Journal of Korean
Data & Information Science Society
2007, Vol. 18, No. 4, pp. 1191 ~ 1203

Bayesian Hypothesis Testing for Homogeneity of the Shape Parameters in the Gamma Populations

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Abstract

In this paper, we consider the hypothesis testing for the homogeneity of the shape parameters in the gamma distributions. The noninformative priors such as Jeffreys' prior or reference prior are usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the objective Bayesian testing procedure for the homogeneity of the shape parameters based on the fractional Bayes factor and the intrinsic Bayes factor under the reference prior. Simulation study and a real data example are provided.

Keywords : Fractional Bayes Factor, Intrinsic Bayes Factor, Reference Prior, Shape Parameter

1. Introduction

In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith

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(1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006).

Consider k independent gamma populations with the shape parameter α_i and the scale parameter $\beta_i, i=1,\cdots,k$. Let $X_{ij}, j=1,\cdots,n_i$, denote observations from the ith gamma population. Then the gamma distribution of X_{ij} is given by

$$f(x_{ij}) = \frac{x_{ij}^{\alpha_i - 1}}{\Gamma(\alpha_i)\beta_i^{\alpha_i}} \exp\left\{-\frac{x_{ij}}{\beta_i}\right\}, i=1,\cdots,k, j=1,\cdots,n_i, \tag{1}$$

where $\alpha_i > 0$ and $\beta_i > 0$. As noted by various authors (Lawless, 1982; Keating et. al., 1990, etc), the gamma distribution is widely used in reliability and survival analysis. In particular, the shape parameter is of special interest because the shape parameter less than 1, equal to 1 and greater than 1 correspond to a decreasing failure rate, a constant failure rate and an increasing failure rate, respectively. The testing problem of the homogeneity of the shape parameters can be motivated within competing risks theory (see Wong and Wu, 1998).

Wong and Wu (1998) compared the accuracy of tail probabilities obtained by various approximate inference procedures for the common shape parameter of the gamma distributions. They concluded that although the first order methods based on the maximum likelihood estimator and signed square root of the likelihood ratio statistic are the most common approximations used by applied statisticians, they sometimes give unsatisfactory or even misleading approximations, and all the third order methods give very similar results but the approximation using the exact conditional log likelihood function seems to be the best. However there is a little work in this problem from the viewpoint of objective Bayesian framework.

This paper focuses on Bayesian testing for the homogeneity of the shape parameters in the gamma distributions. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996, 1998).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factor. In Section 3, Using the reference prior, we provide the Bayesian testing procedure based on the fractional Bayes factor and intrinsic Bayes factor for testing the homogeneity of the shape parameters. In Section 4, simulation study and a real example are given.

2. Intrinsic and Fractional Bayes Factors

Hypotheses H_1, $H_2,\cdots$, H_q are under consideration, with the data $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ having probability density function $f_i(\boldsymbol{x}\mid\boldsymbol{\theta}_i)$ under model H_i. The parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distributions of model H_i, and let p_i be the prior probabilities of model $H_i, i=1,2,\cdots,q$. Then the posterior probability that the model H_i is true is

$$P(H_i\mid\boldsymbol{x})=\left(\sum_{j=1}^{q}\frac{p_j}{p_i}\cdot B_{ji}\right)^{-1}, \tag{2}$$

where B_{ji} is the Bayes factor of model H_j to model H_i defined by

$$B_{ji}=\frac{\int f_j(\boldsymbol{x}\mid\boldsymbol{\theta}_j)\pi_j(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x}\mid\boldsymbol{\theta}_i)\pi_i(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}=\frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{3}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Usually, one can use the noninformative prior such as uniform prior, Jeffreys' prior or reference prior in Bayesian analysis. Denote it as π_i^N. The use of noninformative priors $\pi_i^N(\cdot)$ in (3) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0<m_i^N(\boldsymbol{x}(l))<\infty, i=1,\cdots,q. \tag{4}$$

