_{B,Q_{t,j}} \leq 2^n t$, and*

$$\{x \in \mathbb{R}^n : M_B^d f(x) > t\} = \bigcup_j Q_{t,j},$$

$$\{x \in \mathbb{R}^n : M_B f(x) > 4^n t\} \subset \bigcup_j 3Q_{t,j}.$$

The collection of dyadic cubes $\{Q_{t,j}\}$ is referred to as the Calderón-Zygmund decomposition of f with respect to B at height t.

We fix a constant $a > 2^n$, and for each integer k we let

$$\Omega_k = \{x \in \mathbb{R}^n : M_B f(x) > 4^n a^k\},$$

$$D_k = \{x \in \mathbb{R}^n : M_B^d f(x) > a^k\}.$$

Hence, by Lemma 3.1 with $t = a^k$ there is a family of maximal non-overlapping dyadic cubes $\{Q_{k,j}\}$ for which $D_k = \cup_j Q_{k,j}$, $\Omega_k \subset \cup_j 3Q_{k,j}$, and $a^k < \|f\|_{B,Q_{k,j}} \leq 2^n a^k$.

Lemma 3.2 ([6]). *Suppose $a > 2^n$. For all integers k, j we let $E_{k,j} = Q_{k,j} \setminus (Q_{k,j} \cap D_{k+1})$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfies*

$$|Q_{k,j}| \leq \frac{1}{1 - 2^n/a}|E_{k,j}|.$$

Proof of Theorem 2.3. We set for each $t > 0$,

$$\overline{\Phi}(t) = \sup_{t<|y|\leq 2t} \Phi(y),$$

and

$$\underline{\Phi}(t) = \frac{1}{t^n}\int_{\delta(1-\varepsilon)t<|y|\leq 2\delta(1+\varepsilon)t} \Phi(y)dy,$$

where δ, ε are the numbers in (1). Following [5], we can discretize the operator $T_\Phi^{b,m}$ as follows:

$$\begin{aligned} &|T_\Phi^{b,m} f(y)| \\ = \ & |\sum_{\nu\in\mathbb{Z}} \int_{2^{-\nu-1}<|z-y|\leq 2^{-\nu}} (b(y) - b(z))^m \Phi(y-z) f(z)dz| \\ \leq \ & \sum_{\nu\in\mathbb{Z}} \overline{\Phi}(2^{-\nu-1}) \int_{|z-y|\leq 2^{-\nu}} |b(y) - b(z)|^m |f(z)|dz \\ = \ & \sum_{\nu\in\mathbb{Z}} \sum_{Q:l(Q)=2^{-\nu}} \chi_Q(y)\overline{\Phi}(2^{-\nu-1}) \int_{|z-y|\leq 2^{-\nu}} |b(y) - b(z)|^m |f(z)|dz. \end{aligned}$$

The ball $B(y, 2^{-\nu})$ is covered by the cube $3Q$ if $y \in Q$ and $l(Q) = 2^{-\nu}$. Hence

$$\begin{aligned} &|T_\Phi^{b,m} f(y)| \\ \leq \ & \sum_{Q\in\Delta} \overline{\Phi}(\frac{l(Q)}{2}) \sum_{l=0}^{m} \binom{m}{l} |b(y) - b_Q|^{m-l} \chi_Q(y) \int_{3Q} |b(z) - b_Q|^l |f(z)|dz, \end{aligned}$$

where Δ denotes the family of all dyadic cubes in $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|}\int_Q b(z)dz$.

Now we prove (i). By a limiting argument we may assume that w is bounded with compact support. Then, by the Hölder inequality (3) and the John-Nirenberg theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |T_\Phi^{b,m} f(y)| w(y) dy \\
&\leq \sum_{Q\in\Delta} \sum_{l=0}^{m} \binom{m}{l} \int_Q |b(y)-b_Q|^{m-l} w(y) dy \cdot \overline{\Phi}(\frac{l(Q)}{2}) \int_{3Q} |b(z)-b_Q|^l |f(z)| dz \\
&\leq \sum_{Q\in\Delta} \sum_{l=0}^{m} \binom{m}{l} |Q| \|(b-b_Q)^{m-l}\|_{\exp L^{\frac{1}{(m-l)}},Q} \|w\|_{L(\log L)^{m-l},Q} \\
&\quad \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|(b-b_Q)^l\|_{\exp L^{\frac{1}{l}},3Q} \|f\|_{L(\log L)^l,3Q} \\
&\leq \sum_{Q\in\Delta} \sum_{l=0}^{m} \binom{m}{l} |Q| \|b-b_Q\|_{\exp L,Q}^{m-l} \|w\|_{L(\log L)^{m-l},Q} \\
&\quad \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|b-b_Q\|_{\exp L,3Q}^{l} \|f\|_{L(\log L)^l,3Q} \\
&\leq C\|b\|_{BMO}^m \sum_{Q\in\Delta} \sum_{l=0}^{m} \|w\|_{L(\log L)^{m-l},Q} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^l,3Q} \\
&\leq C\|b\|_{BMO}^m \sum_{Q\in\Delta} \|w\|_{L(\log L)^{m},Q} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m,3Q}.
\end{aligned}
$$

