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THE BONDAGE NUMBER OF $C_3 \times C_n$

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Abstract. The domination number $\gamma(G)$ of a graph $G = (V, E)$ is the minimum cardinality of a subset of V such that every vertex is either in the set or is adjacent to some vertex in the set. The bondage number of $b(G)$ of a graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. In this paper, we calculate the bondage number of the Cartesian product of cycles C_3 and C_n for all n.

1. Introduction

Let G be a finite, undirected, simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A set D of vertices of G is a dominating set if every vertex of $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. A dominating set D with $|D| = \gamma(G)$ is called a minimum dominating set. The Cartesian product $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, and (a, x) is adjacent to (b, y) if and only if $a = b$ and x, y is adjacent in H or $x = y$ and a, b is adjacent in G, where $x, y \in V(G)$ and $a, b \in V(H)$. Let C_n denote the cycle of n vertices. The edge between x and y will be written as xy.

One measure of the stability of the domination number of G under edge removal is the bondage number $b(G)$ defined in [4]. The bondage number $b(G)$ of G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. Dunbar et al. [2] surveyed results on the bondage number. Moreover, there are so many results on the domination number of graphs and an excellent survey on the bondage number can be found in [2]. But, in contrast, there are only a few results on the bondage number of a graph.

Fink et al. [4] computed that the bondage numbers of cycles, paths and complete multipartite graphs and showed that $b(T) \leq 2$ for any tree T. The bondage numbers for other graphs have been studied in several papers (see [5],

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[6], [7]). Recently, Kang et al. [8] showed that the bondage number of the Cartesian product of cycles C_4 and C_n $(n \geq 4)$ is equal to 4.

In this paper, we show that the bondage number of the Cartesian product of cycles C_3 and C_{4k+r} is equal to 2 if $r = 0$; 4 if $r = 1$ or 2; and 5 if $r = 3$, where $k \geq 1$ and $0 \leq r \leq 3$.

2. Bondage number of $C_3 \times C_{4k+r}$ for $r \neq 3$

We consider $C_3 \times C_n$ as a $3 \times n$ array of vertices $\{v_{ij} \mid 1 \leq i \leq 3, 1 \leq j \leq n\}$ where $V_j = \{v_{1j}, v_{2j}, v_{3j}\}$ induces a C_3 for each $1 \leq j \leq n$, and $H_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ induces a C_n for each $1 \leq i \leq 3$.

An edge e in $C_3 \times C_n$ is said to be *vertical* (*horizontal*) depending on whether it belongs to the induced subgraph $\langle V_j \rangle$ ($\langle H_i \rangle$, respectively). The cycles $\langle V_j \rangle$ and $\langle H_i \rangle$ are also called *vertical* and *horizontal*, respectively.

Notice that S. Klavzar et al. determined in [9] the domination number of $C_m \times C_n$ for some m, n, they obtained that $\gamma(C_3 \times C_n) = n - \lfloor \frac{n}{4} \rfloor$ for $n \geq 4$.

To compute $b(C_3 \times C_{4k+r})$ for $0 \leq r \leq 2$ $(k \geq 1)$, we begin with the following lemma.

Lemma 1. *For a positive integer k,*

$$(1) \quad b(C_3 \times C_{4k+1}) \leq 4 \quad \textit{and} \quad b(C_3 \times C_{4k+2}) \leq 4,$$
$$(2) \quad b(C_3 \times C_{4k}) \leq 2.$$

Proof. (1). It suffices to show that $\gamma(C_3 \times C_{4k+1} - v_{34k+1}) \geq 3k + 1 = \gamma(C_3 \times C_{4k+1})$ and $\gamma(C_3 \times C_{4k+2} - v_{34k+2}) \geq 3k + 2 = \gamma(C_3 \times C_{4k+2})$. First let $G' = C_3 \times C_{4k+1} - v_{34k+1}$ and we prove the following assertion. □

Assertion 1. *There is a minimum dominating set D of G' such that $|D \cap V_j| \leq 2$ for any $j = 1, 2, \ldots, 4k + 1$.*

Proof. Let D_0 be a minimum dominating set of G' such that $|D_0 \cap V_{i_j}| = 3$ holds for t V_{i_j} where $1 \leq t \leq 4k$, $1 \leq j \leq t$. We now construct a minimum dominating set D_1 with $|D_1| = |D_0|$ such that only $t - 1$ V_j have 3 vertices in common with D_1. Let $|D_0 \cap V_j| = 3$. Clearly, $j \neq 4k + 1$. We first assume that $j \neq 1, 4k$. If $D_0 \cap V_{j-1} = \emptyset = D_0 \cap V_{j+1}$, we let $D_1 = D_0 - V_j \cup \{v_{1j-1}, v_{2j+1}, v_{3j+1}\}$, then we obtain the desired dominating set of G'. If $D_0 \cap V_{j-1} \neq \emptyset$ or $D_0 \cap V_{j+1} \neq \emptyset$, say $v_{ij+1} \in D_0 \cap V_{j+1} \neq \emptyset$, as D_0 is a minimum dominating set, then $V_{j-1} \cap D_0 = \emptyset$. Let $D_1 = D_0 - \{v_{ij}\} \cup \{v_{ij-1}\}$, we also obtain the desired result. If $j = 1$ or $4k$, we can use the same technique to obtain the desired result. This completes Assertion 1.

Now let D be a minimum dominating set of G' such that $|D| \leq 3k$ and $|D \cap V_j| \leq 2$ for any $j = 1, 2, \ldots, 4k + 1$, we deduce a contradiction. Let s be the number of V_j such that $D \cap V_j = \emptyset$. Since no two V_j with $D \cap V_j = \emptyset$ are adjacent, $k + 1 \leq s \leq 2k$. As each V_j with $D \cap V_j = \emptyset$ is dominated by exactly two V_i which are adjacent to V_j, there are at least $\lfloor \frac{s}{2} \rfloor$ V_i with $|D \cap V_i| = 2$. Thus, $|D| \geq 2\lfloor \frac{s}{2} \rfloor + 4k + 1 - \lfloor \frac{s}{2} \rfloor - s = 4k + 1 - \lceil \frac{s}{2} \rceil \geq 3k + 1$, a contradiction. Hence, $\gamma(C_3 \times C_{4k+1} - v_{34k+1}) \geq 3k+1 = \gamma(C_3 \times C_{4k+1})$. This implies $b(C_3 \times C_{4k+1}) \leq 4$.

Next let $G' = C_3 \times C_{4k+2} - v_{34k+2}$. Similarly, there is a minimum dominating set D of G' such that $|D \cap V_j| \leq 2$ for any $j = 1, 2, \ldots, 4k+2$. Now let D be a minimum dominating set of G' such that $|D| \leq 3k+1$ and $|D \cap V_j| \leq 2$ for any $j = 1, 2, \ldots, 4k+2$. Let s be the number of V_j such that $D \cap V_j = \emptyset$. Similarly, we have that $k+1 \leq s \leq 2k+1$ and $|D| \geq 2\lfloor \frac{s}{2} \rfloor + 4k+2 - \lfloor \frac{s}{2} \rfloor - s = 4k+2 - \lceil \frac{s}{2} \rceil$. If $s \leq 2k$, then $|D| \geq 3k+2$, a contradiction. If $s = 2k+1$ and $D \cap V_{4k+2} \neq \emptyset$, as no two V_j with $D \cap V_j = \emptyset$ are adjacent, then there are at least $k+1$ V_j such that $|D \cap V_j| = 2$, and thus $|D| \geq 2(k+1) + k = 3k+2$, a contradiction. Hence, $s = 2k+1$ and $D \cap V_{4k+2} = \emptyset$. Clearly, we have $D \cap V_{2t} = \emptyset$ for $1 \leq t \leq 2k$ and $|D \cap V_1| = 1$ and $|D \cap V_{4k+1}| = 1$. We may assume that $D \cap V_1 = \{v_{11}\}$, $D \cap V_{4k+1} = \{v_{24k+1}\}$. Then we set

$$D' = \{v_{24k+1}, v_{14k-1}, v_{34k-1}, v_{24k-3}, \ldots, v_{25}, v_{11}\} \subseteq D.$$

Note that $\{v_{22}, v_{23}, v_{13}, v_{23}, v_{33}, v_{14}, v_{34}\}$ are not dominated by D'. As $|D| - |D'| = 2$, we must use two vertices to dominate these vertices. It is straightforward to verify that it is impossible, a contradiction. Hence, $\gamma(C_3 \times C_{4k+2} - v_{3,4k+2}) \geq 3k+2 = \gamma(C_3 \times C_{4k+2})$, implying $b(C_3 \times C_{4k+2}) \leq 4$.