In view (4), the posteriors $\pi_i^N(\theta_i\mid\boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$, treating the remainder of data $\boldsymbol{x}(-l)$ as the likelihood function with $\pi_i^N(\theta_i\mid\boldsymbol{x}(l))$ being as the prior:

$$B_{ji}(l)=\frac{\int f(x(-l)\mid\theta_j,x(l))\pi_j^N(\theta_j\mid x(l))d\theta_j}{\int f(x(-l)\mid\theta_i,x(l))\pi_i^N(\theta_i\mid x(l))d\theta_i}=B_{ji}^N\cdot B_{ij}^N(x(l)), \tag{5}$$

where

$$B_{ji}^N=B_{ji}^N(x)=\frac{m_j^N(x)}{m_i^N(x)} \text{ and } B_{ij}^N(x(l))=\frac{m_i^N(x(l))}{m_j^N(x(l))}$$

are the Bayes factors that would be obtained for the full data x and training samples $x(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(x(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the Arithmetic Intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI}=B_{ji}^N\cdot\frac{1}{L}\sum_{l=1}^{L}B_{ij}^N(x(l)). \tag{6}$$

where L is the number of all possible minimal training samples. Also the Median Intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI}=B_{ji}^N\cdot ME[B_{ij}^N(x(l))], \tag{7}$$

where ME indicates the median. So we can also calculate the posterior probability of H_i using (2), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (6) and (7).

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i)=f_i(x\mid\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of model H_j versus H_i is

$$B_{ji}^F=B_{ji}^N\cdot\frac{\int L^b(\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\theta_j)\pi_j^N(\theta_j)d\theta_j}=B_{ji}^N\cdot\frac{m_i^b(x)}{m_j^b(x)}. \tag{8}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b=m/n$, where m is the size of the minimal training samples, assuming that this number is uniquely defined. (see O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

The IBF approaches for the hypothesis testing are the most generally applicable approach, but the IBF can be computationally intensive because of the minimal training samples. The FBF is easier to compute than the IBF. The FBF is typically available even for very small sample sizes, but their utility in such situations is tempered by the fact that the answer will typically then highly sensitive to the choice of the fraction b and no reasonable automatic choices for this fraction seem possible in such situation. The detailed advantages and disadvantages of IBF and FBF approaches are given in Berger and Pericchi (1996, 1998) and O'Hagan (1997).

3. Bayesian Testing Procedures

The joint probability density function of $\boldsymbol{X}$ by (1) is

$$f(\boldsymbol{x})=\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{x_{ij}^{\alpha_i-1}}{\Gamma(\alpha_i)\beta_i^{\alpha_i}}\exp\left\{-\frac{x_{ij}}{\beta_i}\right\}, \tag{9}$$

where $\boldsymbol{x}=(x_{11},\cdots,x_{1n_1},\cdots,x_{k1},\cdots,x_{kn_k})$, $\alpha_i>0$ and $\beta_i>0$. We want to test the hypothesis

$$H_1:\alpha_1=\cdots=\alpha_k \text{ vs. } H_2:\alpha_1\neq\cdots\neq\alpha_k. \tag{10}$$

Our interest is to develop the Bayesian testing procedure based on the fractional Bayes factor and the intrinsic Bayes factor for the hypothesis (10). Let $\mu_i=\alpha_i\beta_i, i=1,\cdots,k$. With this parameterization, the joint probability density function from the model (9) is given by

$$f(\boldsymbol{x})=\left[\prod_{i=1}^{k}\frac{1}{\Gamma(\alpha_i)^{n_i}}\left(\frac{\alpha_i}{\mu_i}\right)^{n_i\alpha_i}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}x_{ij}^{\alpha_i-1}\right]\exp\left\{-\sum_{i=1}^{k}\frac{\alpha_i x_{i\cdot}}{\mu_i}\right\}, \tag{11}$$

where $x_{i\cdot}=\sum_{j=1}^{n_i}x_{ij}, i=1,\cdots,k.$

3.1 Bayesian Testing Procedure based on the Fractional Bayes Factor

Under the hypothesis H_1, the reference prior for $\alpha(\equiv\alpha_1=\cdots=\alpha_k)$ and $(\mu_1,\cdots,\mu_k)$ is