Since w is bounded with compact support, for the Young function $\Psi_m(t) = t\log(e+t)^m$, we have $\|w\|_{\Psi_m,Q} \to 0$ as $l(Q) \to \infty$. For a constant $a > 2^n$ and an integer k, by Lemma 3.1 with $t = a^k$ there is a family of maximal non-overlapping dyadic cubes $\{Q_{k,j}\}$ for which

$$a^k < \|w\|_{L(\log L)^m,Q_{k,j}} \leq 2^n a^k.$$

For each integer k we let

$$\mathcal{C}_k = \{Q \in \Delta : a^k < \|w\|_{L(\log L)^m,Q} \leq a^{k+1}\}.$$

Every dyadic cube Q for which $\|w\|_{L(\log L)^m,Q} \neq 0$ belongs to exactly one $\mathcal{C}_k$. Furthermore, if $Q \in \mathcal{C}_k$, it follows that $Q \subset Q_{k,j}$ for some j. Then

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |T_\Phi^{b,m} f(y)| w(y) dy \\
&\leq C\|b\|_{BMO}^m \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{C}_k} \|w\|_{L(\log L)^m,Q} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m,3Q} \\
&\leq C\|b\|_{BMO}^m \sum_{k\in\mathbb{Z}} a^{k+1} \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{C}_k : Q\subset Q_{k,j}} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m,3Q}
\end{aligned}
$$

$$\leq C\|b\|_{BMO}^m \sum_{k,j} \|w\|_{L(\log L)^m, Q_{k,j}} \sum_{Q: Q\subset Q_{k,j}} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m, 3Q}$$

$$\leq C\|b\|_{BMO}^m \sum_{k,j} \|w\|_{L(\log L)^m, Q_{k,j}} \widetilde{\Phi}(\delta(1+\varepsilon)l(Q_{k,j})) |3Q_{k,j}| \|f\|_{L(\log L)^m, 3Q_{k,j}},$$

where the last inequality will follow if we show that there is a constant C such that for any dyadic cube Q_0,

$$\begin{aligned} \sum_{Q: Q\subset Q_0} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m, 3Q} \\ \leq C\widetilde{\Phi}(\delta(1+\varepsilon)l(Q_0)) |3Q_0| \|f\|_{L(\log L)^m, 3Q_0}. \end{aligned} \tag{7}$$

However, if $l(Q_0) = 2^{-\nu_0}$, then

$$\begin{aligned} &\sum_{Q: Q\subset Q_0} |Q| \cdot \overline{\Phi}(\frac{l(Q)}{2}) |3Q| \|f\|_{L(\log L)^m, 3Q} \\ &= \sum_{\nu \geq \nu_0} \sum_{\substack{l(Q)=2^{-\nu} \\ Q\subset Q_0}} 2^{-\nu n} \overline{\Phi}(2^{-\nu-1}) |3Q| \|f\|_{L(\log L)^m, 3Q} \\ &= \sum_{\nu \geq \nu_0} 2^{-\nu n} \overline{\Phi}(2^{-\nu-1}) \sum_{\substack{l(Q)=2^{-\nu} \\ Q\subset Q_0}} |3Q| \|f\|_{L(\log L)^m, 3Q}. \end{aligned} \tag{8}$$

By the inequality (2), we have

$$\begin{aligned} &\sum_{\substack{l(Q)=2^{-\nu} \\ Q\subset Q_0}} |3Q| \|f\|_{L(\log L)^m, 3Q} \\ \leq &\sum_{\substack{l(Q)=2^{-\nu} \\ Q\subset Q_0}} |3Q| \inf_{\mu>0} \left\{\mu + \frac{\mu}{|3Q|} \int_{3Q} \Psi_m(\frac{|f(z)|}{\mu}) dz\right\} \\ \leq &\inf_{\mu>0} \left\{|3Q_0|\mu + C\mu \int_{3Q_0} \Psi_m(\frac{|f(z)|}{\mu}) dz\right\} \\ \leq &C|3Q_0| \inf_{\mu>0} \left\{\mu + \frac{\mu}{|3Q_0|} \int_{3Q_0} \Psi_m(\frac{|f(z)|}{\mu}) dz\right\} \\ \leq &C|3Q_0| \|f\|_{L(\log L)^m, 3Q_0}, \end{aligned}$$

because of bounded overlap. Now, by condition (1), we have $\overline{\Phi}(2^{-\nu}) \leq C\underline{\Phi}(2^{-\nu})$ for $\nu \in \mathbb{Z}$, and then

$$\sum_{\nu \geq \nu_0} 2^{-\nu n} \overline{\Phi}(2^{-\nu-1}) \leq C \sum_{\nu \geq \nu_0} 2^{-\nu n} \underline{\Phi}(2^{-\nu-1})$$

$$
\begin{aligned}
&= C\sum_{\nu\geq\nu_0}\int_{\delta(1-\varepsilon)2^{-\nu-1}<|y|\leq\delta(1+\varepsilon)2^{-\nu}}\Phi(y)dy\\
&\leq C\int_{|y|\leq\delta(1+\varepsilon)2^{-\nu_0}}\Phi(y)dy\\
&= C\widetilde{\Phi}(\delta(1+\varepsilon)l(Q_0)),
\end{aligned}
$$

again because the overlap is finite. Combining this and (8) yields inequality (7). Let $\rho=\max\{\delta(1+\varepsilon),3\}$, then