(2). Let $e_1 = v_{11}v_{12}, e_2 = v_{13}v_{14}$ and $G' = C_3 \times C_{4k} - \{e_1, e_2\}$. It is known that $\gamma(C_3 \times C_{4k}) = 3k$. If $\gamma(G') = 3k$, then we can similarly deduce that G' has a minimum dominating set D such that $|D \cap V_j| \leq 2$ for $j = 1, 2, \ldots, 4k$. As shown in the proof of Theorem 2.3 of [9], there are $2k$ of V_j such that $D \cap V_j = \emptyset$ for $j = 1, 2, \ldots, 4k$. Then, there are two of V_1, V_2, V_3, V_4 such that $D \cap V_j = \emptyset$ for $j = 1, 2, 3, 4$. It is straightforward to verify that there is no such dominating set with $3k$ vertices in G'. Hence, we have $\gamma(G') \geq 3k+1$, implying $b(G) \leq 2$. □

Lemma 2. *$b(C_3 \times C_{4k+1}) \geq 4$, $b(C_3 \times C_{4k+2}) \geq 4$, and $b(C_3 \times C_{4k}) \geq 2$.*

Proof. First let $G = C_3 \times C_{4k+1}$ and we show that $\gamma(G - \{e_1, e_2, e_3\}) = 3k+1$ for any three edges e_1, e_2, e_3 of G. Let $G' = G - \{e_1, e_2, e_3\}$, we divide our discussion into four cases.

Case 1. All of e_1, e_2, e_3 are vertical edges of G.

By symmetry, we suppose $e_1 \in \langle V_1 \rangle$. If both e_2, e_3 also belong to $\langle V_1 \rangle$, let

$$D_1 = \cup_{j=0}^{k-1}\{v_{14j+2}, v_{24j+2}, v_{34(j+1)}\} \cup \{v_{34k+1}\},$$

then D_1 is a dominating set of $3k+1$ vertices of G'. If $e_2 \in \langle V_1 \rangle$ and $e_3 \in \langle V_j \rangle$ ($j \neq 1$), then, for $j \neq 4t$ for $1 \leq t \leq k$ or $j \neq 4k+1$, D_1 is still a dominating set of G'. Otherwise, we may suppose that $e_3 = v_{14t}v_{34t}$ for some $1 \leq t \leq k$ or $e_3 = v_{14k+1}v_{34k+1}$. Let

$$D_2 = \cup_{j=0}^{k-1}\{v_{14j+2}, v_{34j+2}, v_{24(j+1)}\} \cup \{v_{24k+1}\}.$$

Then, D_2 is a dominating set of $3k+1$ vertices of G'. For the remaining cases, by symmetry, we suppose that $e_2 \in \langle V_j \rangle$ and $e_3 \in \langle V_l \rangle$ and $j \neq 1, l \neq 1$

and $j \neq l$. If both j and l are not equal to $4t$ for $1 \leq t \leq k$ and $4k+1$, then D_1 dominates G'. Otherwise, we may assume $e_2 = v_{14t}v_{34t}$ for some $1 \leq t \leq k$ or $e_2 = v_{14k+1}v_{34k+1}$. Let

$$D_3 = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{34j+1}, v_{24j+3}\} \cup \{v_{24k}\}.$$

If $l = 4t+3$ for some $0 \leq t \leq k-1$, then D_2 dominates G'. If $l \neq 4t+3$ any $0 \leq t \leq k-1$, then D_3 is a dominating set of $3k+1$ vertices of G'. Hence, in any case we can choose a dominating set of $3k+1$ vertices of G', and thus $\gamma(G') \leq 3k+1$. On other hand, $\gamma(G') \geq \gamma(G) = 3k+1$, implying $\gamma(G') = 3k+1$.

Case 2. Two of e_1, e_2, e_3 are vertical edges and one is horizontal edge of G.

We may suppose $e_3 = v_{31}v_{34k+1}$. Then, we only need to consider the following two subcases, because the set D_2 dominates G' for the remaining cases.

Subcase (2.1) Both of e_1, e_2 belong to $\langle V_{4t+2}\rangle$ for some $0 \leq t \leq k-1$;

Subcase (2.2) At least one of e_1, e_2 belongs to $\langle V_{4t}\rangle$ for some $1 \leq t \leq k$ or $\langle V_{4k+1}\rangle$.

If the subcase (2.1) appears, then

$$D_4 = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\} \cup \{v_{34k}\}$$

dominates G'.

If the subcase (2.2) appears, we first suppose $e_1 \in \langle V_{4t}\rangle$ for some $1 \leq t \leq k$. By symmetry, we may suppose $e_1 = v_{14t}v_{24t}$ for some $1 \leq t \leq k$. Now if $e_2 \in \langle V_{4t+3}\rangle$ for some $0 \leq t \leq k-1$ or $e_2 \in \langle V_{4k}\rangle$, then

$$D_5 = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}\}$$

dominates G'. Otherwise D_4 dominates G'.

Next let $e_1 \in \langle V_{4k+1}\rangle$. If $e_2 = v_{24t+3}v_{34t+3}$ for some $0 \leq t \leq k-1$ or $e_2 = v_{24k}v_{34k}$, then

$$D_6 = \cup_{j=0}^{k-1}\{v_{24j+2}, v_{14(j+1)}, v_{34(j+1)}\} \cup \{v_{21}\}$$

dominates G'. Otherwise D_4 dominates G'. By the same reasoning, we have $\gamma(G') = 3k+1$.

Case 3. Only one edge of e_1, e_2, e_3 is vertical edge and two are horizontal edges of G.

We assume that e_1 is vertical and e_2, e_3 are horizontal. Assume that $e_3 = v_{11}v_{14k+1}$. For convenience, we denote the set of edges between V_j, V_{j+1} by E_j for $1 \leq j \leq 4k$ and E_{4k+1} denote the set of edges between V_{4k+1}, V_1. Note that

$$D_7 = \cup_{j=0}^{k-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup \{v_{14k}\}$$

dominates $G - \{e_3\}$, we can assume that e_1 or e_2 is incident to some vertex of D_7. It divides into two subcases.

Subcase (3.1) e_1 is incident to a vertex of D_7.

Since e_1 is vertical, $e_1 \in \langle V_{4t+1}\rangle \cup \langle V_{4t+3}\rangle$ for some $0 \leq t \leq k-1$ or $e_1 \in \langle V_{4k}\rangle$. If $e_1 \in \langle V_{4t+1}\rangle$ for some $0 \leq t \leq k-1$, then D_7 still dominates $G - \{e_1, e_3\}$, and thus we can assume that e_2 is also incident to a vertex of D_1. First we assume $e_2 \in \langle H_1 \rangle \cap E_{4j+2}$. Now if $e_1 = v_{24t+1}v_{34t+1}$ for some $0 \le t \le k-1$, we choose

$$D_8 = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}\};$$

and if $e_1 = v_{14t+1}v_{24t+1}$ for some $0 \le t \le k-1$, we choose

$$D_8 = \{v_{31}\} \cup_{j=1}^{k} \{v_{34(j-1)+2}, v_{14j}, v_{24j}\};$$

and if $e_1 = v_{14t+1}v_{34t+1}$ for some $0 \le t \le k-1$, we choose

$$D_8 = \{v_{21}\} \cup_{j=1}^{k} \{v_{24(j-1)+2}, v_{14j}, v_{34j}\}.$$

It is straightforward to verify that D_8 is a dominating set of $3k+1$ vertices of G' in each case. Next let $e_2 \in \langle H_1 \rangle \cap E_{4j+3}$ for some $0 \le j \le k-2$ or $e_2 \in \langle H_1 \rangle \cap E_{4k}$. Then,

$$\cup_{j=0}^{k-1}\{v_{14j+2}, v_{24j+2}, v_{34(j+1)}\} \cup \{v_{34k+1}\}$$

dominates G'. Finally we assume that e_2 is incident to a vertex of $H_2 \cup H_3$. By symmetry, we only consider that e_2 is incident to a vertex of H_2. If $e_2 \in \langle H_2 \rangle \cap E_{4j+1}$ for some $0 \le j \le k-1$, then

$$\cup_{j=0}^{k-1}\{v_{14j+2}, v_{34j+2}, v_{24j}\} \cup \{v_{24k+1}\}$$

dominates G'; if $e_2 \in \langle H_2 \rangle \cap E_{4j}$ for some $1 \le j \le k-1$ or $e_2 \in \langle H_2 \rangle \cap E_{4k+1}$, then

$$\cup_{j=0}^{k-1}\{v_{14j+2}, v_{24j+2}, v_{34j}\} \cup \{v_{34k+1}\}$$

dominates G'.