$$\pi_1^N(\alpha,\mu_1,\cdots,\mu_k)\propto[\psi'(\alpha)-\alpha^{-1}]^{1/2}\mu_1^{-1}\cdots\mu^{-1}, \tag{12}$$

where $\psi'(\cdot)$ is the trigamma function. This reference prior is derived by Kang, Kim and Lee (2007). They showed that the posterior under the above reference prior is proper if $n_1+\cdots+n_k-k-2>0$. And the likelihood function under H_1 is

$$L(\alpha,\mu_1,\cdots,\mu_k\mid \boldsymbol{x})=\frac{1}{\Gamma(\alpha)^n}\left[\prod_{i=1}^{k}\left(\frac{\alpha}{\mu_i}\right)^{n_i\alpha}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}x_{ij}^{\alpha-1}\right]\exp\left\{-\sum_{i=1}^{k}\frac{\alpha x_{i\cdot}}{\mu_i}\right\}, \tag{13}$$

where $n=n_1+\cdots+n_k$. Then from the likelihood (13) and the reference prior (12), the element of FBF under H_1 is given by

$$m_1^b(\boldsymbol{x})=\int_0^\infty\int_0^\infty\cdots\int_0^\infty L^b(\alpha,\mu_1,\cdots,\mu_k\mid \boldsymbol{x})\pi_1^N(\alpha,\mu_1,\cdots,\mu_k)d\mu_1\cdots d\mu_k d\alpha$$

$$=\int_0^\infty b^{-nb\alpha}\frac{\prod_{i=1}^{k}\Gamma(n_ib\alpha)}{\Gamma(\alpha)^{nb}}\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{x_{ij}^{b(\alpha-1)}}{x_{i\cdot}^{b\alpha}}\right][\psi'(\alpha)-\alpha^{-1}]^{1/2}d\alpha.$$

For the hypothesis H_2, the reference prior for $(\alpha_1,\cdots,\alpha_k,\mu_1,\cdots,\mu_k)$ is

$$\pi_1^N(\alpha_1,\cdots,\alpha_k,\mu_1,\cdots,\mu_k)\propto\prod_{i=1}^{k}[\psi'(\alpha_i)-\alpha_i^{-1}]^{1/2}\mu_i^{-1}.$$

This reference prior is derived by Liseo (1993) and the posterior is proper if $n_i>2, i=1,\cdots,k$ (Garvan and Ghosh, 1999). The likelihood function under H_2 is

$$L(\alpha_1,\cdots,\alpha_k,\mu_1,\cdots,\mu_k\mid \boldsymbol{x})=\left[\prod_{i=1}^{k}\frac{1}{\Gamma(\alpha_i)^{n_i}}\left(\frac{\alpha_i}{\mu_i}\right)^{n_i\alpha_i}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}x_{ij}^{\alpha_i-1}\right]\exp\left\{-\sum_{i=1}^{k}\frac{\alpha_i x_{i\cdot}}{\mu_i}\right\}.$$

Thus the element of FBF under H_2 gives as follows.

$$m_2^b(\boldsymbol{x})=\int_0^\infty\cdots\int_0^\infty\int_0^\infty\cdots\int_0^\infty L^b(\alpha_1,\cdots,\alpha_k,\mu_1,\cdots,\mu_k\mid \boldsymbol{x})$$

$$\times \pi_2^N(\alpha_1,\cdots,\alpha_k,\mu_1\cdots,\mu_k)d\mu_1\cdots d\mu_k d\alpha_1\cdots d\alpha_k$$

$$=\prod_{i=1}^{k}\left(\int_0^\infty \frac{b^{-n_i b\alpha_i}\Gamma(n_i b\alpha_i)}{\Gamma(\alpha_i)^{n_i b}}\left[\prod_{j=1}^{n_i}\frac{x_{ij}^{b(\alpha_i-1)}}{x_{i\cdot}^{b\alpha_i}}\right]\left[\psi'(\alpha_i)-\alpha_i^{-1}\right]^{1/2}d\alpha_i\right).$$

