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Estimating a Skewed Parameter and Reliability in a Skew-Symmetric Double Rayleigh Distribution

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Abstract

We define a skew-symmetric double Rayleigh distribution by a symmetric double Rayleigh distribution, and derive an approximate maximum likelihood estimator(AML) and a moment estimator(MME) of a skewed parameter in a skew-symmetric double Rayleigh distribution, and hence compare simulated mean squared errors of those two estimators. We also compare simulated mean squared errors of two proposed estimators of reliability in two independent skew-symmetric double Rayleigh distributions.

Keywords: Approximate ML, Reliability, Right-Tail Probability, Skew-Symmetric Double Rayleigh Distribution.

1. Introduction

Johnson et al(1995) has introduced a definition of a symmetric double Weibull distribution. Ali and Woo(2006) and Woo(2006) introduced several skew-symmetric reflected distributions, Azzalini and Capitanio(1999) studied the multivariate skewed normal distribution, and Son and Woo(2007) studied the estimation problem of a skewed parameter and reliability in a skew-symmetric Laplace distribution.

Balakrishnan & Cohen(1991) proposed method of finding the approximate MLE of the scale parameter in the several distributions. Han & Kang(2006) studied the approximate maximum likelihood estimator(AMLE) of parameters in several distributions with censored data.

It's not easy for us to estimate a skewed parameter α in a skew-symmetric

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distribution, so we'd like to consider in estimating the skewed parameter in a symmetric double Rayleigh distribution based on a method of finding an approximate MLE.

In this paper, we define a skew-symmetric double Rayleigh distribution by a symmetric double Rayleigh distribution, and derive an AML and a moment estimator(MME) of a skewed parameter δ in a skew-symmetric double Rayleigh distribution, and hence compare simulated mean squared errors of those estimators. And we compare simulated mean squared errors of two proposed estimators of the reliability in two independent skew-symmetric double Rayleigh distributions.

2. A skew-symmetric double Rayleigh distribution

Let X and Y be two independent identical distributed continuous random variables with the probability density function(pdf) $f(x)=F'(x)$ which is symmetric about $\theta\in R^1$. Then, for $\forall\delta\in R^1$, $\frac{1}{2}=P\{(X-\theta)-\delta(Y-\theta)\leq 0\}$

$$= \int_{-\infty}^{\infty} f(t)F(\theta+\delta(t-\theta))dt.$$

Therefore, $f(z;\delta)\equiv 2f(z)F(\delta(z-\theta)+\theta)$, (see Azzalini(1985)) (2.1)

where δ is called a skewed parameter of the skew-symmetric distribution.

The density $f(z;\delta)$ in (2.1) becomes a skew-symmetric density of a random variable Z derived from a symmetric distribution which is symmetric about $\theta\in R^1$ (see Azzalini(1985)). Especially if $\delta=0$, then f(z;0) becomes the original symmetric density.

From the skewed density (2.1), the cdf of the skewed random variable Z with the density (2.1) is given by(see Azzalini(1985)):

$$F(z;\delta)=2\int_{-\infty}^{z} f(t)\int_{-\infty}^{\delta(t-\theta)+\theta} f(s)\,ds\,dt. \tag{2.2}$$

2.1. A skew-symmetric double Rayleigh

From the density (2.1) and the cdf of double Weibull(especially, double Rayleigh) in Johnson et al (1995, p.198), we obtain a skew-symmetric double Rayleigh density as below:

$$f(z;\delta)=\frac{1}{\beta}|z|\cdot e^{-\frac{z^2}{\beta}}[1+sgn(\delta z)(1-e^{-\frac{\delta^2 z^2}{\beta}})] \quad -\infty<z<\infty \tag{2.3}$$

where, $sgn(x)=\begin{cases}1, & \text{if } x>0\\ -1, & \text{if } x<0\end{cases}$ and $\beta>0$.

From the cdf (2.2) and the density (2.3), and the formula 3.14 in Oberhettinger(1974, p.27), the cdf of the skew-symmetric double Rayleigh random variable Z is given by:

For $\delta>0, F(z;\delta)=I_{[0,\infty)}(z)\cdot(1-e^{-z^2/\beta})+\frac{1}{2(1+\delta^2)}e^{-\frac{1+\delta^2}{\beta}z^2}, \quad -\infty<z<\infty$ (2.4)

where $I_{[0,\infty)}(z)=\begin{cases}1, & \text{if } z\in[0,\infty)\\ 0, & else\end{cases}$.

