J. Korean Math. Soc. **44** (2007), No. 6, pp. 1197–1211

A RELATIONSHIP BETWEEN VERTICES AND QUASI-ISOMORPHISMS FOR A CLASS OF BRACKET GROUPS

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Abstract. In this article, we characterize the quasi-isomorphism classes of bracket groups in terms of vertices using vertex-switches. In particular, if two bracket groups are quasi-isomorphic, then there is a sequence of vertex-switches transforming a collection of vertices of a group to a collection of vertices of the other group.

1. Introduction

Let C be a finite rank completely decomposable torsion-free abelian group and X a rank one pure subgroup of C, then in the terminology of [7] the Butler group C/X is called a $\mathcal{B}^{(1)}$-group. Suppose $A_1, \ldots, A_n$ $(n \geq 2)$ are nonzero subgroups of the additive group of rationals $\mathbf{Q}$, then $\mathcal{G}[A_1, \ldots, A_n] =$ the cokernel of the diagonal embedding $\bigcap_{i=1}^n A_i \to \bigoplus_{i=1}^n A_i$ is a rank $n-1$ Butler group. The groups of the form $\mathcal{G}[A_1, \ldots, A_n]$ are a subclass of $\mathcal{B}^{(1)}$-groups and we will adopt a colloquial terminology and call such a group a *bracket group*. The class of bracket groups has been studied extensively and classified up to numerical quasi-isomorphism invariants [9], and {0,1}-matrices [7], and is also studied by the order structures of typeset called *tent* [5, 6] (certain finite Z_2-representation.) It has been shown that two bracket groups $G = \mathcal{G}[A_1, \ldots, A_n]$ and $H = \mathcal{G}[B_1, \ldots, B_n]$ are quasi-isomorphic if and only if rank $G(\tau) =$ rank $H(\tau)$ for each type τ in the lattice generated by typeset $G \cup$ typeset H [9, Main Theorem]. These numerical invariants are a notable achievement in studying bracket groups recently.

In this article, we will investigate the relationship between A_i's and B_i's for quasi-isomorphic bracket groups $\mathcal{G}[A_1, \ldots, A_n]$ and $\mathcal{G}[B_1, \ldots, B_n]$. The main result in [11] states that two strongly indecomposable bracket groups

Received December 15, 2005.

2000 *Mathematics Subject Classification.* Primary 20K15.

Key words and phrases. Butler groups, $\mathcal{B}^{(1)}$-groups, quasi-decomposable bracket groups, quasi-isomorphisms.

This work was supported in part by a grant from the City University of New York PSC-CUNY Research Award Program.

$\mathcal{G}[A_1, \ldots, A_n]$ and $\mathcal{G}[B_1, \ldots, B_n]$ are quasi-isomorphic if and only if, after cotrimming, A_i's can be obtained from B_i's by a sequence of vertex-switches. (In [11], vertex-switches are called co-two-vertex exchanges.) The purpose of this article is to show that strongly indecomposability condition can be removed for classes of completely decomposable, flower-like, and co-elementary bracket groups.

2. Preliminaries

A group in this article means a torsion-free abelian group. The class of bracket groups $\mathcal{G}[A_1, \ldots, A_n]$ is the dual class of the groups of the form $\mathcal{G}(A_1, \ldots, A_n)$ in the sense of quasi-isomorphism Butler duality of [2], where $\mathcal{G}(A_1, \ldots, A_n)$ = the kernel of the codiagonal map $\bigoplus_{i=1}^n A_i \to \mathbf{Q}$ given by $(a_1, \ldots, a_n) \mapsto \sum a_i$. Thus, the results obtained for the class of groups $\mathcal{G}(A_1, \ldots, A_n)$ will be dualized and used in this article for the class of bracket groups.

Notation. If n is a positive integer, then we write $\overline{n}$ for the set $\{1, \ldots, n\}$. Two groups G and H are *quasi-isomorphic*, $G \doteq H$, if G is isomorphic to a subgroup of finite index in H.

A *type* is an isomorphism class of subgroups of $\mathbf{Q}$. If $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are nonzero subgroups of $\mathbf{Q}$ such that A_i is isomorphic to B_i for each $i \in \overline{n}$, then $\mathcal{G}[A_1, \ldots, A_n]$ is quasi-isomorphic to $\mathcal{G}[B_1, \ldots, B_n]$. Hence the quasi-isomorphisms equivalence class of $\mathcal{G}[A_1, \ldots, A_n]$ is determined by $\tau_1, \ldots, \tau_n$, where τ_i = type A_i for each $i \in \overline{n}$. We shall refer to the group $G = \mathcal{G}[A_1, \ldots, A_n]$ as $G = [\tau_1, \ldots, \tau_n]$, keeping in mind that G is defined up to quasi-isomorphism, and the τ_i's will be called the *vertices of* G.

Let $G = [\tau_1, \ldots, \tau_n]$. A collection of types $\tau_1, \ldots, \tau_n$ is called *cotrimmed* if $\tau_i \geq \bigwedge_{j \neq i} \tau_j$ for each i. From here on, otherwise stated, all collections of types $\tau_1, \ldots, \tau_n$ in bracket groups $[\tau_1, \ldots, \tau_n]$ in this article are cotrimmed.

For a type τ, we define an equivalence relation on $\overline{n}$ as follows: i and j are τ-equivalent if either $i = j$ or else there is a finite sequence of integers $i = i_1, \ldots, i_{k+1} = j$ in $\overline{n}$ such that $\tau_{i_r} \vee \tau_{i_{r+1}} \not\geq \tau$ for each $r = 1, \ldots, k$. We will regard τ-equivalence as on G rather than on the index set $\overline{n}$.

Notation. For nonempty subset $I = \{i_1, \ldots, i_k\}$ of $\overline{n}$, define $\tau_I = \tau_{i_1} \wedge \cdots \wedge \tau_{i_k}$. For a type τ, let $G(\tau) = \{x \in G\colon \text{type}(x) \geq \tau\}$. The subgroup $G(\tau)$ is called the *τ-socle* of G.

Lemma 2.1 (Dual of Lemma 1.4 in [11]). *Let $G = [\tau_1, \ldots, \tau_n]$. Suppose $I_1, \ldots, I_m$ are τ-equivalence classes in G. Then*

(a) *$G(\tau) = [\sigma_1, \ldots, \sigma_m]$ where $\sigma_i = \tau_{I_i} \vee \tau_{J_i}$ and $J_i = \overline{n} \setminus I_i$ for each $i \in \overline{m}$; and rank $G(\tau) = m - 1$.*

(b) *If $\tau_k \geq \tau$ then $I_t = \{k\}$ for some $t \in \overline{m}$.*

Lemma 2.2. $[\tau_1, \tau_2, \ldots, \tau_n] \doteq [\sigma_1, \tau_2, \ldots, \tau_n]$ *if and only if* $\tau_1 = \sigma_1$.

Proof. Let $G = [\tau_1, \tau_2, \ldots, \tau_n]$ and $H = [\sigma_1, \tau_2, \ldots, \tau_n]$. Suppose $G \doteq H$, then rank $G(\tau)$ = rank $H(\tau)$ for each type τ. Let the τ_1-equivalence classes in G

be $\{1\}, I_1, \ldots, I_k$, hence, there are $k+1$ τ_1-equivalence classes in G. Suppose $\sigma_1 \vee \tau_i \not\geq \tau_1$ for some $i \in I_j$, then $\{1\} \cup I_j$ is a subset of a τ_1-equivalence class in H. Hence, in H, there are at most k τ_1-equivalence classes by Lemma 2.1, a contradiction to the fact rank $G(\tau) =$ rank $H(\tau)$ for each type τ. Therefore, $\sigma_1 \vee \tau_i \geq \tau_1$ for each $i = 2, \ldots, n$. Since $\sigma_1, \tau_2, \ldots, \tau_n$ is cotrimmed, it follows that $\sigma_1 = \sigma_1 \vee (\tau_2 \wedge \cdots \wedge \tau_n) \geq \tau_1$. A symmetric argument shows that $\tau_1 = \sigma_1$.
The converse is clear. □

The following lemma is the dual of Theorem 4 in [12].