Therefore the element B_{21}^N of the FBF is given by

$$B_{21}^N=\frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})}, \tag{14}$$

where

$$S_1(\boldsymbol{x})=\int_0^\infty \frac{\prod_{i=1}^{k}\Gamma(n_i\alpha)}{\Gamma(\alpha)^n}\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{x_{ij}^{\alpha}}{x_{i\cdot}^{\alpha}}\right]\left[\psi'(\alpha)-\alpha^{-1}\right]^{1/2}d\alpha$$

and

$$S_2(\boldsymbol{x})=\prod_{i=1}^{k}\left(\int_0^\infty \frac{\Gamma(n_i\alpha_i)}{\Gamma(\alpha_i)^{n_i}}\left[\prod_{j=1}^{n_i}\frac{x_{ij}^{\alpha_i}}{x_{i\cdot}^{\alpha_i}}\right]\left[\psi'(\alpha_i)-\alpha_i^{-1}\right]^{1/2}d\alpha_i\right).$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\boldsymbol{x},\boldsymbol{y})}{m_2^b(\boldsymbol{x},\boldsymbol{y})}=\frac{S_1(\boldsymbol{x},\boldsymbol{y};b)}{S_2(\boldsymbol{x},\boldsymbol{y};b)},$$

where

$$S_1(\boldsymbol{x};b)=\int_0^\infty b^{-nb\alpha}\frac{\prod_{i=1}^{k}\Gamma(n_i b\alpha)}{\Gamma(\alpha)^{nb}}\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{x_{ij}^{b\alpha}}{x_{i\cdot}^{b\alpha}}\right]\left[\psi'(\alpha)-\alpha^{-1}\right]^{1/2}d\alpha$$

and

$$S_2(\boldsymbol{x};b)=\prod_{i=1}^{k}\left(\int_0^\infty \frac{b^{-n_i b\alpha_i}\Gamma(n_i b\alpha_i)}{\Gamma(\alpha_i)^{n_i b}}\left[\prod_{j=1}^{n_i}\frac{x_{ij}^{b\alpha_i}}{x_{i\cdot}^{b\alpha_i}}\right]\left[\psi'(\alpha_i)-\alpha_i^{-1}\right]^{1/2}d\alpha_i\right).$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})} \cdot \frac{S_1(\boldsymbol{x};b)}{S_2(\boldsymbol{x};b)}. \tag{15}$$

Note that the calculation of the FBF of H_2 versus H_1 requires one dimensional integration.

3.2 Bayesian Testing Procedure based on the Intrinsic Bayes Factor

The element B_{21}^N, (14), of the intrinsic Bayes factor is computed in the FBF. So using minimal training sample, we only calculate the marginal densities under H_1 and H_2, respectively. The marginal density of $(X_{1j_1}, X_{1j_2}, X_{1j_3}, \cdots, X_{kl_1}, X_{kl_2}, X_{kl_3})$ is finite for all $1 \le j_1 < j_2 < j_3 \le n_1, \cdots, 1 \le l_1 < l_2 < l_3 \le n_k$ under each hypothesis (Garvan and Ghosh, 1999; Kim, Kang and Lee, 2007). Thus we conclude that any training sample of size $3k$ is a minimal training sample.