If $e_1 \in \langle V_{4t+3} \rangle$ for some $0 \le t \le k-1$ or $e_1 \in \langle V_{4k} \rangle$, then

$$D_9 = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}\}$$

is a dominating set of $3k+1$ vertices of $G - \{e_1, e_3\}$, and thus we can assume that e_2 is incident to some vertices D_9. If e_2 is incident to some vertices of $H_2 \cap D_9$, then $e_2 \in \langle H_2 \rangle \cap (E_{4i+2} \cup E_{4i+3})$ for some $0 \le i \le k-1$. It follows

$$D_{10} = \cup_{j=0}^{k-1}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'; and if e_2 is incident to some vertices of $H_3 \cap D_9$, we can similarly choose a dominating set of $3k+1$ vertices of G'. If e_2 is incident to a vertex of $H_1 \cap D_3$, then D_{10} dominates G'.

Subcase (3.2) e_1 is not incident to any vertex of D_7 and e_2 is incident to some vertices of D_7.

Let

$$D_7' = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}\}.$$

Clearly, D_7' dominates G' unless $e_2 = v_{14k}v_{14k+1}$ or $e_1 = v_{14k+1}v_{24k+1}$ or $e_1 = v_{14k+1}v_{34k+1}$.

First we consider $e_2 = v_{14k}v_{14k+1}$. Now if $e_1 = v_{14k+1}v_{24k+1}$, then

$$\cup_{j=0}^{k-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup \{v_{34k+1}\}$$

dominates G'; if $e_1 = v_{14k+1}v_{34k+1}$, then

$$\cup_{j=0}^{k-1}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'; if $e_1 = v_{24k+1}v_{34k+1}$, then

$$\{v_{12}\} \cup_{j=1}^{k} \{v_{14j-1}, v_{24j+1}, v_{34j+1}\}$$

dominates G'; if $e_1 \notin \langle V_{4k+1}\rangle$, by noting that e_1 is not incident to any vertex of D_7, then

$$\cup_{j=0}^{k-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup \{v_{34k+1}\}$$

dominates G'.

Next we assume that $e_2 \neq v_{14k}v_{14k+1}$ and $e_1 = v_{14k+1}v_{24k+1}$ or $e_1 = v_{14k+1}v_{34k+1}$. By the symmetry of v_{11} and v_{14k+1}, we can similarly obtain a dominating set of $3k+1$ vertices of G' as in (3.1).

Combining (3.1) and (3.2), we obtain $\gamma(G') = 3k+1$ for this case.

Case 4. All of e_1, e_2, e_3 are horizontal edges of G.

We suppose $e_3 = v_{11}v_{14k+1}$. If e_1 is incident to v_{11} and e_2 is incident to v_{14k+1}, then

$$\cup_{j=0}^{k-1}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'. By symmetry, we assume that neither e_1 nor e_2 is incident to v_{14k+1}. Let

$$D_1' = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}\},$$

$$D_2' = \cup_{j=0}^{k-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup \{v_{14k}\}.$$

Note that D_1' is a dominating set of $3k+1$ vertices of $G-\{e_3\}$, at least one of e_1, e_2 is incident to some vertex of $D_1' - v_{14k+1}$. Let e_1 be incident to some vertex of $D_1' - v_{14k+1}$. Then, D_2' dominates $G-\{e_1, e_3\}$, and thus e_2 is incident to some vertex of $D_2' - v_{14k}$. We divide our discussion into three subcases.

Subcase (4.1) Assume $e_1 \in \langle H_1\rangle \cap E_{4t+1}$ for some $0 \leq t \leq k-1$.

If $e_2 \in \langle H_1\rangle \cap E_{4j+2}$ for some $0 \leq j \leq k-1$, then

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\}$$

dominates G'; and if $e_2 \in \langle H_1\rangle \cap E_{4j+3}$ for some $0 \leq j \leq k-2$, then

$$\cup_{j=0}^{t}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup_{j=t+1}^{k-1} \{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}\}$$

dominates G'; and if $e_2 \in \langle H_2\rangle$, then

$$D_3' = \cup_{j=0}^{k-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup \{v_{34k+1}\}$$

dominates G'. If $e_2 \in \langle H_3\rangle$, by symmetry of H_2 and H_3, we can similarly choose a dominating set of $3k+1$ vertices of G'.

Subcase (4.2) Assume $e_1 \in \langle H_1\rangle \cap E_{4t}$ for some $0 \leq t \leq k-1$.

If $e_2 \in \langle H_1\rangle \cap E_{4j+2}$ for some $0 \leq j \leq k-1$, then

$$\{v_{12}\} \cup_{j=1}^{t} \{v_{24j}, v_{34j}, v_{14j+2}\} \cup_{j=t+1}^{k} \{v_{14j-1}, v_{24j+1}, v_{34j+1}\}$$

dominates G'; and if $e_2 \in \langle H_1 \rangle \cap E_{4t+3}$ for some $0 \le t \le k-2$, then

$$\{v_{24k+1}\} \cup_{j=1}^{k} \{v_{14j-2}, v_{34j-2}, v_{24j}\}$$

dominates G'. If $e_2 \in \langle H_2 \rangle$, then D_3' dominate G'. If $e_2 \in \langle H_3 \rangle$, we can similarly choose a dominating set of $3k+1$ vertices of G'.

Subcase (4.3) $e_1 \in \langle H_2 \rangle \cup \langle H_3 \rangle$. By symmetry, we only consider $e_1 \in \langle H_2 \rangle$. We divide two cases.

(i) Assume $e_1 \in \langle H_2 \rangle \cap E_{4t+2}$ for some $0 \le t \le k-1$.

If $e_2 \in \langle H_1 \rangle \cap E_{4i+2}$ for some $0 \le i \le k-1$, then

$$D_4' = \{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\}$$

dominates G'; and if $e_2 \in \langle H_1 \rangle \cap E_{4i+3}$ for some $0 \le i \le k-2$, then

$$\cup_{j=0}^{k-1} \{v_{14j+2}, v_{34j+2}, v_{24j}\} \cup \{v_{24k+1}\}$$

dominates G'.

If $e_2 \in \langle H_2 \rangle \cap E_{4i+1}$ for some $0 \le i \le k-1$, then D_4' dominates G'; if $e_2 \in \langle H_2 \rangle \cap E_{4i}$ for some $1 \le i \le k-1$, then

$$\cup_{j=1}^{t} \{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup_{j=t}^{k-1} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'; and if $e_2 = v_{21}v_{24k+1}$, then

$$\cup_{j=0}^{k-1} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'.

If $e_2 \in \langle H_3 \rangle$, we can similarly choose a dominating set of $3k+1$ vertices of G'.

(ii) $e_1 \in \langle H_2 \rangle \cap E_{4t+3}$ for some $0 \le t \le k-1$.