Lemma 2.3 ("vertex-switch"). *Let $\tau_i \neq \hat{\tau}_j$ for $i, j \in \{1, 2\}$. Then the following statements are equivalent:*

(a) $[\tau_1, \tau_2, \tau_3, \ldots, \tau_n] \doteq [\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n]$;
(b) *There is a partition X, Y of $\{3, \ldots, n\}$ such that $\tau_X \vee \tau_Y \geq \tau_1 \vee \tau_2$ and $\hat{\tau}_1 = (\tau_Y \wedge \tau_1) \vee (\tau_X \wedge \tau_2)$ and $\hat{\tau}_2 = (\tau_X \wedge \tau_1) \vee (\tau_Y \wedge \tau_2)$.*

Let $\hat{\tau}_1, \hat{\tau}_2$ be defined as in Lemma 2.3. If $G = [\tau_1, \tau_2, \tau_3, \ldots, \tau_n]$ and $H = [\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n]$ then we say H *is obtained from G by a vertex-switch.*

Corollary 2.4 (Dual of Corollary 1.2 in [11]). *Let $\tau_1, \ldots, \tau_n$ be cotrimmed and suppose X, Y is a partition of $\{3, \ldots, n\}$ such that $\tau_X \vee \tau_Y \geq \tau_1 \vee \tau_2$ and $\hat{\tau}_1 = (\tau_Y \wedge \tau_1) \vee (\tau_X \wedge \tau_2)$ and $\hat{\tau}_2 = (\tau_X \wedge \tau_1) \vee (\tau_Y \wedge \tau_2)$. Then*

(a) *$\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n$ is cotrimmed,*
(b) *$\tau_1 \vee \tau_2 = \hat{\tau}_1 \vee \hat{\tau}_2$ and $[\tau_1, \tau_2, \tau_3, \ldots, \tau_n] \doteq [\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n]$,*
(c) *$\tau_1 = (\tau_Y \wedge \hat{\tau}_1) \vee (\tau_X \wedge \hat{\tau}_2)$ and $\tau_2 = (\tau_X \wedge \hat{\tau}_1) \vee (\tau_Y \wedge \hat{\tau}_2)$.*

Corollary 2.4 states that if τ_1, τ_2 are replaced by $\hat{\tau}_1, \hat{\tau}_2$ via a vertex-switch partition X, Y, then we can also replace $\hat{\tau}_1, \hat{\tau}_2$ by τ_1, τ_2 using the same vertex-switch partition X, Y.

Let $G = [\tau_1, \ldots, \tau_n]$ and $H = [\sigma_1, \ldots, \sigma_n]$. If $\tau_i = \sigma_j$ for some j then we say τ_i is a *common vertex in G and H.* If there is a permutation ρ of $\overline{n}$ such that $\tau_1 = \sigma_{\rho(1)}, \ldots, \tau_n = \sigma_{\rho(n)}$, then we say G and H are *equivalent.* It is easy to see that if G and H are equivalent then $G \doteq H$. Suppose $\mathcal{T}$ denotes a collection of types $\tau_1, \ldots, \tau_n$, then $[\mathcal{T}]$ means the bracket group $[\tau_1, \ldots, \tau_n]$. Let $G = [\tau_1, \ldots, \tau_n]$ and $H = [\sigma_1, \ldots, \sigma_n]$. We say that *a sequence of vertex-switches $s_1, \ldots, s_m$ transforms G to H* if there is a sequence of collections of types $\mathcal{T}_1, \ldots, \mathcal{T}_{m+1}$ such that

(1) $G = [\mathcal{T}_1]$ and H is equivalent to $[\mathcal{T}_{m+1}]$,
(2) $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by the vertex-switch s_i and s_i fixes vertices common to $[\mathcal{T}_i]$ and H for each $i \in \overline{m}$.

In particular, common vertices in G and H are not replaced by any vertex-switch. Let $G \approx H$ denote that either G and H are equivalent or else there is a sequence of vertex-switches that transforms G to H. By Corollary 2.4, it can be easily shown that $\approx$ is an equivalence relation on a class of bracket groups and if $G \approx H$ then $G \doteq H$.

A group G is *strongly indecomposable* if $G \doteq H \oplus K$ implies $H = 0$ or $K = 0$. And $G = [\tau_1, \ldots, \tau_n]$ is strongly indecomposable if and only if $\{i\}$ and $\overline{n} \setminus \{i\}$ are only two τ_i-equivalence classes in G for each $i \in \overline{n}$. (This is the dual of Theorem 3 in [3].)

Lemma 2.5 (Corollary 3.3 in [11]). *Suppose both $G = [\tau_1, \ldots, \tau_n]$ and $H = [\sigma_1, \ldots, \sigma_n]$ are strongly indecomposable, then $G \approx H$ if and only if $G \doteq H$.*

Let $I = \{i_1, \ldots, i_k\}$ be a nonempty subset of $\overline{n}$, then we denote $[\tau(I)] = [\tau_{i_1}, \ldots, \tau_{i_k}]$.

Definition 2.1. A *splitting triple for G* is a triple (r, I, J) satisfying the three properties:

(1) $r \in \overline{n}$ and $I \cap J = \{r\}$,
(2) $I, J \subset \overline{n}, I \cup J = \overline{n}$ and $\min\{|I|, |J|\} \geq 2$,
(3) $\tau_I \vee \tau_J = \tau_r$.

In this case, $G \doteq [\tau(I)] \oplus [\tau(J)]$ and τ_r will be called a *splitting vertex of* G. Among all splitting triples (r, I, J) for G there is one with $|I|$ as small as possible. Call such a triple a *minimal splitting triple for G.*

A complete discussion of quasi-decomposition of a bracket group can be found in [4, 8]. In the following proposition, we will list few properties of quasi-decompositions of bracket groups which will be used in this article.

Proposition 2.6. *Suppose $G = [\tau_1, \ldots, \tau_n]$ is quasi-decomposable. Then there exist nonempty subsets $I_1, \ldots, I_k$ of $\overline{n}$, $k \geq 2$ which satisfy the following conditions*:

(a) *$|I_j| \geq 2$ for each $j = 1, \ldots, k$ and $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ with each $[\tau(I_j)]$ strongly indecomposable, and $I_1 \cup \cdots \cup I_k = \overline{n}$, and $|I_i \cap I_j| \leq 1$ whenever $i \neq j$,*
(b) *If $i \neq j$ and $I_i \cap I_j = \{r\}$, then $\tau_{I_i} \vee \tau_{I_j} = \tau_r$,*
(c) *For each $i = 1, \ldots, k-1, I_i \cap (I_{i+1} \cup \cdots \cup I_k) = \{s_i\}$, a singleton, and $(s_i, I_i, I_{i+1} \cup \cdots \cup I_k)$ is a minimal splitting triple for $[\tau(I_i \cup I_{i+1} \cup \cdots \cup I_k)]$,*
(d) *If $i \neq j$ and let $G_i = [\tau(I_i)]$, then rank $G_i(\tau_{I_j}) \leq 1$.*

Let $G = [\tau_1, \ldots, \tau_n]$. Then a quasi-decomposition $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ of G that satisfies all the conditions in Proposition 2.6 will be called a *canonical decomposition of G.* Note that (1) a canonical decomposition of G is not uniquely determined; (2) each strongly indecomposable quasi-summand $[\tau(I_i)]$ in the canonical decomposition of G is not necessarily cotrimmed.

A vertex-switch requires the cotrimmed condition on a collection of types, hence we can not directly apply a vertex-switch on quasi-summands in a canonical decomposition of a bracket group. We treat this problem by cotrimming a collection of types as follow: For a collection of types $\tau_1, \ldots, \tau_n$, we define $\tau_i' = \tau_i \vee (\bigwedge_{j \neq i} \tau_j)$ for each $i \in \overline{n}$, then it is easily checked that $\tau_1', \ldots, \tau_n'$ is cotrimmed and we will call it the *cotrimmed version of* $\tau_1, \ldots, \tau_n$. It can be easily shown that $[\tau_1, \ldots, \tau_n] \doteq [\tau'_1, \ldots, \tau'_n]$.