The marginal density $m_1^N(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3})$ under H_1 is given by

$$\begin{aligned} & m_1^N(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3}) \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3} \mid \mu_1, \cdots \mu_k, \alpha) \\ & \qquad \times \pi_1^N(\mu_1, \cdots \mu_k, \alpha) d\mu_1 \cdots d\mu_k d\alpha \\ &= \int_0^\infty \prod_{i=1}^k \Gamma(3\alpha)[\Gamma(\alpha)]^{-3k} [\psi'(\alpha) - \alpha^{-1}]^{\frac{1}{2}} \left[\prod_{i=1}^k \prod_{j=1}^3 \left(x_{ij} / \sum_{j=1}^3 x_{ij} \right)^\alpha \right] d\alpha, \end{aligned}$$

where $1 \le j_1 < j_2 < j_3 \le n_1, \cdots, 1 \le l_1 < l_2 < l_3 \le n_k$. And the marginal density $m_2^N(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3})$ under H_2 is given by

$$\begin{aligned} & m_2^N(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3}) \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1j_1}, x_{1j_2}, x_{1j_3}, \cdots, x_{kl_1}, x_{kl_2}, x_{kl_3} \mid \mu_1, \cdots \mu_k, \alpha_1, \cdots, \alpha_k) \\ & \qquad \times \pi_1^N(\mu_1, \cdots \mu_k, \alpha_1, \cdots, \alpha_k) d\mu_1 \cdots d\mu_k d\alpha_1 \cdots d\alpha_k \end{aligned}$$

$$= \prod_{i=1}^{k}\left(\int_0^\infty \frac{\Gamma(3\alpha_i)}{\Gamma(\alpha_i)^3}\left[\psi'(\alpha_i)-\alpha_i^{-1}\right]^{\frac{1}{2}}\left[\prod_{j=1}^{3}\left(x_{ij}/\sum_{j=1}^{3}x_{ij}\right)^{\alpha_i}\right]d\alpha_i\right)$$
$$\equiv T_2(x_{1j_1},x_{1j_2},x_{1j_3},\cdots,x_{kl_1},x_{kl_2},x_{kl_3}).$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI}=\frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})}\cdot\left[\frac{1}{L}\sum_{j_1,j_2,j_3}\cdots\sum_{l_1,l_2,l_3}\frac{T_1(x_{1j_1},x_{1j_2},x_{1j_3},\cdots,x_{kl_1},x_{kl_2},x_{kl_3})}{T_2(x_{1j_1},x_{1j_2},x_{1j_3},\cdots,x_{kl_1},x_{kl_2},x_{kl_3})}\right]. \quad (16)$$

where $L=[\prod_{i=1}^{k}n_i(n_i-1)(n_i-2)]/6^k$. And the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI}=\frac{S_2(\boldsymbol{x})}{S_1(\boldsymbol{x})}\cdot ME\left[\frac{T_1(x_{1j_1},x_{1j_2},x_{1j_3},\cdots,x_{kl_1},x_{kl_2},x_{kl_3})}{T_2(x_{1j_1},x_{1j_2},x_{1j_3},\cdots,x_{kl_1},x_{kl_2},x_{kl_3})}\right]. \quad (17)$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require one dimensional integration. In Section 4, we investigate our hypothesis testing procedures.

4. Numerical Studies

In order to assess the Bayesian testing procedures, we evaluate the posterior probability for several configurations (α_i,μ_i), n_i, $i=1,\cdots,k$ and k. In particular, for fixed $(\alpha_i,\mu_i), i=1,\cdots,k$, we take 200 random samples in each population. In our simulation, we examine the cases when the k equals 2 and 3.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Table 1 and 2 show the results of the averages and the standard deviations in parentheses of posterior probabilities. In computing the FBF, we use the fraction $b=3k/n$ where $3k$ is the size of the minimal training samples and $n=\Sigma_{i=1}^{k}n_i$. From the results of the Table 1 and 2, the FBF, the AIBF and the

Table 1: The averages and the standard deviations in parentheses of posterior probabilities