If $e_2 \in \langle H_1 \rangle \cap E_{4i+2}$ for some $0 \le i \le k-1$, then

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\}$$

dominates G'; and if $e_2 \in \langle H_1 \rangle \cap E_{4i+3}$ for some $0 \le i \le k-2$, then

$$\cup_{j=0}^{k-1} \{v_{14j+2}, v_{24j+2}, v_{34j}\} \cup \{v_{34k+1}\}$$

dominates G'.

If $e_2 \in \langle H_2 \rangle \cap E_{4i}$ for some $0 \le i \le k-1$, then

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\}$$

dominates G'; and if $e_2 \in \langle H_2 \rangle \cap E_{4i+1}$ for some $0 \le i \le k-1$, then

$$\cup_{j=0}^{t-1} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24t+1}\} \cup_{j=t}^{k-1} \{v_{24j+2}, v_{14(j+1)}, v_{34(j+1)}\}$$

dominates G'; and if $e_2 = v_{21}v_{24k+1}$, then

$$\cup_{j=0}^{k-1} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}\}$$

dominates G'.

If $e_2 \in \langle H_3 \rangle$, we can similarly choose a dominating set of $3k+1$ vertices of G'. This proves $\gamma(G') = 3k+1$ for this case. Summarizing above, we obtain

$\gamma(G') = 3k+1$, implying $b(C_3 \times C_{4k+1}) \geq 4$. Similarly, one can deduce that $b(C_3 \times C_{4k+2}) \geq 4$.

Now, we will show that $b(C_3 \times C_{4k}) \geq 2$. Let $e = v_{11}v_{12}$ or $e = v_{11}v_{21}$. Clearly,

$$\cup_{j=0}^{k-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

is a dominating set $C_3 \times C_{4k} - e$. By symmetry, we have that $\gamma(C_3 \times C_{4k} - e) = 3k$ for any edge $e \in C_3 \times C_{4k}$. Hence $b(C_3 \times C_{4k}) \geq 2$. It completes the proof. □

Now, by Lemmas 1 and 2, we have the following theorem.

Theorem 1. *For any positive integer k, we have*

$$b(C_3 \times C_{4k}) = 2, \quad b(C_3 \times C_{4k+1}) = 4, \quad \text{and} \quad b(C_3 \times C_{4k+2}) = 4.$$

3. Bondage number of $C_3 \times C_{4k+3}$ $(k \geq 1)$

In this section, to complete the computation of $b(C_3 \times C_n)$, we will compute $b(C_3 \times C_{4k+3})$.

Lemma 3. $b(C_3 \times C_{4k+3}) \geq 5$ *for every* $k \geq 1$.

Proof. Let $G = C_3 \times C_{4k+3} (k \geq 1)$. We will prove that $\gamma(G - E') = \gamma(G) = 3k+3$ for any set E' of four edges e_1, e_2, e_3, e_4 of G. Let $G' = G - E'$. We divide our discussion into five cases.

Case 1. All four edges of E' are vertical edges.

First we assume that $\cup_{i=0}^{k} \langle V_{4i+1} \rangle \cup_{i=0}^{k} \langle V_{4i+3} \rangle$ contains at most one edge of E'. By symmetry, we suppose $\cup_{i=0}^{k} \langle V_{4i+3} \rangle \cap E' = \emptyset$. Then,

$$\cup_{j=0}^{k}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\}$$

is a dominating set of $3k+3$ vertices of G'.

Next assume that $\cup_{i=0}^{k} \langle V_{4i+2} \rangle \cup_{i=1}^{k} \langle V_{4i} \rangle$ contains at most two edges of E'. If $\cup_{i=0}^{k} \langle V_{4i+2} \rangle$ contains two edges of E', we assume that $e_1 = v_{14i_1+2}v_{24i_1+2}$, $e_2 = v_{14i_2+2}v_{34i_2+2}$ for some $0 \leq i_1, i_2 \leq k$, then

$$\cup_{j=0}^{k-1}\{v_{14j+2}, v_{34j+2}, v_{24(j+1)}\} \cup \{v_{21}, v_{14k+2}, v_{34k+2}\}$$

is a dominating set of $3k+3$ vertices of G'. For other cases, we can similarly choose a dominating set of $3k+3$ vertices of G'. If $\cup_{i=1}^{k} \langle V_{4i} \rangle$ contains two edges of E', we first assume that $e_1 = v_{14i_1}v_{24i_1}$, $e_2 = v_{14i_2}v_{34i_2}$ for some $0 \leq i_1, i_2 \leq k$, then

$$\cup_{j=1}^{k}\{v_{14j}, v_{34j}, v_{24j+2}\} \cup \{v_{22}, v_{14k+3}, v_{34k+3}\}$$

is a dominating set of $3k+3$ vertices of G'. For other cases, we can similarly choose a dominating set of $3k+3$ vertices of G'. If $\cup_{i=1}^{k} \langle V_{4i} \rangle$ contains at most one edge of E' and $\cup_{i=0}^{k} \langle V_{4i+2} \rangle$ contains at most one edge of E', then it is also easy to choose a dominating set of $3k+3$ vertices of G'.

Case 2. There are three vertical edges and one horizontal edge in E'.

We may suppose that $e_1 = v_{11}v_{14k+3}$ is the horizontal edge of E'. Since there are only three vertical edges in E', either $\cup_{j=1}^{2k+1}\langle V_{2j}\rangle$ contains at most one edge of E', or $\cup_{j=0}^{2k+1}\langle V_{2j+1}\rangle$ contains at most one edge of E'.

For the former case, if $\cup_{i=0}^{k}\langle V_{4i+2}\rangle$ contains one edge of E', then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+2}, v_{34k+3}\}$$

is a dominating set of $3k+3$ vertices of G'; if $\cup_{i=1}^{k}\langle V_{4i}\rangle$ contains one edge of E', then $\cup_{j=1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+3}, v_{34k+3}\}$ is a dominating set of $3k+3$ vertices of G'. For other cases, we can similarly obtain a dominating set of $3k+3$ vertices of G' .

For the latter case, by symmetry, we may suppose $\cup_{i=0}^{k}\langle V_{4i+3}\rangle \cap E' = \emptyset$, then $\cup_{j=0}^{k}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\}$ is a dominating set of $3k+3$ vertices of G'.

Case 3. There are two vertical edges and two horizontal edges in E'.

We assume that $e_1 = v_{11}v_{14k+3}$ is a horizontal edge of E' and e_4 is another horizontal edge of E'. First we assume that $\cup_{i=0}^{k}\langle V_{4i+1}\rangle\ \cup_{i=0}^{k}\langle V_{4i+3}\rangle$ contains at most one edge of E'. By symmetry, let $\cup_{i=0}^{k}\langle V_{4i+1}\rangle \cap E' = \emptyset$. Note that

$$D_1 = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$, we can assume that e_4 is incident to some vertices of D_1. If e_4 is incident to a vertex of $H_1 \cap D_1$ or $H_2 \cap D_1$, then

$$\cup_{j=0}^{k}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\}$$

dominates G'; if e_4 is incident to a vertex of $H_3 \cap D_1$, then

$$\cup_{j=0}^{k}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

dominates G'.

Next we assume $\cup_{j=1}^{2k+1}\langle V_{2j}\rangle \cap E' = \emptyset$. Then,

$$D_2 = \cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+2}, v_{34k+3}\}$$

is a dominating set of $3k+3$ vertices of $G - \{e_1, e_2, e_3\}$. Similarly, if e_4 is incident to a vertex of $H_2 \cap D_2$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+3}, v_{34k+3}\}$$

dominates G'; if e_4 is incident to a vertex of $H_3 \cap D_2$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup \{v_{14k+2}, v_{34k+2}, v_{24k+3}\}$$

dominates G'; if e_4 is incident to a vertex of $H_1 \cap D_2$, then

$$\cup_{j=1}^{k}\{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{14k+3}, v_{34k+3}\}$$

dominates G'.

Case 4. There are one vertical edge and three horizontal edges in E'.

Let $e_1 = v_{11}v_{14k+3}$ and e_2 be the vertical edge. We may assume $e_2 \notin \cup_{i=0}^{k}\langle V_{4i+1}\rangle$. Then,

$$D_3 = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2\}$. Similarly, we can assume that e_3 or e_4 is incident to one vertex of D_3. We distinguish two subcases.