Example 2.1. Let ω_i be the type of subgroup of $\mathbf{Q}$ generated by $\frac{1}{p_i}$ for prime p_i and let $\omega_{ijk} = \omega_i \vee \omega_j \vee \omega_k$. Let

$$G = [\tau_1, \ldots, \tau_7] = [\omega_{457}, \omega_{45}, \omega_{25}, \omega_{15}, \omega_{156}, \omega_{12}, \omega_{13}].$$

It is easy to check that $\tau_1, \ldots, \tau_7$ is cotrimmed and $G \doteq [\tau_1, \tau_2] \oplus [\tau_2, \tau_3, \tau_4] \oplus [\tau_4, \tau_5] \oplus [\tau_4, \tau_6, \tau_7]$ is a canonical decomposition of G. Observe that τ_1, τ_2 in quasi-summand $[\tau_1, \tau_2]$ of G is not cotrimmed, the cotrimmed version of τ_1, τ_2 is τ_1, τ_1.

3. Completely decomposable bracket groups

A group is *completely decomposable* if it is a direct sum of rank one summands. R. Baer showed that two completely decomposable finite rank torsion-free abelian groups are quasi-isomorphic if and only if they have an equal number of quasi-summands of same type for all types. The goal of this section is to show that two completely decomposable bracket groups G and H are quasi-isomorphic if and only if $G \approx H$. In particular, we show that if G and H are quasi-isomorphic, then both G and H can be transformed via a sequence of vertex switches to a bracket group whose vertices are the collection of the types of all rank one quasi-summands.

Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ be a canonical decomposition of G. We call the quasi-summand $[\tau(I_i)]$ an *end-summand* of G if $|I_i \cap (\bigcup_{j \neq i} I_j)| = 1$. Recall τ_k is a splitting vertex if (k, I, J) is a splitting triple for G. The following proposition is clear from the definition of an end-summand.

Proposition 3.1. *Let* $G = [\tau_1, \ldots, \tau_n] \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ *be a canonical decomposition of* G*. If* $[\tau(I_i)]$ *is an end-summand of* G *with the splitting vertex* τ_j*, then* τ_t *is not a splitting vertex of* G *for each* $t \in I_i \setminus \{j\}$*.*

Let $G = [\tau_1, \ldots, \tau_n] \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ and we will call τ_j a *rank-1 vertex of* G if $I_i = \{j, k\}$ and $[\tau(I_i)]$ is an rank one end-summand with type equal to τ_j and $(k, \{j, k\}, \overline{n} \setminus \{j\})$ is a minimal splitting triple of G. Among all canonical decompositions of G, we will call one with the least number of splitting vertices a *basic canonical decomposition*. In the following example, we will show that there are different rank-1 vertices for different canonical decompositions of a quasi-decomposable bracket group G.

Example 3.1. Let τ be a type and let $G = [\tau_1, \tau_2, \tau_3, \tau_4] = [\tau, \tau, \tau, \tau]$.

Case 1. If we let $I_1 = \{1,2\}, I_2 = \{2,3\}, I_3 = \{3,4\}$, then $G \doteq [\tau(I_1)] \oplus [\tau(I_2)] \oplus [\tau(I_3)]$ is a canonical decomposition with the end-summands $[\tau(I_1)]$ and $[\tau(I_3)]$; rank-1 vertices τ_1, τ_4; and splitting vertices τ_2, τ_3.

Case 2. If we let $I'_1 = \{1,2\}, I'_2 = \{1,3\}, I'_3 = \{1,4\}$, then $G \doteq [\tau(I'_1)] \oplus [\tau(I'_2)] \oplus [\tau(I'_3)]$ is a canonical decomposition with the end-summands $[\tau(I'_1)]$,

$[\tau(I'_2)]$ and $[\tau(I'_3)]$; and the rank-1 vertices τ_2, τ_3, τ_4; and one splitting vertex τ_1. In this case, $G \doteq [\tau(I'_1)] \oplus [\tau(I'_2)] \oplus [\tau(I'_3)]$ is a basic canonical decomposition.

Note that a basic canonical decomposition has the most number of rank-1 vertices.

Lemma 3.2. *Let $G = [\tau_1, \dots, \tau_n]$ and $j \in \overline{n}$. Then $\tau_i \leq \tau_j$ for some $i \in \overline{n} \setminus \{j\}$ if and only if τ_j is a rank-1 vertex in a basic canonical decomposition of G. In particular, $\tau_j =$ type of rank one quasi-summand $[\tau_i, \tau_j]$ of G.*

Proof. Suppose $\tau_i \leq \tau_j$ and if we let $I = \{i, j\}, J = \overline{n} \setminus \{j\}$ then $\tau_I \vee \tau_J = (\tau_i \wedge \tau_j) \vee \tau_J = \tau_i \vee \tau_J = \tau_i$. So, (i, I, J) is a minimal splitting triple for G and $G \doteq [\tau_i, \tau_j] \oplus [\tau(\overline{n} \setminus \{j\})]$ and type $[\tau_i, \tau_j] = \tau_i \vee \tau_j = \tau_j$. Hence, τ_j is a rank-1 vertex of G.

Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ be a basic canonical decomposition of G. If τ_j is a rank-1 vertex of G, then there exists $I_r = \{i, j\}$ such that $(i, I_r, \bigcup_{t \neq r} I_t)$ is a minimal splitting triple of G. Let $Y = \bigcup_{t \neq r} I_t = \overline{n} \setminus \{j\}$. Using the fact that $\tau_1, \dots, \tau_n$ is cotrimmed and $(i, \{i, j\}, Y)$ is a minimal splitting triple, we can show that $\tau_i = (\tau_j \wedge \tau_i) \vee \tau_Y = (\tau_j \vee \tau_Y) \wedge (\tau_i \vee \tau_Y) = \tau_j \wedge \tau_i$. Hence, $\tau_i \leq \tau_j$. □

Corollary 3.3. *Let $G = [\tau_1, \dots, \tau_n]$ and suppose $\tau_1, \dots, \tau_k$ are rank-1 vertices of G with $\tau_1 = \cdots = \tau_k$ where $k < n$. Then there is a basic canonical decomposition of G such that for each $i \in \overline{k}$, $(t, \{i, t\}, \overline{n} \setminus \{i\})$ is a minimal splitting triple of G for some $t > k$.*

Proof. Since τ_1 is a rank-1 vertex of G and rank-1 vertex cannot be a splitting vertex by Proposition 3.1, there exists $k < t \leq n$ such that $\tau_t \leq \tau_1$ and $(t, \{1, t\}, \overline{n} \setminus \{1\})$ is a minimal splitting triple of G by Lemma 3.2. Since $\tau_1 = \cdots = \tau_k$, we have $\tau_t \leq \tau_i$ and $(t, \{i, t\}, \overline{n} \setminus \{i\})$ is a minimal splitting triple of G for each $i = 2, \dots, k$. Hence, there is a basic canonical decomposition of G such that $(t, \{i, t\}, \overline{n} \setminus \{i\})$ is a minimal splitting triple of G for each $i \in \overline{k}$. □

Lemma 3.4. *If $G = [\tau_1, \dots, \tau_n]$ has a rank one quasi-summand of type τ and $\tau \neq \tau_i$ for all rank-1 vertices τ_i of G, then*

(a) *$G \approx H = [\hat{\tau}_1, \hat{\tau}_2, \tau_3, \dots, \tau_n]$ where $\hat{\tau}_1 = \tau$ and $\hat{\tau}_2 \leq \tau$,*

(b) *$(2, \{1, 2\}, \{2, \dots, n\})$ is a minimal splitting triple for H and $H \doteq [\tau, \hat{\tau}_2] \oplus [\hat{\tau}_2, \tau_3, \dots, \tau_n]$.*

In particular, τ is a rank-1 vertex of H.

Proof. Since G has a rank one quasi-summand of type τ and $\tau \neq \tau_i$ for all rank-1 vertices τ_i of G, there exists a basic canonical decomposition of $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ as in Proposition 2.6 with some I_i such that $|I_i| = 2$ and note that $[\tau(I_i)]$ is not an end-summand of G.