α_1,α_2	μ_1,μ_2	n_1,n_2	$P^F(M_1 \mid x)$	$P^{AI}(M_1 \mid x)$	$P^{MI}(M_1 \mid x)$
0.5,0.5	1,1	5,5	0.513 (0.097)	0.550 (0.129)	0.537 (0.111)
		5,10	0.585 (0.116)	0.607 (0.125)	0.589 (0.112)
		10,10	0.610 (0.091)	0.679 (0.104)	0.659 (0.102)
		10,15	0.624 (0.119)	0.681 (0.126)	0.662 (0.122)
	1,5	5,5	0.518 (0.099)	0.553 (0.130)	0.539 (0.109)
		5,10	0.579 (0.114)	0.598 (0.127)	0.581 (0.118)
		10,10	0.598 (0.118)	0.665 (0.134)	0.646 (0.129)
		10,15	0.635 (0.110)	0.691 (0.119)	0.671 (0.116)
	0.1,10	5,5	0.531 (0.068)	0.569 (0.095)	0.554 (0.081)
		5,10	0.569 (0.132)	0.587 (0.144)	0.570 (0.130)
		10,10	0.598 (0.116)	0.664 (0.131)	0.644 (0.127)
		10,15	0.619 (0.134)	0.674 (0.145)	0.655 (0.142)
0.5,1.5	1,1	5,5	0.454 (0.119)	0.466 (0.156)	0.470 (0.129)
		5,10	0.452 (0.189)	0.425 (0.221)	0.422 (0.207)
		10,10	0.345 (0.205)	0.378 (0.235)	0.375 (0.224)
		10,15	0.305 (0.246)	0.314 (0.270)	0.308 (0.260)
	1,5	5,5	0.430 (0.140)	0.437 (0.180)	0.442 (0.152)
		5,10	0.437 (0.197)	0.408 (0.231)	0.409 (0.216)
		10,10	0.362 (0.211)	0.395 (0.241)	0.390 (0.230)
		10,15	0.309 (0.230)	0.319 (0.254)	0.314 (0.245)
	0.1,10	5,5	0.446 (0.133)	0.456 (0.174)	0.459 (0.146)
		5,10	0.439 (0.196)	0.408 (0.224)	0.407 (0.209)
		10,10	0.367 (0.205)	0.401 (0.235)	0.396 (0.222)
		10,15	0.336 (0.233)	0.348 (0.260)	0.342 (0.249)
1,1	1,1	5,5	0.530 (0.075)	0.568 (0.105)	0.553 (0.090)
		5,10	0.586 (0.107)	0.604 (0.117)	0.585 (0.107)
		10,10	0.591 (0.124)	0.655 (0.141)	0.637 (0.136)
		10,15	0.616 (0.133)	0.672 (0.143)	0.653 (0.140)
	1,5	5,5	0.525 (0.083)	0.559 (0.112)	0.543 (0.095)
		5,10	0.575 (0.130)	0.592 (0.145)	0.576 (0.134)
		10,10	0.587 (0.120)	0.649 (0.137)	0.631 (0.132)
		10,15	0.619 (0.131)	0.676 (0.143)	0.657 (0.139)
	0.1,10	5,5	0.534 (0.068)	0.576 (0.097)	0.560 (0.081)
		5,10	0.589 (0.111)	0.612 (0.122)	0.595 (0.112)
		10,10	0.592 (0.129)	0.656 (0.146)	0.638 (0.140)
		10,15	0.630 (0.120)	0.684 (0.128)	0.664 (0.125)
1,3	1,1	5,5	0.473 (0.120)	0.489 (0.155)	0.487 (0.134)
		5,10	0.506 (0.177)	0.486 (0.207)	0.479 (0.193)
		10,10	0.367 (0.225)	0.397 (0.254)	0.390 (0.242)
		10,15	0.335 (0.230)	0.343 (0.253)	0.337 (0.244)
	1,5	5,5	0.456 (0.133)	0.464 (0.174)	0.466 (0.145)
		5,10	0.446 (0.196)	0.415 (0.225)	0.413 (0.211)
		10,10	0.378 (0.211)	0.409 (0.239)	0.403 (0.228)
		10,15	0.350 (0.244)	0.359 (0.268)	0.351 (0.258)
	0.1,10	5,5	0.453 (0.135)	0.462 (0.174)	0.465 (0.148)
		5,10	0.469 (0.194)	0.441 (0.222)	0.436 (0.208)
		10,10	0.371 (0.213)	0.402 (0.241)	0.397 (0.229)
		10,15	0.349 (0.235)	0.358 (0.257)	0.350 (0.248)