Subcase (4.1) e_3 is incident to a vertex of $H_1 \cap D_3$.

Let $D_4 = \cup_{j=0}^{k}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\}$, then D_4 dominates $G - \{e_1, e_2, e_3\}$. If e_4 is incident to a vertex of $H_2 \cap D_4$ or $H_3 \cap D_4$, then

$$\cup_{j=0}^{k}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

dominates G'. If e_4 is incident to a vertex of $H_1 \cap D_4$, we distinguish the following four cases.

(i) $e_3 = v_{14i_1+1}v_{14i_1+2}, e_4 = v_{14i_2+2}v_{14i_2+3}$ for some $0 \le i_1, i_2 \le k$. Clearly,

$$\cup_{j=1}^{k}\{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{21}, v_{24k+2}, v_{24k+3}\}$$

or

$$\cup_{j=1}^{k}\{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{31}, v_{34k+2}, v_{34k+3}\}$$

dominates G' unless $e_2 = v_{24i+2}v_{34i+2}$ for some $0 \le i \le k$. But if $e_2 = v_{24i+2}v_{34i+2}$ for some $0 \le i \le k$, then, for $i_2 = 0$,

$$\cup_{j=1}^{k}\{v_{14j-1}, v_{24j+1}, v_{34j+1}\} \cup \{v_{31}, v_{12}, v_{14k+3}\}$$

dominates G'; for $i_2 = k$,

$$\cup_{j=0}^{k-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup \{v_{34k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates G'; for $1 \le i_2 \le k-1$,

$$\cup_{j=0}^{i_2-1}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup \{v_{34i_2+1}, v_{14i_2+2}, v_{14i_2+3}\}$$
$$\cup_{j=i_2+1}^{k}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$$

dominates G'.

(ii) $e_3 = v_{14i_1+1}v_{14i_1+2}, e_4 = v_{14i_2+3}v_{14(i_2+1)}$ for some $0 \le i_1 \le k$, $0 \le i_2 \le k-1$. We first assume $1 \le i_1 \le k-1$, then

$$\cup_{j=0}^{i_1-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14i_1+1}\} \cup_{j=i_1+1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\}$$
$$\cup \{v_{14k+2}, v_{14k+3}\}$$

dominates G' unless $e_2 \in \langle V_{4i+2}\rangle$ for some $1 \le i \le k$ or $e_2 \in \langle V_{4k+3}\rangle$. If $e_2 \in \langle V_{4k+3}\rangle$, we can easily choose a dominating set of $3k+3$ vertices of G'. If $e_2 \in \langle V_{4i+2}\rangle$ for some $1 \le i \le k$, then for $i_1 \le i_2$,

$$\cup_{j=0}^{i_2}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\} \cup_{j=i_2+1}^{k}\{v_{34j}, v_{14j+2}, v_{24j+2}\}$$

dominates G'; for $i_1 > i_2$,

$$\{v_{12}, v_{22}, v_{34}\} \cup_{j=2}^{i_1}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup_{j=i_1+1}^{k}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\}$$

or

$$\{v_{12}, v_{22}, v_{34}\} \cup_{j=1}^{i_1}\{v_{14j+3}, v_{24j+3}, v_{34j+1}\} \cup_{j=i_1+1}^{k}\{v_{34j+1}, v_{24j+2}, v_{34j+3}\}$$

dominates G'. For $i_1 = 0$ or $i_1 = k$, we can similarly choose a dominating set of $3k+3$ vertices of G'.

(iii) $e_3 = v_{14i_1}v_{14i_1+1}, e_4 = v_{14i_2+3}v_{14(i_2+1)}$ for some $0 \leq i_1 \leq k$, $0 \leq i_2 \leq k-1$. If $e_2 \notin \cup_{i=0}^{k} \langle V_{4i+2} \rangle$, then

$$\{v_{11}\} \cup_{j=1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{14k+3}\}$$

dominates G'; and if $e_2 \in \langle V_{4i+2} \rangle$ for some $0 \leq i \leq k$, then

$$\{v_{12}, v_{22}\} \cup_{j=1}^{k} \{v_{34j}, v_{14j+2}, v_{24j+2}\} \cup \{v_{34k+3}\}$$

dominates G'.

(iv) $e_3 = v_{14i_1}v_{14i_1+1}, e_4 = v_{14i_2+2}v_{14i_2+3}$ for some $1 \leq i_1 \leq k, 0 \leq i_2 \leq k$. First let $i_2 = 0$. If $e_2 \in \langle V_{4i} \rangle$ for some $1 \leq i \leq k$, then

$$\{v_{11}, v_{12}, v_{13}\} \cup_{j=1}^{k} \{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$$

dominates G'; if $e_2 \notin \langle V_{4i} \rangle$, then

$$\{v_{31}, v_{s2}, v_{s3}\} \cup_{j=1}^{k} \{v_{34j}, v_{14j+2}, v_{24j+2}\}$$

dominates G', where s is $1, 2$ or 3 according to e_2. For $i_2 > 0$, we can similarly choose a dominating set of $3k+3$ vertices of G'.

Subcase (4.2) e_3 is incident to a vertex of $H_2 \cap D_3$ or $H_3 \cap D_3$. By a symmetry, we suppose that e_3 is incident to a vertex of $H_2 \cap D_3$. Then,

$$D_5 = \cup_{j=0}^{k} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$.

First suppose that e_4 is incident to a vertex of $H_1 \cap D_5$. Now, if $e_2 \notin \cup_{i=0}^{k} \langle V_{4i+3} \rangle$, then

$$\cup_{j=0}^{k} \{v_{14j+1}, v_{24j+1}, v_{34j+3}\}\}$$

dominates G'. If $e_2 \in \langle V_{4i+3} \rangle$ for some $0 \leq i \leq k$, then $e_2 \neq v_{24k+3}v_{34k+3}$. Thus, if $e_3 = v_{14i_1+2}v_{14i_1+3}, e_4 = v_{24i_2+3}v_{24(i_2+1)}$ for some $0 \leq i_1 \leq k$, $0 \leq i_2 \leq k-1$, then

$$\{v_{21}\} \quad \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{24k+3}\}$$

dominates G'. For other cases, we can similarly choose a dominating set of $3k+3$ vertices of G'.

For that e_4 is incident to a vertex of $H_3 \cap D_4$, we can similarly choose a dominating set of $3k+3$ vertices of G' as above.

Case 5. All edges of E' are horizontal edges.

By symmetry, we may suppose that

$$|\langle H_1 \rangle \cap E'| \geq |\langle H_2 \rangle \cap E'| \geq |\langle H_3 \rangle \cap E'|.$$

Then, $|\langle H_1 \rangle \cap E'| \geq 2$. We always assume $e_1 = v_{11}v_{14k+3}$. Since $\langle H_1 \rangle$ contains at least two edges of E', we may assume that $e_2 = v_{14i+1}v_{14i+2}$ or $e_2 = v_{14i+2}v_{14i+3}$ for some $0 \leq i \leq k$, or $e_2 = v_{14i-1}v_{14i}$ or $e_2 = v_{14i}v_{14i+1}$ for some $1 \leq i \leq k$. By symmetry, we only suppose that $e_2 = v_{14i+1}v_{14i+2}$ for $0 \leq i \leq k$ or $e_2 = v_{14i-1}v_{14i}$ for some $1 \leq i \leq k$. In the follow, it divides into three subcases by the number of $|\langle H_1 \rangle \cap E'|$.

Subcase (5.1) $|\langle H_1\rangle \cap E'| = 2$. Then, either $|\langle H_2\rangle \cap E'| = |\langle H_3\rangle \cap E'| = 1$ or $|\langle H_2\rangle \cap E'| = 2$ and $H_3 \cap E' = \emptyset$.