Remark 1. If $\delta<0$, then $F(z;\delta)=1-F(-z;-\delta)$, from Lemma 2(b) in Ali & Woo(2006).

2.2. Estimating the skewed parameter

We only consider estimation of the skewed parameter δ in the skew-symmetric double Rayleigh density (2.3) with a known β .

Assume $Z_1, Z_2, \cdots, Z_n$ be iid random variables having the density (2.3) with known β of Z. Then, by method of finding AML of a parameter of a distribution in Balakrishnan and Cohen (1991), from log-likelihood function of δ and Taylor series, we can obtain the AML $\hat{\delta}$ of δ as the following:

For $Z_1=z_1, Z_2=z_2,\dots,Z_n=z_n$, the likelihood function is given by:

$$f(\delta: z_1,\cdots z_n)=\frac{1}{\beta^n}e^{-\sum_{i=1}^{n} z_i^2/\beta}\cdot\prod_{i=1}^{n}|z_i|[1+sgn(\delta z_i)(1-e^{-\delta^2 z_i^2/\beta})],$$

and hence,

$$\frac{d\ln f(\delta: z_1,\cdots,z_n)}{d\delta}=\sum_{i=1}^{n}\frac{sgn(\delta z_i)(2z_i^2\beta^{-1}\delta\cdot e^{-\delta^2 z_i^2/\beta})}{1+sgn(\delta z_i)(1-e^{-\delta^2 z_i^2/\beta})}=0,$$

and hence, as taking first two terms in an expansion of Taylor series of $R(\delta\mid z_i)$ about $\delta=c$, which c is any real number, we obtain the following:

$$R(\delta|z_i)\approx R(c|z_i)+R'(c\mid z_i)(\delta-c)=0, \quad i=1,2,\dots,n, \tag{2.5}$$

where $R(c|z_i)=\dfrac{sgn(cz_i)(2z_i^2\beta^{-1}ce^{-c^2z_i^2/\beta})}{1+sgn(cz_i)(1-e^{-c^2z_i^2/\beta})}$,

and

$$R'(c|z_i)=\frac{2sgn(cz_i)z_i^2\beta^{-1}e^{-c^2z_i^2/\beta}\cdot\left\{(1+sgn(cz_i))(1-2z_i^2\beta^{-1}c^2)-sgn(cz_i)e^{-c^2z_i^2/\beta}\right\}}{\left[1+sgn(cz_i)(1-e^{-c^2z_i^2/\beta})\right]^2}$$

From the equation (2.5), the AML $\hat{\delta}$ of δ is given as below:

$$\hat{\delta}=c-\sum_{i=1}^{n}R(c\mid Z_i)\,/\,\sum_{i=1}^{n}\frac{d}{dc}R(c\mid Z_i). \quad (2.6)$$

And also, from density (2.3), we obtain the moment estimator(MME) $\tilde{\delta}$ of δ by:

$$1+(sgn(\delta)\tilde{\delta})^2=\beta\left(\beta^{3/2}-sgn(\delta)\frac{2\beta}{n\sqrt{\pi}}\sum_{i=1}^{n}Z_i\right)^{-2/3}, \quad (2.7)$$

which is well-defined.

Remark 2. Because the AML having $c=\delta$ performs better than the AML having $c\neq\delta$ in a skew-symmetric Laplace distribution(Son & Woo(2007)), we shall take simulations of MSE of AML only when any number "c" of AML equals to the given skewed parameter.

To simulate mean squared error(MSE) of two estimators (2.6) & (2.7), by the cdf (2.4) with $\beta=1$ it can be obtained that a transformed random variable U follows a uniform distribution over (0.1). By the computer 6000-simulations based on the uniform distribution, Table 1 in the Appendix shows 6000-simulated averages and mean squared errors of AML $\hat{\delta}$ and MME $\tilde{\delta}$ when n=10(10)30, $\delta=c=\pm 1/2, \pm 1.0, \pm 2$. From Table 1 in the Appendix, we observe the following Fact 1:

Fact 1. Let the density (2.3) have $\beta=1$. If the true value $\delta=\pm 1/2, \pm 1, \pm 2$ in (2.3), the AML performs better than the MME in a sense of simulated mean squared error.