Table 2: The averages and the standard deviations in parentheses of posterior probabilities

$\alpha_1,\alpha_2,\alpha_3$	μ_1,μ_2,μ_3	n_1,n_2,n_3	$P^F(M_1 \mid x)$	$P^{AI}(M_1 \mid x)$	$P^{MI}(M_1 \mid x)$
0.5,0.5,0.5	1,1,1	5,5,5	0.537 (0.140)	0.609 (0.178)	0.627(0.161)
		5,5,7	0.579 (0.127)	0.649 (0.152)	0.665(0.133)
		5,7,10	0.629 (0.148)	0.695 (0.156)	0.714(0.141)
	1,1,5	5,5,5	0.543 (0.134)	0.616 (0.171)	0.634(0.152)
		5,5,7	0.567 (0.147)	0.641 (0.176)	0.658(0.156)
		5,7,10	0.617 (0.160)	0.680 (0.178)	0.698(0.162)
	0.1,1,10	5,5,5	0.543 (0.122)	0.613 (0.157)	0.629(0.138)
		5,5,7	0.563 (0.150)	0.631 (0.179)	0.649(0.161)
		5,7,10	0.635 (0.152)	0.700 (0.159)	0.716(0.140)
0.5,0.5,1.5	1,1,1	5,5,5	0.439 (0.159)	0.484 (0.202)	0.519(0.188)
		5,5,7	0.436 (0.184)	0.451 (0.237)	0.494(0.224)
		5,7,10	0.425 (0.210)	0.421 (0.249)	0.457(0.240)
	1,1,5	5,5,5	0.444 (0.155)	0.490 (0.201)	0.530(0.180)
		5,5,7	0.429 (0.182)	0.441 (0.235)	0.484(0.219)
		5,7,10	0.456 (0.219)	0.457 (0.257)	0.489(0.250)
	0.1,1,10	5,5,5	0.459 (0.155)	0.504 (0.202)	0.541(0.175)
		5,5,7	0.442 (0.170)	0.462 (0.221)	0.509(0.212)
		5,7,10	0.432 (0.225)	0.427 (0.265)	0.460(0.258)
1,1,1	1,1,1	5,5,5	0.552 (0.116)	0.621 (0.152)	0.635(0.136)
		5,5,7	0.581 (0.131)	0.660 (0.157)	0.677(0.140)
		5,7,10	0.629 (0.146)	0.695 (0.158)	0.711(0.140)
	1,1,5	5,5,5	0.550 (0.109)	0.618 (0.143)	0.633(0.128)
		5,5,7	0.576 (0.128)	0.651 (0.158)	0.670(0.141)
		5,7,10	0.624 (0.171)	0.689 (0.184)	0.705(0.166)
	0.1,1,10	5,5,5	0.548 (0.110)	0.623 (0.142)	0.636(0.130)
		5,5,7	0.588 (0.124)	0.665 (0.150)	0.683(0.133)
		5,7,10	0.618 (0.162)	0.679 (0.177)	0.697(0.163)
1,1,3	1,1,1	5,5,5	0.456 (0.152)	0.500 (0.195)	0.531(0.173)
		5,5,7	0.470 (0.167)	0.487 (0.215)	0.516(0.197)
		5,7,10	0.466 (0.235)	0.470 (0.275)	0.499(0.266)
	1,1,5	5,5,5	0.474 (0.154)	0.520 (0.199)	0.552(0.177)
		5,5,7	0.445 (0.180)	0.460 (0.233)	0.497(0.220)
		5,7,10	0.479 (0.207)	0.481 (0.242)	0.511(0.232)
	0.1,1,10	5,5,5	0.473 (0.154)	0.519 (0.199)	0.548(0.177)
		5,5,7	0.472 (0.181)	0.492 (0.225)	0.521(0.212)
		5,7,10	0.445 (0.220)	0.445 (0.259)	0.475(0.249)
3,3,3	1,1,1	5,5,5	0.554 (0.121)	0.599 (0.166)	0.616 (0.143)
		5,5,7	0.565 (0.146)	0.611 (0.186)	0.628 (0.169)
		5,7,10	0.642 (0.154)	0.683 (0.176)	0.694 (0.164)
	1,1,5	5,5,5	0.554 (0.108)	0.605 (0.148)	0.623 (0.135)
		5,5,7	0.584 (0.119)	0.629 (0.153)	0.646 (0.136)
		5,7,10	0.633 (0.157)	0.674 (0.178)	0.689 (0.165)
	0.1,1,10	5,5,5	0.549 (0.127)	0.597 (0.171)	0.614 (0.151)
		5,5,7	0.575 (0.134)	0.628 (0.168)	0.642 (0.156)
		5,7,10	0.607 (0.173)	0.650 (0.187)	0.666 (0.175)
0.5,1,3	1,1,1	5,5,5	0.357 (0.179)	0.374 (0.227)	0.422 (0.207)
		5,5,7	0.329 (0.199)	0.316 (0.239)	0.357 (0.232)
		5,7,10	0.289 (0.229)	0.273 (0.256)	0.307 (0.257)
	1,1,5	5,5,5	0.375 (0.168)	0.400 (0.212)	0.447 (0.192)
		5,5,7	0.348 (0.198)	0.341 (0.242)	0.384 (0.236)
		5,7,10	0.312 (0.230)	0.296 (0.258)	0.330 (0.259)
	0.1,1,10	5,5,5	0.369 (0.173)	0.392 (0.221)	0.440 (0.204)
		5,5,7	0.333 (0.182)	0.324 (0.227)	0.373 (0.223)
		5,7,10	0.320 (0.229)	0.300 (0.256)	0.337 (0.253)