First we assume $|\langle H_2\rangle \cap E'| = |\langle H_3\rangle \cap E'| = 1$ and $e_2 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$. Note that $D_1' = \cup_{j=0}^{k}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$ dominates $G - \{e_1, e_2\}$, e_3 or e_4 is incident to $H_2 \cap D_1$ or $H_3 \cap D_1$. By symmetry, we assume that e_3 is incident to a vertex v_{24i+1} for some $0 \le i \le k$. Then,

$$D_2' = \cup_{j=0}^{k}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, we only need to consider that e_4 is incident to v_{34i+1} for some $0 \le i \le k$.

If $e_3 = v_{24i_1+1}v_{24i_1+2}$ for some $0 \le i_1 \le k$ and $e_4 = v_{31}v_{32}$, then

$$\{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates G'. If $e_3 = v_{24i_1+1}v_{24i_1+2}$ for some $0 \le i_1 \le k$ and $e_4 = v_{34i_2+1}v_{34i_2+2}$ for some $1 \le i_2 \le k$, then

$$\{v_{31}\} \cup_{j=1}^{i_2} \{v_{14j-1}, v_{24j-1}, v_{34j+1}\} \cup_{j=i_2+1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates G'. If $e_3 = v_{24i_1+1}v_{24i_1+2}$ for some $0 \le i_1 \le k$ and $e_4 = v_{34i}v_{34i+1}$ for some $0 \le i \le k$, then

$$\{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates G'. If $e_3 = v_{24i_1}v_{24i_1+1}$ for some $1 \le i_1 \le k$ and $e_4 = v_{34i_2}v_{34i_2+1}$ for some $1 \le i_2 \le k$, then, if $e_2 = v_{14k+1}v_{14k+2}$,

$$\cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates G'; otherwise,

$$\{v_{21}, v_{31}, v_{13}\} \cup_{j=1}^{k-1} \{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}, v_{14k+1}, v_{14k+2}\}$$

dominates G'. By symmetry, we can similarly choose a dominating set of $3k+3$ vertices of G' for another case of e_3, e_4.

Secondly, we assume that $|\langle H_2\rangle \cap E'| = 2$ and $\langle H_3\rangle \cap E' = \emptyset$ and $e_2 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$. Also note that

$$D_1' = \cup_{j=0}^{k}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$$

dominates G', by assuming that e_3 is incident to v_{24i+1} for some $0 \le i \le k$, we obtain that

$$D_3' = \cup_{j=0}^{k}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

dominates G'. Similarly, we assume that e_4 is incident to v_{24i+3} for some $0 \le i \le k$.

Now if $e_3 = v_{24i_1+1}v_{24i_1+2}$ for some $0 \le i_1 \le k$ and $e_4 = v_{24i_2-1}v_{24i_2}$ for some $1 \le i_2 \le k$, then, for $i_1 = 0$,

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{24k+3}\}$$

dominates G'; for $1 \le i_1 \le k-1$,

$$\{v_{21}\} \cup_{j=1}^{i_1} \{v_{14j-1}, v_{34j-1}, v_{24j+1}\} \cup_{j=i_1+1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{24k+3}\}$$

dominates G'; for $i_1 = k$,

$$\cup_{j=0}^{k-1}\{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}, v_{24k+2}, v_{24k+3}\}$$

dominates G'.

If $e_3 = v_{24i_1+1}v_{24i_1+2}$ for some $0 \le i_1 \le k$ and $e_4 = v_{24i_2+2}v_{24i_2+3}$ for some $0 \le i_2 \le k$, then

$$\{v_{31}, v_{32}, v_{14}, v_{24}, \dots, v_{14k}, v_{24k}, v_{34k+2}, v_{34k+3}\}$$

dominates G'.

If $e_3 = v_{24i_1}v_{24i_1+1}$ for some $1 \le i_1 \le k$ and $e_4 = v_{24i_2-1}v_{24i_2}$ for some $1 \le i_2 \le k$, then

$$\{v_{21}, v_{22}, v_{14}, v_{34}, \dots, v_{14k}, v_{34k}, v_{24k+2}, v_{24k+3}\}$$

dominates G'.

If $e_3 = v_{24i_1}v_{24i_1+1}$ for some $1 \le i_1 \le k$ and $e_4 = v_{24i_2+2}v_{24i_2+3}$ for some $0 \le i_2 \le k$, then

$$\cup_{j=0}^{i_1-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup \{v_{24i_1}\} \cup_{j=i_1}^{k-1} \{v_{24j+1}, v_{14j+3}, v_{34j+3}\} \cup \{v_{24k+1}, v_{14k+2}\}$$

dominates G'.

Thirdly, we assume that $|\langle H_2\rangle \cap E'| = |\langle H_3\rangle \cap E'| = 1$ and $e_2 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$. Note that

$$D_4' = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2\}$, we can assume that e_3 or e_4 is incident to a vertex $H_2 \cap D_1$ or $H_3 \cap D_1$. By symmetry, we assume that e_3 is incident to v_{24i+3} for some $0 \le i \le k$. Then,

$$D_5' = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{34i+3} for some $0 \le i \le k$.

If $e_3 = v_{24i+2}v_{24i+3}$ for some $0 \le i \le k$ and $e_4 = v_{34i+2}v_{34i+3}$ for some $0 \le i \le k$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+3}, v_{34k+3}\}$$

dominates G'. If $e_3 = v_{24i+2}v_{24i+3}$ for some $0 \le i \le k$ and $e_4 = v_{34i+3}v_{34(i+1)}$ for some $0 \le i \le k-1$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup \{v_{14k+2}, v_{34k+2}, v_{24k+3}\}$$

dominates G'. If $e_3 = v_{24i+3}v_{24(i+1)}$ for some $0 \le i \le k-1$ and $e_4 = v_{34i+2}v_{34i+3}$ for some $0 \le i \le k$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+2}, v_{34k+3}\}$$

dominates G'. If $e_3 = v_{24i+3}v_{24(i+1)}$ for some $0 \le i \le k-1$ and $e_4 = v_{34i+3}v_{34(i+1)}$ for some $0 \le i \le k-1$, then, by assuming $e_2 = v_{14i_1-1}v_{14i_1}$ for some $1 \le i_1 \le k$,

$$\cup_{j=0}^{i_1-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup_{j=i_1}^{k-1} \{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}, v_{14k+1}, v_{14k+2}\}$$

dominates G'.

Finally, we assume that $|\langle H_2\rangle \cap E'| = 2$ and $e_2 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$. Note that

$$D_6' = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2\}$, we assume that e_3 is incident to v_{24i+3} for some $0 \le i \le k$. Note that

$$D_7' = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i+1} for some $0 \le i \le k$.

If $e_3 = v_{24i+2}v_{24i+3}$ for some $0 \le i \le k$ and $e_4 = v_{24i+1}v_{24i+2}$ for some $0 \le i \le k$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup \{v_{14k+2}, v_{34k+2}, v_{24k+3}\}$$

dominates G'. If $e_3 = v_{24i_1+2}v_{24i_1+3}$ for some $0 \le i_1 \le k$ and $e_4 = v_{24i}v_{24i+1}$ for some $1 \le i \le k$, then

$$\cup_{j=0}^{i_1-1} \{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$
$$\cup \{v_{14i_1+1}, v_{24i_1+2}, v_{24i_1+3}\} \cup_{j=i_1+1}^{k} \{v_{14j+1}, v_{34j+1}, v_{24j+3}\}$$

dominates G'. If $e_3 = v_{24i_1+3}v_{24(i_1+1)}$ for some $0 \le i_1 \le k$ and $e_4 = v_{24i+1}v_{24i+2}$ for some $0 \le i \le k$, then

$$\{v_{21}\}\cup_{j=1}^{i_1}\{v_{14j-2}, v_{24j-2}, v_{34j}\}\cup\{v_{14i_1+2}, v_{24i_1+3}\}\cup_{j=i_1+1}^{k}\{v_{24j}, v_{14j+2}, v_{34j+2}\}$$

dominates G'. If $e_3 = v_{24i+3}v_{24(i+1)}$ for some $0 \le i \le k-1$ and $e_4 = v_{24i}v_{24i+1}$ for some $1 \le i \le k$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+2}, v_{34k+3}\}$$

dominates G'.