Remark 3 (Application) In practical data problem, the AML $\hat{\delta}$ in (2.6) would be recommended more favorable estimator of δ than the MME $\tilde{\delta}$ in (2.7), when the value c in (2.6) is substituted by the moment estimate $\tilde{\delta}$ in (2.7).

2.3. Reliability

In this section, we consider estimation of the right-tail probability of the skew-sym-
metric double Rayleigh random variable having the density (2.3) with known β, and consider estimation of reliability of two independent skew-symmetric double Rayleigh random variables with different skewed parameters δ_1 and δ_2.

First we consider estimation of the right-tail probability of the skew-symmetric double Rayleigh random variable. From the cdf (2.4) of the skew-symmetric double Rayleigh random variable Z, the right-tail probability $R(t;\delta)=P(Z>t)$ of Z is given by: For $\delta>0$

$$R(t;\delta)=1-[I_{[0,\infty)}(t)\cdot(1-e^{-t^2/\beta})+\frac{1}{2(1+\delta^2)}e^{-\frac{1+\delta^2}{\beta}t^2}],\quad -\infty<t<\infty \tag{2.8}$$

Since $dR(t;\delta)/d\delta$ is positive for $\delta>0$, the right-tail probability $R(t;\delta)$ is a monotone increasing function of $\delta>0$, and hence, because inference on $R(t;\delta)$ is equivalent to inference on $\delta>0$(see McCool(1991)), from AML (2.9), MME (2.10) of δ, Fact 2, and for $\delta<0$, $R(t;\delta)=I_{[0,\infty)}(-t)\cdot(1-e^{-t^2/\beta})+\frac{1}{2(1+\delta^2)}e^{-\frac{1+\delta^2}{\beta}t^2}$, $-\infty<t<\infty$, is also monotone function of $\delta<0$, we obtain the following:

Fact 3. Let parameter β be known in the density (2.3). Then the MLE $\hat{R}(t;\delta)\equiv R(t;\hat{\delta})$ performs better than another estimator $\tilde{R}(t;\delta)\equiv R(t;\tilde{\delta})$ in a sense of approximate MSE.

Now we consider estimation of reliability of two independent skew-symmetric double Rayleigh random variables. Let Z and W be independent skew-symmetric double Rayleigh random variables each having the density (2.3) with two parameters (δ_1,β_1) and (δ_2,β_2), respectively. Then, from the density $f(x;\delta)$ in (2.3), the cdf $F(x;\delta)$ in (2.4), and the formula 3.14 in Oberhettinger(1974, p.27) we obtain the reliability P(Z<W) :

For $\delta_i>0,\ i=1,2$,

$$R(\delta_1,\delta_2)\equiv P(Z<W)=\int_0^\infty(f(x;\delta_2)F(x;\delta_1)+f(-x;\delta_2)F(-x;\delta_1))dx$$

$$=1-\frac{1}{1+\beta_2/\beta_1}-\frac{1}{2(1+\delta_2^2)}+\frac{1}{2(1+\delta_2^2+\beta_2/\beta_1)}+\frac{1}{2(1+\delta_1^2)(1+(1+\delta_1^2)\beta_2/\beta_1)}$$

$$=\frac{1}{2}+\rho, \tag{2.9}$$

where

$$\rho=\frac{1}{2}-\frac{1}{1+\beta_2/\beta_1}-\frac{1}{2(1+\delta_2^2)}+\frac{1}{2(1+\delta_2^2+\beta_2/\beta_1)}+\frac{1}{2(1+\delta_1^2)(1+(1+\delta_1^2)\beta_2/\beta_1)}$$

Remark 4. For $\delta_i<0(i=1,2)$ or $\delta_i<0$ and $\delta_j>0(i\neq j)$, we can evaluate the reliability P(Y<X) similarly by use of (2.3), (2.4) and Remark 1.

Especially if Z and Ware identical random variables each having $\delta_1=\delta_2$ and $\beta_1=\beta_2$, then it is no wonder that the reliability is 1/2.