(a) For notational convenience, let $I_i = \{1, 2\}$ with $\tau_1 \vee \tau_2 = \tau$. As in Proposition 2.6(c), without loss of generality, we may assume $I_i \cap (I_{i+1} \cup \cdots \cup I_k) = \{1\}$. Let $I = I_{i+1} \cup \cdots \cup I_k$, then $2 \notin I$. If we let $X = I \setminus \{1\}$

and $Y = \{3,\dots,n\} \setminus X$, then (i) X, Y is a partition of $\{3,\dots,n\}$ (ii) $Y = (I_1 \cup \cdots \cup I_{i-1}) \setminus \{2\}$ and (iii) $X \cup \{1\}, Y \cup \{2\}$ is a partition of $\overline{n}$.

Observe that $(1, X \cup \{1\}, Y \cup \{1,2\})$ and $(2, X \cup \{1,2\}, Y \cup \{2\})$ are splitting triples of G. Hence $\tau_X \vee \tau_Y \geq \tau_{X\cup\{1\}} \vee \tau_{Y\cup\{1,2\}} = \tau_1$ and similarly, $\tau_X \vee \tau_Y \geq \tau_2$. Thus, it follows that $\tau_X \vee \tau_Y \geq \tau_1 \vee \tau_2$. If we let $\hat{\tau}_1 = \tau_{X\cup\{1\}} \vee \tau_{Y\cup\{2\}}$ and $\hat{\tau}_2 = \tau_{X\cup\{2\}} \vee \tau_{Y\cup\{1\}}$, then by Lemma 2.3, $G \approx H = [\hat{\tau}_1, \hat{\tau}_2, \tau_3, \dots, \tau_n]$. We will next show $\hat{\tau}_1 = \tau$ and $\hat{\tau}_1 \geq \hat{\tau}_2$.

Since $(1, X \cup \{1\}, Y \cup \{1,2\})$ is a splitting triple for G, it follows that $\tau_1 \vee \tau_2 = (\tau_{X\cup\{1\}} \vee \tau_{Y\cup\{1,2\}}) \vee \tau_2 = \tau_{X\cup\{1\}} \vee (\tau_{Y\cup\{1,2\}} \vee \tau_2) \leq \tau_X \vee \tau_2$. Similarly, we can show that $\tau_1 \vee \tau_2 \leq \tau_1 \vee \tau_Y$. Hence, $\hat{\tau}_1 = \tau_{X\cup\{1\}} \vee \tau_{Y\cup\{2\}} = (\tau_X \vee \tau_Y) \wedge (\tau_1 \vee \tau_Y) \wedge (\tau_X \vee \tau_2) \wedge (\tau_1 \vee \tau_2) = \tau_1 \vee \tau_2 = \tau$. By Corollary 2.4, we have $\hat{\tau}_1 \vee \hat{\tau}_2 = \tau_1 \vee \tau_2 = \tau$ and this implies $\hat{\tau}_2 \leq \tau = \hat{\tau}_1$ and $\hat{\tau}_1 \wedge \hat{\tau}_2 = \hat{\tau}_2$.

(b) Let $I = \{1,2\}$ and $J = \{2,3,\dots,n\}$. Using the fact $\hat{\tau}_1, \hat{\tau}_2, \tau_3, \dots, \tau_n$ is cotrimmed we can show $\tau_I \vee \tau_J = (\hat{\tau}_1 \vee \tau_J) \wedge (\hat{\tau}_2 \vee \tau_J) = \hat{\tau}_1 \wedge \hat{\tau}_2 = \hat{\tau}_2$. Thus, $(2, I, J)$ is a minimal splitting triple for H and $H \doteq [\tau, \hat{\tau}_2] \oplus [\hat{\tau}_2, \tau_3, \dots, \tau_n]$. And it is clear that $\hat{\tau}_1$ is a rank-1 vertex of H. □

Lemma 3.5. *Let $G = [\tau_1, \dots, \tau_n]$ be completely decomposable. If $\omega_1, \dots, \omega_{n-1}$ is the collection of the types of all rank one quasi-summands of G then $G \approx [\omega_1, \dots, \omega_{n-1}, \omega_n]$, where $\omega_n = \omega_1 \wedge \cdots \wedge \omega_{n-1}$.*

Proof. Let $\omega_1, \dots, \omega_{n-1}$ be the collection of types of the rank one quasi-summands in a canonical decomposition of G. Let $\tau_1, \dots, \tau_k$ be all rank-1 vertices of G and, without loss of generality, assume $\tau_1 = \omega_1, \dots, \tau_k = \omega_k$ for $k \leq n-1$. Let us assume $k \neq n-1$. Since a rank-1 vertex cannot be the splitting vertex for a splitting triple by Proposition 3.1, it follows that $G \approx G' = [\omega_1, \dots, \omega_k, \omega_{k+1}, \hat{\tau}_{k+2}, \tau_{k+3}, \dots, \tau_n]$ by Lemma 3.4 and $\omega_1, \dots, \omega_k, \omega_{k+1}$ are rank-1 vertices of G'. We continue the same process until $G' \approx [\omega_1, \dots, \omega_{n-1}, \omega_n]$. Hence, $G \approx [\omega_1, \dots, \omega_{n-1}, \omega_n]$ and by Corollary 2.4 $\omega_1, \dots, \omega_{n-1}, \omega_n$ is cotrimmed and $\omega_n = \omega_1 \wedge \cdots \wedge \omega_{n-1}$. □

Theorem 3.6. *Let $G = [\tau_1, \dots, \tau_n]$ and $H = [\sigma_1, \dots, \sigma_n]$ be completely decomposable. Then $G \doteq H$ if and only if $G \approx H$.*

Proof. Let $\omega_1, \dots, \omega_{n-1}$ be the collection of types of all rank one quasi-summands of G. Suppose $G \doteq H$, then $\omega_1, \dots, \omega_{n-1}$ is also the collection of types of all rank one quasi-summands of H. Hence, by Lemma 3.5, $G \approx [\omega_1, \dots, \omega_{n-1}, \omega_n]$ and $H \approx [\omega_1, \dots, \omega_{n-1}, \omega_n]$, where $\omega_n = \omega_1 \wedge \cdots \wedge \omega_{n-1}$. Thus, $G \approx H$.

Conversely, by Lemma 2.3, if $G \approx H$ then $G \doteq H$. □

Corollary 3.7. *Let $G = [\tau_1, \dots, \tau_n]$ and $H = [\sigma_1, \dots, \sigma_n]$ and suppose $n \leq 4$. Then $G \doteq H$ if and only if $G \approx H$.*

Proof. Suppose $G \doteq H$ and G is neither strongly indecomposable (Lemma 2.5) nor completely decomposable (Theorem 3.6). Then $G \doteq A \oplus B$ with rank $A = 1$ and rank $B = 2$. Let $\omega = \text{type}\, A$, then $G \approx G_1 = [\omega, \xi_2, \xi_3, \xi_4]$ and $H \approx H_1 = [\omega, \eta_2, \eta_3, \eta_4]$ by Lemma 3.4 and assume ξ_2, η_2 are splitting vertices for ω

in G, H respectively. Write $G_1 \doteq [\omega, \xi_2] \oplus [\gamma, \xi_3, \xi_4]$ and $H_1 \doteq [\omega, \eta_2] \oplus [\delta, \eta_3, \eta_4]$, where $\gamma = (\omega \wedge \xi_2) \vee (\xi_3 \wedge \xi_4)$ and $\delta = (\omega \wedge \eta_2) \vee (\eta_3 \wedge \eta_4)$. Then both γ, ξ_3, ξ_4 and δ, η_3, η_4 are cotrimmed. By the dual of Lemma 1.8 in [3], $[\gamma, \xi_3, \xi_4]$ and $[\delta, \eta_3, \eta_4]$ are equivalent, hence $\xi_i = \eta_j$ for some $i, j \in \{3, 4\}$. So, there are at least two common vertices in G_1 and H_1. If there are two common vertices in G_1 and H_1 then $G_1 \approx H_1$ by Lemma 2.3 or G_1 and H_1 are equivalent by Lemma 2.2. Therefore, we have shown $G \approx H$.

The converse is clear. □

4. Flower-like bracket groups

We say $G = [\tau_1, \dots, \tau_n]$ is *treated* if there is one-to-one correspondence between the collection of types of all rank one quasi-summands of G and a subcollection of $\tau_1, \dots, \tau_n$. In particular, if G is treated and τ is a type of a rank one quasi-summand of G then $\tau_i = \tau$ for some i.