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|T_\Phi^{b,m}f(y)|w(y)dy\\
&\leq C\|b\|_{BMO}^m\sum_{k,j}\|w\|_{L(\log L)^m,Q_{k,j}}\widetilde{\Phi}(\rho l(Q_{k,j}))\|f\|_{L(\log L)^m,3Q_{k,j}}|3Q_{k,j}|\\
&\leq C\|b\|_{BMO}^m\sum_{k,j}\|f\|_{L(\log L)^m,3Q_{k,j}}\widetilde{\Phi}(\rho l(Q_{k,j}))\|w\|_{L(\log L)^m,\rho Q_{k,j}}|E_{k,j}|\\
&\leq C\|b\|_{BMO}^m\sum_{k,j}\int_{E_{k,j}}M_{L(\log L)^m}f(y)M_{\Psi_m,\widetilde{\Phi}}w(y)dy\\
&\leq C\|b\|_{BMO}^m\int_{\mathbb{R}^n}M_{L(\log L)^m}f(y)M_{\Psi_m,\widetilde{\Phi}}w(y)dy.
\end{aligned}
$$

This concludes the proof of (i).

Now we prove (ii). Let $p>1$. Our argument will be based on duality and the result in (i). In fact we will prove something sharper than (6): if $\delta>0$, there is a constant C such that for any weight w and all f,

$$
(9)\quad \int_{\mathbb{R}^n}|T_\Phi^{b,m}f(y)|^pw(y)dy\leq C\int_{\mathbb{R}^n}|f(y)|^pM(M_{L(\log L)^{(m+1)p-1+\delta},\widetilde{\Phi}^p})w(y)dy,
$$

where $M_{L(\log L)^{(m+1)p-1+\delta},\widetilde{\Phi}^p}$ denotes the maximal operator associated to $\widetilde{\Phi}^p$ and

$$
B(t)=t\log(e+t)^{(m+1)p-1+\delta},\quad t>0.
$$

Selecting $\delta>0$ such that $(m+1)p-1+\delta=[(m+1)p]$ we get (6).

By (5), there is a constant C so that for all $g\geq 0$,

$$
\int_{\mathbb{R}^n}|T_\Phi^{b,m}f(y)|g(y)w(y)^{1/p}dy\leq C\int_{\mathbb{R}^n}M_{L(\log L)^m}f(y)M_{\Psi_m,\widetilde{\Phi}}(gw^{1/p})(y)dy.
$$

We choose $A(t)\approx t^{p'}(\log t)^{-1-(p'-1)\delta}$ and $C(t)\approx t^p(\log t)^{(m+1)p-1+\delta}$ for large t. It is easy to check that $A\in B_{p'}$ and

$$
\Psi_m^{-1}(t)\approx A^{-1}(t)\times C^{-1}(t).
$$

Noticing that $\Psi_m \in B_p$, by the generalized Hölder inequality (4) and Lemma 2.2 (iii), we can continue with

$$\begin{aligned}
&\leq C \int_{\mathbb{R}^n} M_{L(\log L)^m} f(y) M_{C,\tilde{\Phi}}(w^{1/p})(y) M_A g(y) dy \\
&\leq C\Big(\int_{\mathbb{R}^n} (M_{L(\log L)^m} f(y))^p (M_{C,\tilde{\Phi}}(w^{1/p})(y))^p dy\Big)^{1/p} \Big(\int_{\mathbb{R}^n} M_A g(y)^{p'} dy\Big)^{1/p'} \\
&\leq C\Big(\int_{\mathbb{R}^n} (M_{L(\log L)^m} f(y))^p M_{L(\log L)^{(m+1)p-1+\delta},\tilde{\Phi}^p}(w)(y) dy\Big)^{1/p} \\
&\qquad \times \Big(\int_{\mathbb{R}^n} g(y)^{p'} dy\Big)^{1/p'} \\
&\leq C\Big(\int_{\mathbb{R}^n} |f(y)|^p M(M_{L(\log L)^{(m+1)p-1+\delta},\tilde{\Phi}^p}) w(y) dy\Big)^{1/p} \Big(\int_{\mathbb{R}^n} g(y)^{p'} dy\Big)^{1/p'}.
\end{aligned}$$

Thus (9) is true. This concludes the proof of Theorem 2.3. □

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