From the reliability (2.9) with known $\beta_i's$, since the reliability $R(\delta_1,\delta_2)$ is a monotone function of ρ, and hence, because inference on $R(\delta_1,\delta_2)$ is equivalent to inference on ρ (see McCool(1991)). it's sufficient for us to consider estimation of ρ instead of estimating reliability $R(\delta_1,\delta_2)$ when β_1 and β_2 are known.

Assume $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_m$ be two independent samples each having the density $f(z;\delta_1)$ and $f(z;\delta_2)$ in the density (2.3) with known β_i, i=1,2, respectively. Then, by using the AML $\hat{\delta}$ and MME $\tilde{\delta}$ of δ in (2.6) and (2.7), the following two proposed estimators of ρ in the reliability $R \equiv R(\delta_1,\delta_2)$ are defined by:

For $\delta_i > 0$, i=1 and 2

$$\hat{\rho} = \frac{1}{2} - \frac{1}{1+\beta_2/\beta_2} - \frac{1}{2(1+\hat{\delta}_2^2)} + \frac{1}{2(1+\hat{\delta}_2^2+\beta_2/\beta_1)} + \frac{1}{2(1+\hat{\delta}_1^2)(1+\beta_2/\beta_1+\beta_2\hat{\delta}_1^2/\beta_1)},$$

$$\tilde{\rho} = \frac{1}{2} - \frac{1}{1+\beta_2/\beta_2} - \frac{1}{2(1+\tilde{\delta}_2^2)} + \frac{1}{2(1+\tilde{\delta}_2^2+\beta_2/\beta_1)} + \frac{1}{2(1+\tilde{\delta}_1^2)(1+\beta_2/\beta_1+\beta_2\tilde{\delta}_1^2/\beta_1)} \tag{2.10}$$

where $\hat{\delta}_1 = c_1 - \dfrac{\sum_{i=1}^{n} R(c_1 \mid Z_i)}{\sum_{i=1}^{n} R'(c_1 \mid Z_i)}$, $\hat{\delta}_2 = c_2 - \dfrac{\sum_{i=1}^{m} R(c_2 \mid W_i)}{\sum_{i=1}^{m} R'(c_2 \mid W_i)}$, $c_i \in R^1$, i=1,2.

$$\tilde{\delta}_1^2 = \beta_1\left(\beta_1^{3/2} - \frac{2\beta_1}{n\sqrt{\pi}}\sum_{i=1}^{n} Z_i\right)^{-2/3} - 1, \text{ and } \tilde{\delta}_2^2 = \beta_2\left(\beta_2^{3/2} - \frac{2\beta_2}{m\sqrt{\pi}}\sum_{i=1}^{m} W_i\right)^{-2/3} - 1.$$

To simulate MSEs of two estimators (2.13), by the cdf (2.4) with $\beta = 1$ it can be obtained that a transformed random variable U follows a uniform distribution over (0.1). By the computer 6000-simulations based on the uniform distribution. Table 3 in the Appendix shows asymptotic mean squared errors of two estimators $\hat{\rho}$ and $\tilde{\rho}$ in the reliability $R \equiv R(\delta_1,\delta_2)$ in the skew-symmetric double Rayleigh distribution with $\beta = 1$ when n and m are 10(10)30, $(\delta_1, \delta_2) = (4,2), (4,1), (4,1/2), (4,1/4), (1/2,1/4)$, and $c_i = \delta_i$ for each corresponding i=1 and 2 on account of the result in Fact 2. From Table 3 in the Appendix, since equivalence between inferences on $R(\delta_1,\delta_2)$ and ρ in McCool(1991), we observe the following Fact 4:

Fact 4. Assume $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_m$ be two independent samples each having the density $f(z;\delta_1)$ and $f(z;\delta_2)$ in the density (2.3) with known β_i, i=1,2, respectively and $\delta_i > 0$. When $(\delta_1, \delta_2) = (4,2), (4,1), (4,1/2), (4,1/4), (1/2,1/4)$ in (2.3), then the $\hat{R} = R(\hat{\delta}_1, \hat{\delta}_2)$ of the reliability $R \equiv R(\delta_1, \delta_2)$ performs better than another estimator $\tilde{R} = R(\tilde{\delta}_1, \tilde{\delta}_2)$ in a sense of simulated mean squared error.