Proposition 4.1. *Any bracket group can be transformed to a treated bracket group via a sequence of vertex-switches.*

Proof. See Lemma 3.4. □

From here on in this article, otherwise stated, we assume all bracket groups are treated and all considered canonical decompositions are basic canonical decompositions. Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$, then we say $[\tau(I_i)]$ is a *petal-summand of* G if $|I_i| > 2$ and $|I_i \cap (\bigcup_{j \neq i, |I_j| > 2} I_j)| \leq 1$.

Example 4.1. Let $G = [\tau_1, \dots, \tau_7] = [\omega_{457}, \omega_{45}, \omega_{25}, \omega_{15}, \omega_{156}, \omega_{12}, \omega_{13}]$, where vertices are defined as in Example 2.1. It is easy to check that (i) G is treated and $G \doteq [\tau_1, \tau_2] \oplus [\tau_2, \tau_3, \tau_4] \oplus [\tau_4, \tau_6, \tau_7] \oplus [\tau_4, \tau_5]$; (ii) $[\tau_2, \tau_3, \tau_4]$ and $[\tau_4, \tau_6, \tau_7]$ are petal-summands of G.

For each type τ, let $G[\tau] = \bigcap \{\ker f \mid f : G \to X_\tau\}$, where X_τ is a fixed rank one group of type τ. The subgroup $G[\tau]$ is called the *τ-radical* of G.

Lemma 4.2 (Dual of Corollary 2.1 in [1]). *Let $G = [\tau_1, \dots, \tau_n]$, then rank $G/G[\tau] = |\{i \colon \tau_i \leq \tau\}| - 1$.*

A quasi-decomposable bracket group $G = [\tau_1, \dots, \tau_n]$ is *flower-like* if there is a basic canonical decomposition of G such that if (r, I, J) and (s, K, L) are any two splitting triples of G then $r = s$. That is, there is exactly one splitting vertex in some basic canonical decomposition of G. In particular, if G is flower-like then G is already treated. Note that if G is flower-like then each strongly indecomposable quasi-summand $[\tau(I_i)]$ with $|I_i| > 2$ is a petal-summand.

Lemma 4.3. *Let $G = [\tau_1, \dots, \tau_n]$ be flower-like. Then there is a basic canonical decomposition of G such that*

(a) *Every quasi-summand in the canonical decomposition is an end-summand,*

(b) *If G has a rank one quasi-summand of type τ, then there is a rank-1 vertex τ_i such that $\tau_i = \tau$ for some $i \in \overline{n}$,*
(c) *If τ_i is a rank-1 vertex of G, then $\tau_k \nleq \tau_i$ for all $k \in \overline{n} \setminus \{1, i\}$,*
(d) *Any two rank-1 vertices have incomparable types.*

Proof. Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ be a basic canonical decomposition for flower-like group G and assume τ_1 is the splitting vertex of G.

(a) and (b) are clear from the definition of flower-like group.

(c) Let τ_i be a rank-1 vertex of G. Then, by Lemma 3.2, $\tau_k \leq \tau_i$ for some k and τ_k is a splitting vertex for τ_i. Since G is flower-like, we have that $k = 1$. Thus, we must have $\tau_k \nleq \tau_i$ for all $k \in \overline{n} \setminus \{1, i\}$.

(d) Since $\tau_k \nleq \tau_i$ for all $k \in \overline{n} \setminus \{1, i\}$, it follows that any two rank-1 vertices must have incomparable types. □

Lemma 4.4. *Let both $G = [\tau_1, \ldots, \tau_n]$ and $H = [\sigma_1, \ldots, \sigma_n]$ be flower-like. Suppose G has only one petal-summand and if $G \doteq H$ then $G \approx H$.*

Proof. Suppose $G \doteq H$ and let $(1, I, J)$ be a splitting triple for G such that $G \doteq [\tau(I)] \oplus [\tau(J)]$ with $[\tau(I)]$ the petal-summand of G and $[\tau(J)]$ completely decomposable. Similarly, $H \doteq [\sigma(I')] \oplus [\sigma(J')]$ and let $[\tau(I)] \doteq [\sigma(I')]$. Note that τ_1 is the splitting vertex of G. Let τ_i be a rank-1 vertex and $k \in I \setminus \{1\}$. Then since $\tau_1 \vee \tau_i = \tau_i \ngeq \tau_k$ by Lemma 4.3(c) and $[\tau(I)]$ is strongly indecomposable, there are exactly two τ_k-equivalence classes in G and in H for each $k \in I \setminus \{1\}$. By Lemma 2.3, assume the number of common vertices in G with H is less than $n - 2$. Hence, there always exists vertex $\tau_i, i \in I \setminus \{1\}$, which is not a common vertex in G with H. Since each rank-1 vertex of G is a common vertex in G with H, G is "almost" strongly indecomposable. Thus, the proof of the lemma follows with the same arguments as in the proof of Theorem 2.3 in [11] for strongly indecomposable groups.

Let $n(G, H)$ = the number of common vertices of G and H. Assume $n(G, H) = m_0 < n - 2$ and $\sigma_2 \in I' \setminus \{1\}$ is not a common vertex of G and H. Let X and Y be the σ_2-equivalence classes in G with $1 < |X| < n - 1$ and $n([\tau(X)], H) \leq |X| - 2$, then $G(\tau_X) = [\sigma_2, \tau(X)]$. Let $\{2\}$ and each $V_i, i \in X$, be the τ_X-equivalence classes in H and define $\omega_i = \sigma_{V_i} \vee \sigma_{V_i'}$, where $V_i' = \overline{n} \setminus V_i$ for each $i \in X$. Then, $H(\tau_X) = [\sigma_2, \omega(X)]$ by Lemma 2.1. Since $[\sigma_2, \tau(X)] \doteq [\sigma_2, \omega(X)]$ it follows that $G = [\tau(X), \tau(Y)] \approx G' = [\omega(X), \tau(Y)]$ by Lemma 4.6 and the induction on rank of G.

Note that $n(G', H) \geq m_0$. If $n(G', H) > m_0$, then $G' \approx H$ by the induction hypothesis on $n(G', H)$. Hence $G \approx G' \approx H$ and the proof of the theorem follows by the induction hypothesis on rank G. So, assume $n(G', H) = m_0$.

Since $n(G, H) = n(G', H)$, it follows that $n([\omega(X)], H) = n([\tau(X)], H) \leq |X| - 2$. Without loss of generality, assume that $\{2, 3\} \subseteq X$ and both ω_2, ω_3 are not common vertices of G' and H. Since $G' \doteq H$, let V_2 and $W = \overline{n} \setminus V_2$ be the ω_2-equivalence classes in H. Observe that (i) $\omega_2 = \sigma_{V_2} \vee \sigma_W$; (ii)

$H(\sigma_W) = [\omega_2, \sigma(W)]$; and (iii) $\sigma_W \leq \omega_i$ for each $i \in X$. Thus, the σ_W-equivalence classes in G' consist of singleton sets $\{i\}$, for each $i \in X$, and a partition $I_1, \ldots, I_r$ of Y. For a notational convenience, we let $\omega_i = \tau_i$ for each $i \in Y$ and define $\delta_i = \omega_{I_i} \vee \omega_{I'_i}$ where $I'_i = \overline{n} \setminus I_i$ for each $i = 1, \ldots, r$ then $G'(\sigma_W) = [\omega(X), \delta_1, \ldots, \delta_r]$ by Lemma 2.1. Since $H(\sigma_W) \approx G'(\sigma_W)$ it follows that $H \approx H' = [\omega(X \setminus \{2\}), \sigma(V_2), \delta_1, \ldots, \delta_r]$. Since ω_3 is a common vertex in G' with H' but not with H, it follows that $n(G', H') > m_0$. Therefore, $G' \approx H'$ by the induction hypothesis on $n(G', H')$ and the proof of theorem is complete. □

Definition 4.1. A splitting triple (r, I, J) for G is an *isolating splitting triple at I* if

(1) $[\tau(I)]$ has exactly one petal-summand $[\tau(X)]$ of G and $r \in X$,
(2) If $k \in J$ and $\tau_k \leq \tau_j$, then $j \in J$.