Subcase (5.2) $|\langle H_1\rangle \cap E'| = 3$ and $\langle H_2\rangle \cap E'| = 1$. We distinguish two cases.

(i) $e_2 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$.

Note that $D_1' = \cup_{j=0}^{k}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$ dominates $G - \{e_1, e_2\}$, e_3 or e_4 is incident to a vertex of $H_1 \cap D_1$ or $H_2 \cap D_1$.

We first assume that e_3 is incident to a vertex v_{14i+3} of $H_1 \cap D_1$. Now if $e_3 = v_{14i+2}v_{14i+3}$ for some $0 \le i \le k$, then

$$D_2' = \{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i} for some $1 \le i \le k$, and it follows

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{24k+3}\}$$

dominates G'.

If $e_3 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$, and $e_2 = v_{11}v_{12}$, then

$$\{v_{11}\} \cup_{j=1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{14k+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i} for some $1 \le i \le k$. and it follows

$$\{v_{31}, v_{32}\} \cup_{j=1}^{k} \{v_{34j-1}, v_{14j+1}, v_{24j+1}\} \cup \{v_{34k+3}\}$$

dominates G' for $e_4 = v_{24i-1}v_{24i}$ for some $1 \le i \le k$; and

$$\{v_{21}, v_{22}\} \cup_{j=1}^{k} \{v_{24j-1}, v_{14j+1}, v_{34j+1}\} \cup \{v_{24k+3}\}$$

dominates G' for $e_4 = v_{24i}v_{24i+1}$ for some $1 \le i \le k$.

If $e_2 = v_{14i_1+1}v_{14i_1+2}$ for some $1 \le i_1 \le k-1$, then

$$\{v_{11}\} \cup_{j=1}^{i_1} \{v_{24j-1}, v_{34j-1}, v_{14j+1}\} \cup_{j=i_1+1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{14k+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i-1} for some $1 \le i \le i_1 - 1$ or v_{24i} for some $i_1 \le i \le k$. For the former case, if $e_4 = v_{24i-2}v_{24i-1}$ for some $1 \le i \le i_1 - 1$, then

$$\begin{aligned} &\cup_{j=1}^{i_1} \{v_{14j-2}, v_{34j-2}, v_{24j}\} \\ &\cup \{v_{14i_1+1}\} \cup_{j=i_1+1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+3}\} \end{aligned}$$

dominates G'; if $e_4 = v_{24i-1}v_{24i}$ for some $1 \le i \le i_1 - 1$, then

$$\begin{aligned} &\cup_{j=1}^{i_1} \{v_{14j-2}, v_{24j-2}, v_{34j}\} \\ &\cup \{v_{34i_1+1}, v_{34i_1+2}, v_{34i_1+3}\} \cup_{j=i_1+1}^{k} \{v_{14j+1}, v_{24j+1}, v_{34k+3}\} \end{aligned}$$

dominates G'. For the latter case, if $e_4 = v_{24i-1}v_{24i}$ for some $i_1 \le i \le k$, then

$$\begin{aligned} \cup_{j=0}^{i_1-1} \{v_{14j+1}, v_{24j+3}, v_{34j+3}\} &\cup \{v_{14i_1+1}, v_{14i_1+2}, v_{34i_1+3}\} \\ &\cup \ \cup_{j=i_1+1}^{k} \{v_{14j+1}, v_{24j+1}, v_{34j+3}\} \end{aligned}$$

dominates G'; if $e_4 = v_{24i}v_{24i+1}$ for some $i_1 \le i \le k$, then

$$\begin{aligned} \cup_{j=1}^{i_1-1} \{v_{14j+1}, v_{24j+3}, v_{34j+3}\} &\cup \{v_{14i_1+1}, v_{14i_1+2}, v_{24i_1+3}\} \\ &\cup \ \cup_{j=i_1+1}^{k} \{v_{14j+1}, v_{34j+1}, v_{24j+3}\} \end{aligned}$$

dominates G'.

If $e_2 = v_{14k+1}v_{14k+2}$, then

$$\cup_{j=0}^{k} \{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i+3} for some $0 \le i \le k-1$, and it follows

$$\cup_{j=1}^{k} \{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup \{v_{24k+1}, v_{24k+2}, v_{24k+3}\}$$

dominates G' for $e_4 = v_{24i+2}v_{24i+3}$ for some $0 \le i \le k-1$; and

$$\cup_{j=1}^{k} \{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{34k+1}, v_{34k+2}, v_{34k+3}\}$$

dominates G' for $e_4 = v_{24i+3}v_{24(i+1)}$ for some $0 \le i \le k-1$.

Secondly we assume that e_3 is incident to a vertex of $H_2 \cap D_1$. Then,

$$D'_3 = \cup_{j=0}^{k}\{v_{34j+1}, v_{14j+3}, v_{24j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{14i+3} for some $0 \le i \le k$. If $e_4 = v_{14i+2}v_{14i+3}$ for some $0 \le i \le k$, then

$$\{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates G' for $e_3 = v_{24i+1}v_{24i+2}$ for some $1 \le i \le k$, and it follows

$$\{v_{21}\} \cup_{j=1}^{k} \{v_{24j-2}, v_{14j}, v_{34j}\} \cup \{v_{24k+2}, v_{24k+3}\}$$

dominates G' for $e_3 = v_{24i}v_{24i+1}$ for some $1 \le i \le k$.

If $e_4 = v_{14i+3}v_{14(i+1)}$ for some $0 \le i \le k-1$, and $e_3 = v_{24i+1}v_{24i+2}$ for some $0 \le i \le k$, then, for $e_2 = v_{11}v_{12}$,

$$\{v_{11}\} \cup_{j=1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{14k+3}\}$$

dominates G'; for $e_2 = v_{14i_1+1}v_{14i_1+2}$ for some $1 \le i_1 \le k-1$,

$$\{v_{11}\} \cup_{j=1}^{i_1} \{v_{24j-1}, v_{34j-1}, v_{14j+1}\} \cup_{j=i_1+1}^{k} \{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{14k+3}\}$$

dominates G'; for $e_2 = v_{14k+1}v_{14k+2}$,

$$\cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates G'. If $e_3 = v_{24i}v_{24i+1}$ for some $1 \le i \le k$, and $e_2 = v_{11}v_{12}$,

$$\{v_{21}, v_{22}, v_{24k+3}\} \cup_{j=1}^{k} \{v_{24j-1}, v_{14j+1}, v_{34j+1}\}$$

dominates G'; if $e_2 = v_{14i_1+1}v_{14i_1+2}$ for some $1 \le i_1 \le k-1$,

$$\cup_{j=0}^{i_1-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14i_1+1}, v_{14i_1+2}, v_{24i_1+3}\} \cup_{j=i_1}^{k} \{v_{14j+1}, v_{34j+1}, v_{24j+3}\}$$

dominates G'; if $e_2 = v_{14k+1}v_{14k+2}$,

$$\cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates G'.

(ii) $e_2 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$. Note that

$$D'_4 = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2\}$, e_3 is incident to a vertex of D_1. First we assume that e_3 is incident to v_{14i+1} for some $0 \le i \le k$. For the case $e_3 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$, it has the same situation as some case in (i) which has been verified. If $e_3 = v_{14i}v_{14i+1}$ for some $1 \le i \le k$, then

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+2}, v_{34k+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{24i+2} for some $0 \le i \le k$. Then,

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{34j-2}, v_{24j}\} \cup \{v_{14k+2}, v_{34k+2}, v_{24k+3}\}$$

dominates G'. Next we assume that e_3 is incident to v_{24i+3} for some $0 \le i \le k$. Then,

$$\cup_{j=0}^{k}\{v_{14j+1}, v_{24j+1}, v_{34j+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{14i+1} for some $0 \le i \le k$. If $e_4 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$, the same situation has appeared in (i) which has been verified. If $e_4 = v_{14i}v_{14i+1}$ for some $1 \le i \le k$, then it is also easy to choose a dominating set of $3k+3$ vertices of G' as above.

Subcase (5.3) $|\langle H_1\rangle \cap E'| = 4$, we also distinguish two cases.