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Appendix

<Table 1> Simulated mean squared errors of the AMLE and MME of the skewed parameter in the skew-symmetric double Rayleigh distribution with $\beta = 1$.

(a) $\delta = 0.5 = c$

	Average value		MSE	
n	AMLE	MME	AMLE	MME
10	0.516266	0.414580	0.000726	0.023904
20	0.514006	0.426977	0.000344	0.017422
30	0.513442	0.439959	0.000280	0.011745

(b) $\delta = -0.5 = c$

	Average value		MSE	
n	AMLE	MME	AMLE	MME
10	-0.516358	-0.424890	0.000553	0.020913
20	-0.503352	-0.430857	0.000539	0.020196
30	-0.502413	-0.447658	0.000483	0.016774

(c) $\delta = 1.0 = c$

	Average value		MSE	
n	AMLE	MME	AMLE	MME
10	1.014512	0.950524	0.000663	0.030196
20	1.009887	0.961787	0.000377	0.018012
30	1.005098	0.983413	0.000194	0.009876

(d) $\delta = -1.0 = c$

	Average value		MSE	
n	AMLE	MME	AMLE	MME
10	-0.980552	-0.948917	0.001072	0.027063
20	-0.987863	-0.960016	0.000779	0.018456
30	-0.990364	-0.971203	0.000432	0.010102

(e) $\delta = 2.0 = c$

	Average value		MSE	
	AMLE	MME	AMLE	MME
10	2.060427	1.900103	0.002640	0.029907
20	2.054082	1.925918	0.001249	0.016901
30	2.030396	1.930839	0.000901	0.011096

(f) $\delta = -2.0 = c$

	Average value		MSE	
n	AMLE	MME	AMLE	MME
10	-2.057208	-1.893024	0.004203	0.014260
20	-2.022069	-1.922990	0.002236	0.014178
30	-2.017469	-1.939120	0.001176	0.013985

<Table 2> Simulated mean squared errors of $\hat{\rho}$ and $\tilde{\rho}$ in the skew-symmetric double Rayleigh distribution with $\beta = 1$.

(a) $\delta_1 = 4.0$, $\delta_2 = 2.0$ (ρ = 0.484967)

		Average value		MSE	
n	m	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.492878	0.514086	0.001021	0.001511
	20	0.490988	0.513976	0.000788	0.001504
	30	0.489455	0.508968	0.000188	0.000909
20	10	0.491112	0.513447	0.000909	0.001440
	20	0.488946	0.509954	0.000270	0.000983
	30	0.488487	0.508769	0.000186	0.000899
30	10	0.488462	0.509068	0.000182	0.000927
	20	0.487957	0.506662	0.000065	0.000697
	30	0.487010	0.506611	0.000044	0.000684

(b) $\delta_1 = 4.0$, $\delta_2 = 0.5$ ($\rho = 0.323856$)

		Average value		MSE	
n	m	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.329746	0.403759	0.001016	0.007045
	20	0.327948	0.403115	0.000762	0.006909
	30	0.327548	0.398646	0.000207	0.005925
20	10	0.329622	0.403653	0.000885	0.007028
	20	0.327508	0.399626	0.000267	0.006099
	30	0.327124	0.397131	0.000203	0.005614
30	10	0.329379	0.398447	0.000225	0.005895
	20	0.327399	0.396936	0.000175	0.005479
	30	0.327010	0.395984	0.000159	0.005460

(c) $\delta_1 = 0.5$, $\delta_2 = 0.25$ ($\rho = 0.449614$)

		Average value		MSE	
n	m	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.423165	0.433299	0.005626	0.006049
	20	0.431164	0.435581	0.005448	0.006004
	30	0.435471	0.436830	0.004084	0.005627
20	10	0.428441	0.435936	0.004130	0.005963
	20	0.439038	0.438324	0.003755	0.005864
	30	0.440466	0.440358	0.003698	0.005456
30	10	0.435140	0.436947	0.002792	0.005466
	20	0.439624	0.439241	0.002628	0.005252
	30	0.441419	0.441976	0.002312	0.005144