Observe that in Example 4.1, the splitting triple

$$(4, I, J) = (4, \{1, 2, 3, 4\}, \{4, 5, 6, 7\})$$

is an isolating splitting triple at I for G.

Lemma 4.5. *Let $G = [\tau_1, \ldots, \tau_n]$ be treated. Suppose (r, I, J) is an isolating splitting triple at I of G and if $\zeta = \tau_{I\setminus\{r\}} \vee \tau_r$. Then*

(a) $G(\tau_J) = [\zeta, \tau(J)] \doteq [\zeta, \tau_r] \oplus [\tau(J)]$,
(b) $\tau_{I\setminus\{r\}} \vee \tau_J = \zeta$.

Proof. (a) Let (r, I, J) be an isolating splitting triple at I and τ_r the splitting vertex for I. Then $G \doteq G' \oplus G'' = [\tau(I)] \oplus [\tau(J)]$ and assume $[\tau(X)]$ is the petal-summand of G contained in $[\tau(I)]$. Since $[\tau(X)]$ is strongly indecomposable, it follows that $\{r\}, X \setminus \{r\}$ are the τ_r-equivalence classes in $[\tau(X)]$. We will show $\{r\}, I \setminus \{r\}$ are the τ_r-equivalence classes in G'. Since (r, I, J) is an isolating splitting triple at I and $r \in J$, it follows that $\tau_r \leq \tau_i$ implies $\tau_i \in G''$. So, if $(j, \{i, j\}, \overline{n} \setminus \{i\})$ is a splitting triple with $\{i, j\} \subset I$ and $\tau_i \in G'$, then $\tau_i = \tau_i \vee \tau_j \not\geq \tau_r$. Therefore, $\{r\}, I \setminus \{r\}$ are the τ_r-equivalence classes in G' and rank $G'(\tau_r) = 1$ and type $G'(\tau_r) = \tau_{I\setminus\{r\}} \vee \tau_r = \zeta$.

Since $G' = [\tau(I)]$, it follows that $G' = G'(\tau_I)$. Hence, we have

$$G'(\tau_J) = G' \cap G'(\tau_J) = G'(\tau_I) \cap G'(\tau_J) = G'(\tau_I \vee \tau_J) = G'(\tau_r).$$

Thus, we have rank $G'(\tau_J) = 1$ and type $G'(\tau_J) = \zeta$, hence $G(\tau_J) \doteq G'(\tau_J) \oplus G''(\tau_J) \doteq [\zeta, \tau_r] \oplus [\tau(J)] = [\zeta, \tau(J)]$. It can be easily checked that the collection of types $\zeta, \tau(J)$ is cotrimmed.

(b) $\zeta = \tau_{I\setminus\{r\}} \vee \tau_r = \tau_{I\setminus\{r\}} \vee (\tau_I \vee \tau_J) = (\tau_{I\setminus\{r\}} \vee \tau_I) \vee \tau_J = \tau_{I\setminus\{r\}} \vee \tau_J$. □

Lemma 4.6 (Dual of Lemma 2.2 in [11]). *Let both $\tau_1, \ldots, \tau_n$ and $\sigma_1, \ldots, \sigma_n$ be cotrimmed. Suppose $\gamma = \tau_I \vee \tau_J$, where I, J is a partition of $\overline{n}$ and if $[\gamma, \tau(J)] \approx [\gamma, \sigma(J)]$, then $[\tau(I), \tau(J)] \approx [\tau(I), \sigma(J)]$.*

Lemma 4.4 provides an initial step for induction hypothesis to prove the main result of this section.

Theorem 4.7. *Let both $G = [\tau_1, \dots, \tau_n]$ and $H = [\sigma_1, \dots, \sigma_n]$ be flower-like. Then $G \doteq H$ if and only if $G \approx H$.*

Proof. Let us induct on rank G. It is clear for rank $G \leq 3$ (or $n \leq 4$) by Corollary 3.7. So, assume the theorem is true for all rank $G < n-1$. Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ be a basic canonical decomposition with τ_1 the splitting vertex of G. Suppose $G \doteq H$ and G and H are non-equivalent. Let $H \doteq [\sigma(I_1')] \oplus \cdots \oplus [\sigma(I_k')]$ be a canonical decomposition such that $[\tau(I_i)] \doteq [\sigma(I_i')]$ for each $i = 1, \dots, k$ and let σ_1 be the splitting vertex of H. By Lemma 4.4, let us assume G has at least two petal-summands. Without loss of generality, let $|I_1| > 2$ and $J = (\overline{n} \setminus I_1) \cup \{1\}$, then $(1, I_1, J)$ is an isolating splitting triple at I_1 for G and $[\tau(I_1)]$ is a petal-summand of G. By Lemma 4.5(a), $G(\tau_J) = [\zeta, \tau(J)] \doteq [\zeta, \tau_1] \oplus [\tau(J)]$, where $\zeta = \tau_{I_1 \setminus \{1\}} \vee \tau_1$. Since $G \doteq H$, it follows that $G(\tau_J) \doteq H(\tau_J)$ and $H(\tau_J) = [\zeta, \sigma(J')]$ and $[\zeta, \sigma(J')] \doteq [\zeta, \sigma_1] \oplus [\sigma(J')]$, where $J' = (\overline{n} \setminus I_1') \cup \{1\}$. Hence, $[\zeta, \tau(J)] \doteq [\zeta, \sigma(J')]$.

We claim that $G(\tau_J) = [\zeta, \tau(J)]$ is flower-like by showing τ_1 is the only splitting vertex in $G(\tau_J)$. If $\zeta \geq \tau_k$ for some $k \in J$ and if we let $L = (I_1 \setminus \{1\}) \cup \{k\}$. Then, it is easy to check (i) $J \cup L = \overline{n}$ and $J \cap L = \{k\}$ and (ii) $\tau_L \vee \tau_J = (\tau_{I_1 \setminus \{1\}} \wedge \tau_k) \vee \tau_J = (\tau_{I_1 \setminus \{1\}} \vee \tau_J) \wedge (\tau_k \vee \tau_J) = \zeta \wedge \tau_k = \tau_k$ by Lemma 4.5(b). Hence, (k, J, L) is a splitting triple for G but since G is flower-like, $k = 1$. So, τ_1 is the only splitting vertex for ζ and $[\zeta, \tau(J)]$ is flower-like.

By the induction hypothesis on rank G, $[\zeta, \tau(J)] \approx [\zeta, \sigma(J')]$ and $G \approx G_1 = [\tau(I_1 \setminus \{1\}), \sigma(J')]$ by Lemma 4.6. We choose $|I_i'| > 2$ such that $[\sigma(I_i')]$ and $[\tau(I_1)]$ are not quasi-isomorphic. Let $M = (\overline{n} \setminus I_i') \cup \{1\}$, then $(1, I_i', M)$ is an isolating splitting triple at I_i' for H. By Lemma 4.5, $H(\sigma_M) = [\eta, \sigma(M)]$, where $\eta = \sigma_1 \vee \sigma_{I_i' \setminus \{1\}}$. Since $H \doteq G_1$, it follows that

$$H(\sigma_M) \doteq G_1(\sigma_M) = [\eta, \tau(I_1 \setminus \{1\}), \sigma((J' \setminus I_i') \cup \{1\})].$$

And by Lemma 4.6,

$$H \approx H_1 = [\sigma(I_i' \setminus \{1\}), \tau(I_1 \setminus \{1\}), \sigma((J' \setminus I_i') \cup \{1\})] = [\tau(I_1 \setminus \{1\}), \sigma(J')]$$

and note that G_1 and H_1 are equivalent. Hence, we have that $G \approx G_1 \approx H$. The converse is clear and the proof is complete. □

5. Co-elementary group

We say $G = [\tau_1, \dots, \tau_n]$ is *co-elementary* if $|\{i : \tau_i \leq \tau\}| > 2$ implies $|\{i : \tau_i \leq \tau\}| = n$ for all types τ in the typeset of G.