(i) $e_2 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$. Note that

$$D_1' = \cup_{j=0}^{k}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\}$$

dominates $G - \{e_1, e_2\}$, e_3 is incident to v_{14i+3} for some $0 \le i \le k$. First let $e_3 = v_{14i+2}v_{14i+3}$ for some $0 \le i \le k$, then

$$\{v_{31}\} \cup_{j=1}^{k} \{v_{34j-2}, v_{14j}, v_{24j}\} \cup \{v_{34k+2}, v_{34k+3}\}$$

dominates $G - \{e_1, e_2, e_3\}$. Now if $e_4 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$, then

$$\{v_{21}, v_{22}, v_{24k+3}\} \cup \ \cup_{j=1}^{k}\{v_{24j-1}, v_{14j+1}, v_{34j+1}\}$$

dominates G' for $e_2 = v_{11}v_{12}$;

$$\{v_{11}, v_{23}, v_{33}, \dots, v_{14i_1+1}, v_{34i_1+2}, v_{34i_1+3}, v_{14(i_1+1)+1}, v_{24(i_1+1)+1}, \dots, v_{34k+3}\}$$

dominates G' if $e_2 = v_{14i_1+1}v_{14i_1+2}$ for some $1 \le i_1 \le k-1$;

$$\cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates G' for $e_2 = v_{14k+1}v_{14k+2}$. If $e_4 = v_{14i}v_{14i+1}$ for some $1 \le i \le k$, then, by symmetry, we can similarly choose a dominating set of $3k+3$ vertices of G' as above.

Next let $e_3 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$. Now if $e_2 = v_{11}v_{12}$, then

$$D_2' = \{v_{11}, v_{14k+2}, v_{14k+3}\} \cup_{j=1}^{k} \{v_{14j-2}, v_{24j}, v_{34j},\}$$

dominates $G - \{e_1, e_2, e_3\}$. Thus, e_4 is incident to v_{14i+2} for some $0 \le i \le k$. Then, for $e_4 = v_{14i+2}v_{14i+3}$ for some $0 \le i \le k-1$,

$$\{v_{21}, v_{22}, v_{24k+3}\} \cup_{j=1}^{k} \{v_{24j-1}, v_{14j+1}, v_{34j+1}\}$$

dominates G'; for $e_4 = v_{14i+1}v_{14i+2}$ for some $1 \le i \le k$, by assuming $e_3 = v_{14i_2-1}v_{14i_2}$ for some $1 \le i_2 \le k$,

$$\cup_{j=0}^{i_2-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup_{j=i_2}^{k-1} \{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}, v_{14k+1}, v_{14k+2}\}$$

dominates G'.

If $e_2 = v_{14k+1}v_{14k+2}$, then

$$D_3' = \cup_{j=0}^{k-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14k+1}, v_{14k+2}, v_{14k+3}\}$$

dominates $G-\{e_1,e_2,e_3\}$. Thus, e_4 is incident to v_{14i+1} for some $0 \le i \le k-1$. By assuming $e_3 = v_{14i_2-1}v_{14i_2}$ for some $1 \le i_2 \le k$, and $e_4 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k-1$, it follows

$$\cup_{j=0}^{i_2-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup \cup_{j=i_2}^{k-1}\{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}, v_{14k+1}, v_{14k+2}\}$$

dominates G'; for that $e_4 = v_{14i}v_{14i+1}$ for some $1 \le i \le k-1$,

$$\cup_{j=1}^{k}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{34k+1}, v_{34k+2}, v_{34k+3}\}$$

dominates G'.

If $e_2 = v_{14i_1+1}v_{14i_1+2}$ for some $1 \le i_1 \le k-1$, then

$$D'_4 = \cup_{j=0}^{i_1-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14i_1+1}, v_{14i_1+2}, v_{14k+3}\} \cup_{j=i_1}^{k}\{v_{24j}, v_{34j}, v_{14j+2}\}$$

dominates $G-\{e_1,e_2,e_3\}$. Thus, e_4 is incident to v_{14i+1} for some $0 \le i \le i_1$ or v_{14i+2} for some $i_1 \le i \le k$. If $e_4 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le i_1-1$, then

$$\cup_{j=0}^{i_2-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup \{v_{14i_2}, v_{14k+1}, v_{14k+2}\} \cup_{j=i_2+1}^{k}\{v_{24j-2}, v_{34j-2}, v_{14j}\}$$

dominates G'; if $e_4 = v_{14i}v_{14i+1}$ for some $1 \le i \le i_1$, then

$$\cup_{j=1}^{i_1}\{v_{14j-2}, v_{24j-2}, v_{34j}\} \cup \{v_{14i_1+1}, v_{14k+2}, v_{34k+3}\} \cup_{j=i_1+1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\}$$

dominates G'; if $e_4 = v_{14i+2}v_{14i+3}$ for some $i_1 \le i \le k$, then

$$\cup_{j=0}^{i_1-1}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\} \cup \{v_{14i_1+1}, v_{14i_1+2}, v_{34k+3}\} \cup_{j=i_1+1}^{k}\{v_{34j-1}, v_{14j+1}, v_{24j+1}\}$$

dominates G'; if $e_4 = v_{14i+1}v_{14i+2}$ for some $i_1+1 \le i \le k$, then

$$\cup_{j=0}^{i_2-1}\{v_{24j+1}, v_{34j+1}, v_{14j+3}\} \cup_{j=i_2}^{k-1}\{v_{14j}, v_{24j+2}, v_{34j+2}\} \cup \{v_{14k}, v_{14k+1}, v_{14k+2}\}$$

dominates G'.

(ii) $e_2 = v_{14i-1}v_{14i}$ for some $1 \le i \le k$. Note that $D'_5 = \cup_{j=0}^{k}\{v_{14j+1}, v_{24j+3}, v_{34j+3}\}$ dominates $G-\{e_1,e_2\}$, e_3 is incident to v_{14i+1} for some $0 \le i \le k$. If $e_3 = v_{14i+1}v_{14i+2}$ for some $0 \le i \le k$, the same situation has appeared in (i) which has been verified. If $e_3 = v_{14i}v_{14i+1}$ for some $1 \le i \le k$, then

$$D'_6 = \cup_{j=1}^{k}\{v_{14j-2}, v_{24j}, v_{34j}\} \cup \{v_{14k+2}, v_{24k+3}, v_{34k+3}\}$$

dominates $G-\{e_1,e_2,e_3\}$. Thus, e_4 is incident to v_{14i+2} for some $0 \le i \le k$. By symmetry, the same situation also has appeared in (i) which has been verified. This completes the proof. □

Theorem 2. *$b(C_3 \times C_{4k+3}) = 5$ for every $k \ge 1$.*

Proof. By Lemma 3, it suffices to show that $b(C_3 \times C_{4k+3}) \leq 5$. Let $E' = \{v_{11}v_{14k+3}, v_{31}v_{34k+3}, v_{13}v_{14}, v_{23}v_{24}, v_{24}v_{25}\}$. We will show that $\gamma(G - E') \geq 3k + 4 > \gamma(G)$. Let G_1 be the induced subgraph of $G - E'$ by $V_1 \cup V_2 \cup V_3$ and $G_2 = (G - E') - (V_1 \cup V_2 \cup V_3)$. It is easy to see that at least three vertices are needed to dominate the vertices $V(G_1) - \{v_{21}, v_{33}\}$. On other hand, by using the idea of Lemma 1, we can show that at least $3k + 1$ vertices are needed to dominate $V(G_2) - \{v_{34}, v_{24k+3}\}$. Let D be a dominating set of $G - E'$. Since there are only two edges $v_{21}v_{24k+3}, v_{33}v_{34}$ between $V(G_1)$ and $V(G_2)$ in $G - E'$, $|D \cap (V(G_1) - \{v_{21}, v_{33}\})| \geq 3$ and $|D \cap (V(G_2) - \{v_{34}, v_{24k+3}\})| \geq 3k + 1$. By the construction of G_1 and G_2, we can see that $\gamma(G - E') \geq 3k + 4 > \gamma(G)$. It completes the proof. □

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