MIBF give fairly reasonable answers for all configurations, $(\alpha_i,\mu_i), i=1,\cdots,k$. Also all of Bayes factors give a similar behavior for all sample sizes, but the MIBF favors the hypothesis H_1 than the FBF and the AIBF for the case $(\alpha_1,\alpha_2,\alpha_3) = (0.5,0.5,1.5)$ and $(1,1,3)$ in small sample size.

Example. This example is given as Example T with the Proschan data in Cox and Snell (1981), and Wong and Wu (1998) obtained 90%, 95% and 99% confidence intervals of the common shape parameter based on the third order approximation methods. The confidence intervals by the third order methods are almost indistinguishable and the 90% confidence interval by the approximation method using the exact conditional log-likelihood function is (0.829, 1.114). We select two data sets (aircraft 1 and 6) from Example T in Cox and Snell (1981) for our illustration. For aircraft 1 and 6, the maximum likelihood estimates of the mean and the shape parameter are 95.7 and 0.97, and 76.8 and 1.13, respectively.

The values of the Bayes factor and the posterior probability of H_2 versus H_1 are given Table 3. We assume that the prior probabilities are equal. From the results of Table 3, the values of the FBF, the AIBF and the MIBF give the evidence of H_1 and coincide with the frequentist result (Wong and Wu, 1998).

Table 3 : The Bayes Factors and Posterior Probabilities

BF_{21}^F	BF_{21}^{AI}	BF_{21}^{MI}	$P^F(H_1 \mid x)$	$P^{AI}(H_1 \mid x)$	$P^{MI}(H_1 \mid x)$
0.306	0.211	0.234	0.766	0.826	0.810

5. Concluding Remark

In gamma populations, we developed the objective Bayesian testing procedures based on the fractional Bayes factor and intrinsic Bayes factor for testing the homogeneity of the shape parameters under the reference prior. From our numerical results, the developed testing procedures give fairly reasonable answers for all parameter configurations. In practical application with moderate sample size and population number, we recommend to use the FBF because of computing marginal densities based on minimal training sample in AIBF and MIBF need much time.

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[received date : Sep. 2007, accepted date : Oct. 2007]