Lemma 5.1. *Let $G = [\tau_1, \dots, \tau_n]$. Suppose $(1, \{1, n\}, \{1, ..., n-1\})$ is a splitting triple for rank-1 vertex τ_n in G and let $\zeta_1 = \tau_1 \vee \tau_{\{2,\dots,n-1\}}$ and* *if* $[\zeta_1, \tau_2, \tau(P)] \approx [\delta_1, \delta_2, \tau(P)]$ *by a vertex-switch partition* X, Y *of* $P = \{3, \ldots, n-1\}$. *Then,* $G \approx [\tau_n \wedge \delta_1, \delta_2, \tau_3, \ldots, \tau_n]$ *if and only if* $\tau_n \vee \tau_Y \geq \tau_1 \vee \tau_2$.

Proof. Let $\zeta_1 = \tau_1 \vee \tau_{\{2,\ldots,n-1\}}$ and $P = \{3, \ldots, n-1\}$. Since $[\zeta_1, \tau_2, \tau(P)] \approx [\delta_1, \delta_2, \tau(P)]$, it follows by Lemma 2.3 there is a partition X, Y of P such that $\tau_X \vee \tau_Y \geq \zeta_1 \vee \tau_2, \delta_1 = (\zeta_1 \wedge \tau_Y) \vee (\tau_2 \wedge \tau_X)$ and $\delta_2 = (\tau_2 \wedge \tau_Y) \vee (\zeta_1 \wedge \tau_X)$.

Note that $\tau_n \geq \tau_1$ and $\zeta_1 \vee \tau_J = \tau_1 \vee \tau_J$ for each $J \subseteq \{2, \ldots, n-1\}$. Suppose $\tau_n \vee \tau_Y \geq \tau_1 \vee \tau_2$. Let $X' = X \cup \{n\}$ and $Y' = Y$, then X', Y' is a partition of $\{3, \ldots, n\}$ with $\tau_{X'} \vee \tau_{Y'} = (\tau_X \vee \tau_Y) \wedge (\tau_n \vee \tau_Y) \geq (\zeta_1 \vee \tau_2) \wedge (\tau_1 \vee \tau_2) = \tau_1 \vee \tau_2$. Observe that (i) $\tau_n \wedge \tau_1 = \tau_1$ (ii) $\tau_n \wedge \tau_Y \wedge \zeta_1 = (\tau_n \wedge \tau_Y \wedge \tau_1) \vee (\tau_n \wedge \tau_Y) = \tau_1 \wedge \tau_Y$. So, we have $(\tau_1 \wedge \tau_{Y'}) \vee (\tau_2 \wedge \tau_{X'}) = \tau_n \wedge ((\zeta_1 \wedge \tau_Y) \vee (\tau_2 \wedge \tau_X)) = \tau_n \wedge \delta_1$ and $(\tau_2 \wedge \tau_{Y'}) \vee (\tau_1 \wedge \tau_{X'}) = (\tau_2 \wedge \tau_Y) \vee (\zeta_1 \wedge \tau_X) = \delta_2$. So, by Corollary 2.4, $G \approx [\tau_n \wedge \delta_1, \delta_2, \tau_3, \ldots, \tau_n]$. Conversely, with the same vertex-switch partition X', Y' above we have $\tau_n \vee \tau_Y \geq \tau_{X'} \vee \tau_{Y'} \geq \tau_1 \vee \tau_2$. □

Lemma 5.2. *Let* $G = [\tau_1, \ldots, \tau_n]$ *be co-elementary and* τ_n *a rank-1 vertex. Suppose* $(1, \{1, n\}, \{1, ..., n-1\})$ *is a splitting triple for* τ_n *in* G *then* $\tau_n \vee \tau_J \geq \tau_i$ *for each* $i \in \overline{n}$ *and* $J \subseteq \{2, \ldots, n-1\}$.

Proof. Let $\omega_j = \tau_n \vee \tau_j$ for $j = 2, \ldots, n-1$, then $\omega_j \geq \tau_j, \tau_n, \tau_1$ since $\tau_n \geq \tau_1$. Since G is co-elementary, it follows that $\omega_j \geq \tau_i$ for each $i \in \overline{n}$ and $2 \leq j \leq n-1$. So, in particular $\tau_n \vee \tau_J \geq \tau_i$ for each $i \in \overline{n}$ and $J \subseteq \{2, \ldots, n-1\}$. □

Theorem 5.3. *Let* $G = [\tau_1, \ldots, \tau_n], H = [\sigma_1, \ldots, \sigma_n]$ *be co-elementary. Then* $G \doteq H$ *if and only if* $G \approx H$.

Proof. Suppose $G \doteq H$. We prove the theorem by the induction on rank G. By Corollary 3.7, assume the theorem is true for G with rank $G < n-1$. Let rank $G = n-1$. We will consider two cases: G with at least one rank one quasi-summand and G without any rank one quasi-summands.

Case 1 : Suppose G has at least one rank one quasi-summand, we can let τ_1, σ_1 be splitting vertices for rank-1 vertices τ_n, σ_n of G, H respectively and $\tau_n = \sigma_n$. Let $G \doteq G_0 \oplus G_1 = [\tau_1, \tau_n] \oplus [\zeta_1, \tau(I)]$ and $H \doteq G_0 \oplus H_1 = [\sigma_1, \sigma_n] \oplus [\delta_1, \sigma(I)]$, where $I = \{2, \ldots, n-1\}$ and $\zeta_1 = (\tau_n \wedge \tau_1) \vee \tau_I$ and $\delta_1 = (\tau_n \wedge \sigma_1) \vee \sigma_I$. Note that the collections of types $\zeta_1, \tau(I)$ and $\delta_1, \sigma(I)$ are cotrimmed and both G_1 and H_1 are co-elementary. Since $G \doteq H$, $G_1 \doteq H_1$, hence by the induction hypothesis on rank G we have $G_1 \approx H_1$. We will show that $G_1 \approx H_1$ implies $G \approx H$. Let $s_1, \ldots, s_m$ be the sequence of vertex-switches transforming G_1 to H_1 and X_i, Y_i the vertex-switch partition associated with each vertex-switch s_i. Let $G_1 \approx G_2$ by the vertex-switch s_1 and suppose $\tau_i, \tau_j (i < j)$ are replaced by λ_i, λ_j respectively by s_1. For notational convenience we donote

$$\tau(k, \lambda_i, l) = \tau_k, \tau_{k+1}, \ldots, \tau_{i-1}, \lambda_i, \tau_{i+1}, \ldots, \tau_l \text{ and}$$
$$\tau(k, \lambda_i, \lambda_j, l) = \tau_k, \tau_{k+1}, \ldots, \tau_{i-1}, \lambda_i, \tau_{i+1}, \ldots, \tau_{j-1}, \lambda_j, \tau_{j+1}, \ldots, \tau_l,$$

where $1 \leq k < i < j < l \leq n$.

Suppose $\tau_i \neq \zeta_1$ and $\tau_j \neq \zeta_1$, then, by Lemma 4.6, $G \approx G^1 = [\tau(1, \lambda_i, \lambda_j, n)]$ and $G^1 \doteq G_0 \oplus G_2 = [\tau_1, \tau_n] \oplus [\zeta_1, \tau(2, \lambda_i, \lambda_j, n-1)]$. Note that the collection of types $\zeta_1, \tau(2, \lambda_i, \lambda_j, n-1)$ is cotrimmed.

Suppose $\tau_i = \zeta_1$ then $G_1 \approx G_2 = [\lambda_i, \tau(2, \lambda_j, n-1)]$. By Lemma 5.2, we have $\tau_n \vee \tau_{Y_1} \geq \tau_j \vee \tau_i = \tau_j \vee \zeta_1$ and now by Lemma 5.1, we have $G \approx G^1 = [\tau_n \wedge \lambda_i, \tau(2, \lambda_j, n)]$ and $G^1 \doteq [\tau_n, \tau_n \wedge \lambda_i] \oplus [\tau_n \wedge \lambda_i, \tau(2, \lambda_j, n-1)]$. Next, we show that $\lambda_i, \tau(2, \lambda_j, n-1)$ is the cotrimmed version of $\tau_n \wedge \lambda_i, \tau(2, \lambda_j, n-1)$. Let $\Omega = \bigwedge\{\tau(2, \lambda_j, n-1)\}$. Since $\lambda_i, \tau(2, \lambda_j, n-1)$ is cotrimmed, $\lambda_i \geq \Omega$. And since G^1 is co-elementary, $\tau_n \vee \Omega \geq \lambda_i$ by Lemma 5.2, hence we have $(\tau_n \wedge \lambda_i) \vee \Omega = \lambda_i$. It is now easy to check $\lambda_i, \tau(2, \lambda_j, n-1)$ is the cotrimmed version of $\tau_n \wedge \lambda_i, \tau(2, \lambda_j, n-1)$. Thus, $G^1 \doteq [\tau_1, \tau_n] \oplus G_2$.

In either cases, s_1 induces a vertex-switch on G, namely $G \approx G^1$. If we let $G_1 \approx G_2 \approx \cdots \approx G_{m+1} = H_1$, where G_{i+1} is obtained from G_i by the vertex-switch s_i for each $1 \leq i \leq m$, then by a recursive application of above procedure we can show $G \approx G^i \doteq [\tau_1, \tau_n] \oplus G_{i+1}$ for each $1 \leq i \leq m$. Therefore, $G \approx G^m = [\tau_1, \tau_n] \oplus H_1$ and $G^m \doteq [\tau_n, \tau_n \wedge \delta_1, \sigma_2, \ldots, \sigma_{n-1}]$. Since G^m and H differ at most two places, $G^m \approx H$ by Lemma 2.3 and the proof of Case 1 is complete.

Case 2 : Note that the proof of this case uses the same arguments as in the proof of Theorem 4.7.

Let $G \doteq [\tau(I_1)] \oplus \cdots \oplus [\tau(I_k)]$ be a canonical decomposition with each $|I_i| > 2$. Suppose $G \doteq H$ and G and H are non-equivalent. Since $G \doteq H$, without loss of generality, let $H \doteq [\sigma(I'_1)] \oplus \cdots \oplus [\sigma(I'_k)]$ be a canonical decomposition such that $[\tau(I_i)] \doteq [\sigma(I'_i)]$ for each $i \in \overline{k}$ and assume $[\tau(I_1)]$ is a petal-summand. Let $J = (\overline{n} \setminus I_1) \cup \{r\}$. Then (r, I_1, J) is an isolating splitting triple at I_1 for G, where τ_r is the splitting vertex. By Lemma 4.5, $G(\tau_J) = [\zeta, \tau(J)]$ and $G(\tau_J)$ has a rank one quasi-summand of type ζ, where $\zeta = \tau_{I_1 \setminus \{r\}} \vee \tau_r$. Since $G \doteq H$, it follows that $G(\tau_J) \doteq H(\tau_J)$ and $H(\tau_J) = [\zeta, \sigma(J')]$, where $J' = (\overline{n} \setminus I'_1) \cup \{r'\}$ and $\sigma_{r'}$ is a splitting vertex in H. Hence, $[\zeta, \tau(J)] \doteq [\zeta, \sigma(J')]$. By the induction hypothesis on rank G, $[\zeta, \tau(J)] \approx [\zeta, \sigma(J')]$ and by Lemma 4.6 $G \approx G_1 = [\tau(I \setminus \{r\}), \sigma(J')]$. We choose $|I'_i|, i \neq 1$, such that $[\sigma(I'_i)]$ is a petal-summand of H and let $M = (\overline{n} \setminus I'_i) \cup \{t\}$, then (t, I'_i, M) is an isolating splitting triple at I'_i for H and σ_t a splitting vertex. Note that $[\sigma(I'_i)]$ and $[\tau(I_1)]$ are not quasi-isomorphic. By Lemma 4.5, $H(\sigma_M) = [\eta, \sigma(M)]$, where $\eta = \sigma_t \vee \sigma_{I'_i \setminus \{t\}}$. Since $H \doteq G_1, H(\sigma_M) \doteq G_1(\sigma_M) = [\eta, \tau(I \setminus \{r\}), \sigma((J' \setminus I'_i) \cup \{t\})]$. By Lemma 4.6, $H \approx H_1 = [\sigma(I'_i \setminus \{t\}), \tau(I \setminus \{r\}), \sigma((J' \setminus I'_i) \cup \{t\})]$ and note that G_1 and H_1 are equivalent. Hence, we have that $G \approx G_1 \approx H$ and the proof of Case 2 is complete and the proof of theorem is complete. □

Remark 5.1. A method of using an induction on the rank of quasi-summands of a bracket group does not always work because a class of bracket groups is not closed under direct sums. For example, with the same notation as in Example 2.1, if we let $G = [\tau_1, \tau_2, \tau_3, \tau_4, \tau_5] = [\omega_{12}, \omega_{12}, \omega_{34}, \omega_{13}, \omega_{24}]$ then $G \doteq G_0 \oplus G_1 = [\tau_1, \tau_2] \oplus [\tau_2, \tau_3, \tau_4, \tau_5]$. Choose $X = \{4\}$ and $Y = \{5\}$, then

X, Y is a vertex-switch partition for τ_2, τ_3 in G_1 since $\tau_X \vee \tau_Y = \omega_{1234} \geq \tau_2 \vee \tau_3 = \omega_{1234}$. Hence $G_1 \approx [\hat{\tau}_2, \hat{\tau}_3, \tau_4, \tau_5]$ where $\hat{\tau}_2 = \omega_{23}$ and $\hat{\tau}_3 = \omega_{14}$. Thus, we have that $G \doteq [\tau_1, \tau_2] \oplus [\hat{\tau}_2, \hat{\tau}_3, \tau_4, \tau_5]$. But there is no bracket group whose quasi-decomposition is $[\tau_1, \tau_2] \oplus [\hat{\tau}_2, \hat{\tau}_3, \tau_4, \tau_5]$ since type $(\hat{\tau}_2 \wedge \hat{\tau}_3 \wedge \tau_4 \wedge \tau_5) =$ type$(Z) \neq \tau_2$. We could also show the same result using Lemma 5.1 since $\tau_1 \vee \tau_Y = \omega_{124} \not\geq \tau_2 \vee \tau_3 = \omega_{1234}$.

6. Applications

Define $\mathcal{G}(\tau_1, \ldots, \tau_n) =$ the kernel of the codiagonal map $\bigoplus_{i=1}^n A_i \to \mathbf{Q}$ given by $(a_1, \ldots, a_n) \mapsto \sum a_i$ where $\tau_i =$ type A_i for each i. Then the class of groups $\mathcal{G}(\tau_1, \ldots, \tau_n)$ is the dual class of the bracket groups in the sense of quasi-isomorphism Butler duality of [2]. A collection of types $\tau_1, \ldots, \tau_n$ is called *trimmed* if $\tau_i \leq \bigvee_{j \neq i} \tau_j$ for each $i \in \overline{n}$. For a nonempty subset $I = \{i_1, \ldots, i_k\}$ of $\overline{n}$, we write $\tau^I = \tau_{i_1} \vee \cdots \vee \tau_{i_k}$. The following corollaries can be obtained using quasi-isomorphism Butler duality of [2].

Corollary 6.1 ("two-vertex exchange"). *Let $\tau_1, \ldots, \tau_n$ be trimmed and $\tau_i \neq \sigma_j$ for $i, j \in \{1, 2\}$. Then the following statements are equivalent:*

(a) $\mathcal{G}(\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \doteq \mathcal{G}(\sigma_1, \sigma_2, \tau_3, \ldots, \tau_n)$;

(b) *There is a partition X, Y of $\{3, \ldots, n\}$ such that $\tau^X \wedge \tau^Y \leq \tau_1 \wedge \tau_2$ and $\sigma_1 = (\tau_1 \vee \tau^Y) \wedge (\tau_2 \vee \tau^X)$ and $\sigma_2 = (\tau_1 \vee \tau^X) \wedge (\tau_2 \vee \tau^Y)$.*

Corollary 6.2. *Suppose $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$ are either completely decomposable, flower-like or elementary. Then $G \doteq H$ if and only if there is a sequence of two-vertex exchanges transforming G to H.*

As an application, vertex switches could provide a method of coding and decoding information. If we look at $\tau_1, \ldots, \tau_n$ in a bracket group $[\tau_1, \ldots, \tau_n]$ without group-theoretic properties but just as a lattice of types with distributive property under sup ($\vee$) and inf ($\wedge$) then we can store information in each type and we can encode information as a collection of vertices using vertex switches and decode them using the same vertex switches.

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